QUANTIZATION OF NON-ABELIAN GAUGE THEORY (QCD): PATH INTEGRAL FORMALISM

⇒ RECALL: PATH INTEGRAL QUANTIZATION FOR SCALAR FIELD THEORY

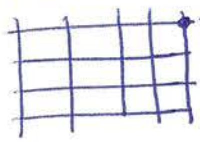
- GENERATING FUNCTIONAL FOR GREEN'S FUNCTIONS

$$Z[J] = \underset{\substack{\uparrow \\ \text{NORMALIZATION} \\ \text{FACTOR}}}{N} \int [d\phi] \exp\left\{ i \int d^4x \left(\mathcal{L} + \phi(x) J(x) \right) \right\}$$

NORMALIZATION FACTOR ⟶ DETERMINED FROM $Z[0]$

↳ $[d\phi]$ IMPLIES AN INTEGRAL OVER ALL VALUES $\phi(x)$ CAN TAKE IN EVERY SPACE-TIME POINT
⇒ INFINITE DIM $\int$

↳ TO EVALUATE SUCH INTEGRALS NUMERICALLY: DISCRETIZE SPACE TIME (EUCLIDEAN) ⇒ LATTICE



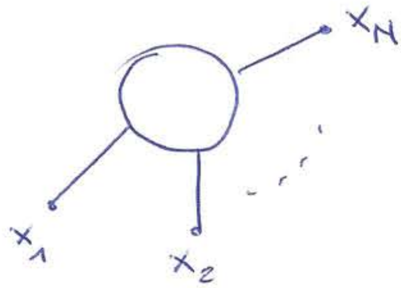
DEGREES OF FREEDOM $\phi(x_i)$ ON EACH LATTICE POINT

↳ e^{iS} ACTION $S = \int d^4x \, \mathcal{L}$

EACH FIELD CONFIGURATION IS WEIGHTED BY e^{iS}

↳ ϕJ TERM : COUPLING TO EXTERNAL SOURCES
↓
EXTERNAL PARTICLES IN PROCESS

- N-POINT GREEN'S FUNCTION (CORRELATION FUNCTION)



e.g. $N=2$

PROPAGATOR

↳ IN SECOND QUANTIZATION

$$G^{(N)}(x_1 \dots x_N) = \langle 0 | T\left[\hat{\phi}(x_1) \dots \hat{\phi}(x_N)\right] | 0 \rangle$$

$\hat{\phi}(x_i)$: FIELD OPERATORS CREATES/ANNIHILATES SCALAR PARTICLE IN SPACE-TIME POINT x_i

↳ IN PATH INTEGRAL QUANTIZATION

$$G^{(N)}(x_1 \dots x_N) = \frac{1}{Z[0]} \cdot (-i)^N \frac{\delta^N Z[J]}{\delta J(x_1) \dots \delta J(x_N)}\bigg|_{J=0}$$

$$= \frac{\int [d\phi]\, \phi(x_1) \dots \phi(x_N)\, e^{iS}}{\int [d\phi]\, e^{iS}}$$

$\phi(x)$: REAL FUNCTION (OR COMPLEX FOR CHARGED SPIN-0 FIELD)

- RELATION $Z[J]$ AND $G^{(N)}(x_1 \dots x_N)$

$$\frac{Z[J]}{Z[0]} = \sum_N \frac{i^N}{N!} \int d^4x_1 \dots d^4x_N \; G^{(N)}(x_1 \dots x_N)\, J(x_1) \dots J(x_N)$$

⟹ EXTENSION OF PATH INTEGRAL QUANTIZATION TO NON-ABELIAN GAUGE THEORY : ISSUES

↳ $$\mathcal{L}_{QCD} = \bar{q}(i\gamma^\mu D_\mu - m)q - \frac{1}{4}F_{a\mu\nu}F_a^{\mu\nu}$$

$$D_\mu = \partial_\mu + ig\frac{\lambda_a}{2}A_{a\mu} \qquad a = 1\ldots 8$$

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f_{abc} A_b^\mu A_c^\nu$$

↳ THEORY HAS $SU(3)_c$ (COLOR) GAUGE SYMMETRY

$\mathcal{L}_{QCD}$ INV. UNDER LOCAL TF :

$$q(x) \xrightarrow{SU(3)_c} \exp\left\{i\Theta_a(x)\frac{\lambda_a}{2}\right\} q(x)$$

$$A_a^\mu(x) \xrightarrow{SU(3)_c} A_a^\mu - \frac{1}{g}\partial^\mu\Theta_a - f_{abc}\Theta_b A_c^\mu$$

↳ $$\mathcal{L}_{YM} = -\frac{1}{4}F_{a\mu\nu}F_a^{\mu\nu}$$ YANG-MILLS TERM

CONTAINS $3g$ AND $4g$ INTERACTIONS

LET'S CONSIDER THIS TERM

AND NAIVELY APPLY PATH INTEGRAL FORMALISM

↳ $\tilde{Z}[\mathbf{0}] \sim \int [dA] \exp\left\{ i \int d^4x \, \mathcal{L}_{YM} \right\}$

$$[dA] \equiv \prod_{\mu=0}^{3} \prod_{a=1}^{8} [dA_a^\mu]$$

WHERE $\int [dA_a^\mu]$ IMPLIES AN INTEGRAL OVER ALL VALUES WHICH COMPONENT μ, a OF $A_a^\mu(x)$ CAN TAKE IN EVERY SPACE-TIME POINT

(IN COMPARISON TO SCALAR FIELD CASE $4 \cdot 8 = 32$ DEGREES OF FREEDOM IN EVERY SPACE-TIME POINT).

↳ $\mathcal{L}_{YM}$ IS INV. UNDER GAUGE TF

$$A_a^\mu \xrightarrow{SU(3)_c} A_a'^\mu \equiv A_a^{\mu(\theta)} = A_a^\mu - \frac{1}{g} \partial^\mu \theta_a - f_{abc} \theta_b A_c^\mu$$

START FROM ONE FIELD CONFIGURATION IN PATH INTEGRAL

$$A_a^{\mu(\theta=0)} \xrightarrow{SU(3)_c} A_a^{\mu(\theta)}$$

θ DEFINES SUBSET OF FIELD CONFIGURATIONS

ACTION S IS CONSTANT FOR ALL $A_a^{\mu(\theta)}$ IN THIS SUBSET OF FIELD CONFIGURATION

∴ ISSUE: FUNCTIONAL INTEGRAL OVER THIS SUBSET DIVERGES AS REGION OF INTEGRAL OVER SUBSET IS INFINITE

ORIGIN OF PROBLEM

$\forall a$: A^μ_a : ONLY 2 TRANSVERSE COMPONENTS ARE DYNAMICAL DEGREES OF FREEDOM

(cf. SECOND QUANTIZATION OF PHOTON FIELD)

BUT IN PATH INTEGRAL:

WE INTEGRATE OVER ALL FIELD CONFIGURATIONS FOR 4 COMPONENTS $\mu = 0, 1, 2, 3$

$\Downarrow$

ILL DEFINED INTEGRAL

$\Rightarrow$ SOLUTION : A MATHEMATICAL ANALOGY

↳ CONSIDER A WELL-DEFINED (CONVERGENT) n-DIM INTEGRAL OVER VARIABLES $z_{k+1}, \dots, z_{k+n}$

$$Z = \int_{-\infty}^{+\infty} dz_{k+1} \cdots \int_{-\infty}^{+\infty} dz_{k+n} \; e^{-X}$$

WITH 'ACTION' $X = X(z_{k+1}, \dots, z_{k+n})$

e.g. $X = \frac{1}{2}\left(z_{k+1}^2 + \dots + z_{k+n}^2\right)$ GAUSSIAN ACTION

↳ ADD k VARIABLES $z_1 \dots z_k$

X DOES NOT DEPEND UPON $z_1 \dots z_k$

INTEGRAL $$\tilde{Z} = \underbrace{\int_{-\infty}^{+\infty} dz_1 \cdots \int_{-\infty}^{+\infty} dz_k}_{\substack{\infty \\ \text{DIVERGENT} \\ \text{RANGE}}} \; \underbrace{\int_{-\infty}^{+\infty} dz_{k+1} \cdots \int_{-\infty}^{+\infty} dz_{k+n} \, e^{-X}}_{Z}$$

↳ QUESTION : CAN WE FIND A WELL-DEFINED $k+n$ DIM INTEGRAL WHICH IS EQUIVALENT TO Z ?

INTRODUCE k CONSTRAINTS :

$$f_i(z_1, \dots, z_{k+n}) = 0 \qquad i = 1 \dots k$$

$$Z = \int_{-\infty}^{+\infty} df_1 \cdots \int_{-\infty}^{+\infty} df_k \int_{-\infty}^{+\infty} dz_{k+1} \cdots \int_{-\infty}^{+\infty} dz_{k+n} \, e^{-X} \underbrace{\prod_{i=1}^{k} \delta(f_i)}_{k\ \delta\text{-FUNCTIONS}}$$

↳ CHANGE VARIABLES $f_1 \dots f_k \Rightarrow z_1 \dots z_k$

$$Z = \int dz_1 \dots dz_k \, dz_{k+1} \dots dz_{k+m} \underbrace{\det\left(\frac{\partial f}{\partial z}\right)}_{\text{JACOBIAN}} e^{-X} \prod_{i=1}^{k} \delta(f_i)$$

↳ USE GRASSMANN AUXILIARY VARIABLES TO EXPONENTIATE THE DETERMINANT

GRASSMANN VARIABLES $\chi_i \, , \, \bar{\chi}_i \qquad i = 1 \dots k$

$$\chi_i^2 = 0 \, , \quad \bar{\chi}_i^2 = 0$$

$$\int d\chi_i = 0 \, , \quad \int d\chi_i \, \chi_j = \delta_{ij}$$

$$\int d\bar{\chi}_i = 0 \, , \quad \int d\bar{\chi}_i \, \bar{\chi}_j = \delta_{ij}$$

$$\exp\left\{ - \bar{\chi}_i A_{ij} \chi_j \right\}$$

$$= 1 - (\bar{\chi}_i A_{ij} \chi_j) + \frac{1}{2!} (\bar{\chi}_i A_{ij} \chi_j)^2 + \dots + \frac{(-1)^k}{k!} (\bar{\chi}_i A_{ij} \chi_j)^k$$

SERIES TERMINATES AS $\chi_i^2 = 0, \ \bar{\chi}_i^2 = 0$

<u>IDENTITY</u>:

$$\det A = \int \prod_{\ell=1}^{k} \left(d\bar{\chi}_\ell \, d\chi_\ell \right) \exp\left\{ - \bar{\chi}_i A_{ij} \chi_j \right\}$$

$$\hookrightarrow \quad \det\left(\frac{\partial f}{\partial z}\right) = \int d\bar{\chi}_1 \, d\chi_1 \dots d\bar{\chi}_k \, d\chi_k \, \exp\left\{-\bar{\chi}_i \left(\frac{\partial f_i}{\partial z_j}\right) \chi_j\right\}$$

$$Z = \int \left(\prod_{i=1}^{k+m} dz_i\right) \left(\prod_{j=1}^{k} d\bar{\chi}_j \, d\chi_j\right) e^{-\tilde{X}} \underbrace{\prod_{l=1}^{k} \delta(f_l)}_{k\ \delta\text{-FUNCTIONS}}$$

INTEGRAL OVER $k+m$ VARIABLES

INTEGRAL OVER AUXILIARY GRASSMANN (GHOST) VARIABLES

$\tilde{X}$: MODIFIED ACTION

$$\tilde{X} = X + \sum_{i,j=1}^{k} \bar{\chi}_i \left(\frac{\partial f_i}{\partial z_j}\right) \chi_j$$

⇒ SOLUTION : FADDEEV-POPOV PROCEDURE

↳ CONSTRAINTS : GAUGE-FIXING CONDITIONS

$$f_a\left(A^{(\theta)}_{(x)}\right) = B_a(x) \qquad a = 1 \dots 8$$

EXAMPLES :

1) COULOMB GAUGE

$$f_a = \bar{\nabla} \cdot \bar{A}_a \quad , \quad B_a = 0$$

2) LORENZ GAUGE

$$f_a = \partial_\mu A^\mu_a \quad , \quad B_a = 0$$

3) AXIAL GAUGE

$$f_a = n_\mu A^\mu_a \quad , \quad B_a = 0$$

$$n^\mu (0,0,0,1)$$

4) TEMPORAL GAUGE

$$f_a = \tilde{n}_\mu A^\mu_a \quad , \quad B_a = 0$$

$$\tilde{n}^\mu (1,0,0,0)$$

$$\hookrightarrow \quad \tilde{Z}[0] \sim \int [dA] \exp\{i \int d^4x \, \mathcal{L}_{YM}\}$$

$$\downarrow \quad \int \prod_{a=1}^{8} [d\Theta_a] \, \delta\left(f_a(A^{(\Theta)}) - B_a\right) = \frac{1}{\det M_f}$$

$$\text{WITH } \underline{\underline{(M_f)_{ab}^{(x,y)} = \frac{\partial f_a(x)}{\partial \Theta_b(y)}}}$$

$$\tilde{Z}[0] \sim \int \prod_a [d\Theta_a] \int [dA] \det M_f \left(\prod_a \delta\left(f_a(A^{(\Theta)}) - B_a\right)\right) \cdot \exp\{i \int d^4x \, \mathcal{L}_{YM}\}$$

$\mathcal{L}_{YM}$, $\det M_f$, $[dA]$ INV. UNDER GAUGE TF

$\Rightarrow$ IN δ - FUNCTION : WE MAY REPLACE

$$A_a^{\mu(\Theta)} \to A_a^{\mu}$$

$$\tilde{Z}[0] \sim \underbrace{\left(\int \prod_a [d\Theta_a]\right)}_{\text{INFINITE CONSTANT}} \int [dA] \det M_f \left(\prod_{a,x} \delta\left(f_a(A) - B_a\right)\right) \cdot \exp\{i \int d^4x \, \mathcal{L}_{YM}\}$$

INSTEAD OF USING $\tilde{Z}[0]$

WE DEFINE A FUNCTIONAL $Z[0]$

BY REMOVING INFINITE CONSTANT

$$Z[0] = \int [dA] \det M_f \prod_{a,x} \delta(f_a(A) - B_a)\, e^{i\int d^4x\, \mathcal{L}_{YM}}$$

AND LIKEWISE IN PRESENCE OF EXTERNAL SOURCES

$$Z[J] = \int [dA] \det M_f \left(\prod_{a,x} \delta(f_a(A) - B_a) \right) \cdot \exp\left\{ i \int d^4x \left[\mathcal{L}_{YM} + A_a^\mu J_{a\mu} \right] \right\}$$

↳ AVERAGE OVER $B_a(x)$: GAUGE FIXING

SINCE $B_a(x)$ IS ARBITRARY

WE AVERAGE $Z[J]$ OVER B_a

WITH EXPONENTIAL WEIGHT

$$\exp\left\{ -\frac{i}{2\xi} \int d^4x \left(B_a(x) \right)^2 \right\}$$

WHERE GAUGE PARAMETER ξ IS AN ARBITRARY CONSTANT

$$\therefore \quad Z[J] = \int [dA] \det M_f \cdot \exp\left\{ i \int d^4x \left[\mathcal{L}_{YM} \underbrace{- \frac{1}{2\xi} \left(f_a(A) \right)^2}_{\text{GAUGE-FIXING TERM}} + A_a^\mu J_{a\mu} \right] \right\}$$

↳ FADDEEV - POPOV GHOST FIELDS

- IN COVARIANT GAUGE

e.g. LORENZ GAUGE

$$f_a(A^{(\theta)}) = \partial_\mu A_a^{(\theta)\mu} = \partial_\mu \left[A_a^\mu - \frac{1}{g} \partial^\mu \theta_a - f_{abc} \theta_b A_c^\mu \right]$$

$$(M_f)_{ab}(x,y) = \frac{\partial f_a(x)}{\partial \theta_b(y)}$$

$$(M_f)_{ab}(x,y) = -\frac{1}{g} \partial_\mu \left[\partial^\mu \delta_{ab} + g f_{abc} A_c^\mu \right] \delta^4(x-y)$$

$$\det M_f \sim \int [d\bar{\chi}\, d\chi] \exp\{ \underbrace{ig}_{\substack{\text{YIELDS CONSTANT} \\ \text{OVERALL FACTOR (IRRELEVANT)}}} \int d^4x \int d^4y\, \bar{\chi}_a(x) M_f(x,y) \chi_b(y) \}$$

$$\boxed{\det M_f \simeq \int \underbrace{[d\bar{\chi}\, d\chi]}_{\substack{\text{AUXILIARY} \\ \text{GHOST FIELDS} \\ \text{(GRASSMANN VARIABLES)}}} \exp\left\{ i \int d^4x \left[(\partial_\mu \bar{\chi}_a)(\partial^\mu \chi_a) + (\partial_\mu \bar{\chi}_a)\, g f_{abc} \chi_b A_c^\mu \right] \right\}}$$

- <u>NOTICE</u> : IN ABELIAN CASE $f_{abc} = 0$

$\Rightarrow \det M_f$ IS CONSTANT $\Rightarrow$ NO GHOSTS !

↳ IN TOTAL.

INTRODUCE ALSO SOURCES σ FOR $\bar{\chi}$

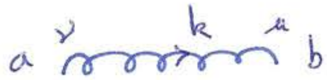
$\bar{\sigma}$ FOR χ

$$\boxed{\begin{aligned} &Z[J,\sigma,\bar{\sigma}] \\ &= \int [dA] \int [d\bar{\chi}\, d\chi] \exp\Big\{ i \int d^4x \Big[\mathcal{L}_{YM} + \mathcal{L}_{GF} + \mathcal{L}_{GHOST} \\ &\qquad + J_a^{\mu} A_{a\mu} + \bar{\sigma}\chi + \bar{\chi}\sigma \Big]\Big\} \end{aligned}}$$

$\Rightarrow$ $\mathcal{L}_{GF}$: GAUGE-FIXING

FEYNMAN RULE GLUON PROP.

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} \left(\partial_\mu A_a^\mu \right)^2 \quad \Rightarrow$$

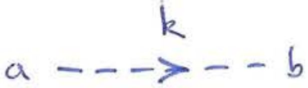


$$\boxed{\delta_{ab} \frac{i}{k^2+i\varepsilon} \left[-g^{\mu\nu} + (1-\xi) \frac{k^\mu k^\nu}{k^2} \right]}$$

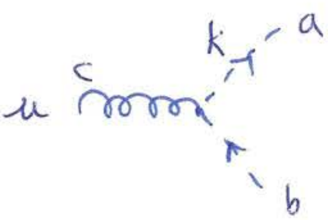
$\Rightarrow$ $\mathcal{L}_{GHOST}$: GHOST TERM

$$\mathcal{L}_{GHOST} = \left(\partial_\mu \bar{\chi}_a \right) \left(\partial^\mu \chi_a \right) + \left(\partial_\mu \bar{\chi}_a \right) g f_{abc} \chi_b A_c^\mu$$

FEYNMAN RULES GHOSTS



$$\boxed{\frac{i}{k^2+i\varepsilon} \delta_{ab}}$$



$$\boxed{-g f_{abc} k^\mu}$$

↳ ABSENCE OF GHOSTS IN AXIAL / TEMPORAL GAUGES

e.g. AXIAL GAUGE

$$n_\mu A_a^\mu = 0$$

$$f_a(A^{(\theta)}) = n_\mu A_a^{\mu(\theta)} = n_\mu \left[A_a^\mu - \frac{1}{g} \partial^\mu \theta_b - f_{abc} \theta_b A_c^\mu \right]$$

$$(M_f)_{ab}(x,y) = \frac{\partial f_a(x)}{\partial \theta_b(y)}$$

$$= -\frac{1}{g} \Big[\delta_{ab} \; n_\mu \partial^\mu + g f_{abc} \underbrace{n_\mu A_c^\mu}_{=0} \Big] \delta^4(x-y)$$

$$= -\frac{1}{g} \delta_{ab} \left(n_\mu \partial^\mu \right) \delta^4(x-y)$$

$\therefore$ $\det M_f$ DOES NOT DEPEND ON A^μ

↳ $\det M_f$ IS CONSTANT

$\Downarrow$

NO NEED FOR GHOSTS IN AXIAL GAUGE

⇝ IN PRACTISE ONE OFTEN USES COVARIANT GAUGES (SUCH AS LORENZ GAUGE) IN CALCULATIONS AND ADDS GHOSTS