Quantum Field Theory II:

Introduction to path-integral quantization of gauge theories

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Fachbereich 08 — Physik, Mathematik und Informatik

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Lecture 1

Originally the concept of a field as physical reality was introduced in the 19th century by Michael Faraday in his description of electricity and magnetism. It did not receive much attention until the works of J. C. Maxwell on laws of electromagnetism, now known as "Maxwell equations".

In contemporary physics the concept of a field is used not only to describe electromagnetism but basically everything that "lives" in spacetime — particles of matter, forces between them, collective excitations, even the vacuum.

1.1 The field concept

A *field* is a generic physical entity that "lives" in space $\vec{x}$ and time t, can carry momentum $\vec{p}$ and energy E, and, possibly, has intrinsic degrees of freedom such as spin. Mathematically, a field is a function of space and time coordinates:

$$\varphi = \varphi_k(\vec{x}, t). \tag{1-1}$$

The index k runs over the intrinsic degrees of freedom, e.g., spin polarizations.

Since we are going to deal with relativistic theories, where the time are space are treated equally, we adopt the four-vector notation for the space-time coordinates:

$$x^\mu = (c\,t, \vec{x})\,, \quad \mu = 0, 1, 2, 3. \tag{1-2}$$

and assume that the fields live in flat Minkowski space-time with metric:

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1-3}$$

a scalar product,

$$x \cdot y = g_{\mu\nu} x^\mu y^\nu = x^\mu y_\mu = x_\mu y^\mu = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \,. \tag{1-4}$$

and an infinitesimal interval:

$$dx^2 = c^2 dt^2 - d\vec{x}^2. \tag{1-5}$$

A field is then simply written as $\varphi_k(x)$, and can be classified according to how it transforms under Lorentz transformations. The most notable examples of relativistic fields are the electromagnetic and gravitational fields, denoted as $A_\mu(x)$ and $g_{\mu\nu}(x)$. They are, respectively, a Lorentz vector and a rank-2 tensor.

1.2 Analogies with classical mechanics

The dynamics, the evolution of field's energy, momentum and intrinsic degrees of freedom, can be described by a Hamiltonian — a functional of the field and its *conjugate momentum* $\pi(x)$:

$$H = H[\varphi(x), \pi(x)]. \tag{1-6}$$

An expression for the Hamiltonian is all we need to specify a particular field theory (FT).

Recall that in classical mechanics (CM) the Hamiltonian is a functional of coordinates and momenta of the system: $H = H[q(t), p(t)]$. Much can be taken over from CM to FT by replacing

$$q(t) \longrightarrow \varphi(x), \tag{1-7}$$

$$p(t) \longrightarrow \pi(x). \tag{1-8}$$

Similarly,

$$\frac{dq(t)}{dt} \equiv \dot{q}(t) \longrightarrow \frac{\partial \varphi(x)}{\partial x^\mu} \equiv \partial_\mu \varphi(x). \tag{1-9}$$

In analogy one can also introduce the Lagrangian,

$$L[\varphi,\, \partial_\mu \varphi] = \int d\vec{x}\, \pi(x)\, \partial_0 \varphi(x) - H, \tag{1-10}$$

and give a definition to the conjugate momentum:

$$\pi(x) = \frac{\delta L}{\delta\, \partial_0 \varphi(x)}. \tag{1-11}$$

Furthermore, the minimal action principle, the Euler-Lagrange equations, Noether theorem, and so on can be taken over in a similar way. However, before considering these topics in detail, we need first to understand the properties of fields under coordinate transformations.

1.3 The concept of path-integral quantization

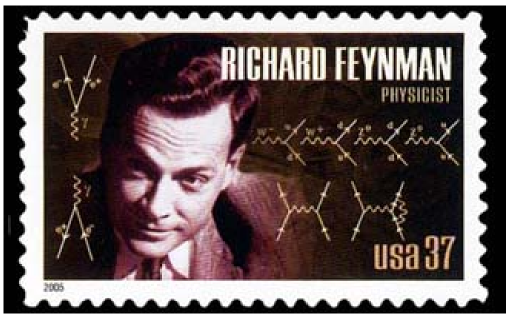
Around the year of 1933 Paul Dirac conjectured that a quantum-mechanical transition of a system from position a to position b, can be expressed in terms of the action of the corresponding classical system:

$$K(a \to b) \sim e^{(i/\hbar) S[q_{a\to b}^{\text{(class.)}}]}, \tag{1-12}$$

where $q_{a\to b}^{\text{(class.)}}$ is classical trajectory, or the path, the system undergoes from a to b.

Richard Feynman derived the exact formula in 1948. Schematically, it can be written as:

$$K(a \to b) = \sum_{\text{all paths } q_{a\to b}} e^{(i/\hbar) S[q_{a\to b}]}. \tag{1-13}$$



The difference with Dirac's formula is obvious: the answer is formed as a *superposition of all possible paths*, not just the classical one. It's quite amazing though, that in some cases the Dirac's and Feynman's formulas yield identical results.

The path-integral formulation of quantum mechanics proved to be a valuable alternative to Heisenberg's and Schrödinger's formulation. This method can readily be taken over to QFT, using, as we discussed earlier, the transition from the classical mechanics to classical field theory (i.e., replacing trajectories with fields, $q(t) \to \varphi(x)$, etc.). As the result we will obtain a very powerful and concise formulation of QFT.

1.4 Path-integral formulation of quantum mechanics: a 1-dimensional system

In classical mechanics we operate with trajectories $q(t)$; they describe how the coordinates of particles (or, 'material points') evolve with time. In quantum mechanics we deal with wave functions $\Psi(q,t)$, which give the probability, $P = |\Psi(q,t)|^2$, of finding the system at position q and time t.

In the limit of vanishing Planck's constant ($\hbar \to 0$), the quantum-mechanical picture should reduce to the classical description. Note $\hbar$ is not dimensionless, it's value is 6.58211899(16)$\times 10^{-16}$ eV·s. To define a quasi-classical regime we need identify a quantity would characterize the dynamics of the system and have the same dimension as $\hbar$. Such quantity is the classical action:

$$S = \int_{t_i}^{t_f} dt\, L[q(t), \dot{q}(t)] = \int_{t_i}^{t_f} dt\, \Big(p(t)\, \dot{q}(t) - H[p(t), q(t)] \Big) \tag{1-14}$$

Indeed, the Lagrangian, or momentum$\times$velocity, or Hamiltonian, all have the dimension of energy, while the integration over time gives the dimension of time. The classical regime is then $\hbar \ll S$, or

$$\frac{\hbar}{S} \ll 1. \tag{1-15}$$

In this regime, the two pictures (classical and quantum) should reconcile. To see this, it would be good to formulate the quantum-mechanical probabilities in terms of the classical action. This is precisely what the path-integral formulation does. Let us see how it can all be derived using the concepts of quantum mechanics.

The key concept in the path-integral formulation is the amplitude of transition from one position state to another in some interval of time:

$$K(q_f, q_i; t_f - t_i)\,. \tag{1-16}$$

In Dirac's 'bra'–'ket' notation for the position state $|q\rangle$, this amplitude can be written as:

$$K(q_f, q_i; t_f - t_i) = \langle q_f | e^{-(i/\hbar)\hat{H}(t_f - t_i)} | q_i \rangle \tag{1-17}$$

where the Hamiltonian operator $\hat{H} = H(\hat{p}, \hat{q})$, consists of operators of momentum p and position q, which satisfy the commutation relation: $[\hat{p}, \hat{q}] = i\hbar$. One can also introduce the time-dependent states as

$$|qt\rangle = e^{(i/\hbar)\hat{H}t} |q\rangle \tag{1-18}$$

and then simply write:

$$K(q_f, q_i; t_f - t_i) = \langle q_f t_f | q_i t_i \rangle\,. \tag{1-19}$$

Now, note that, due the completeness of the position state basis,

$$\int\limits_{-\infty}^{\infty} dq\, |qt\rangle\langle qt| = \int\limits_{-\infty}^{\infty} dq\, |q\rangle\langle q| = 1 \tag{1-20}$$

and hence the transition amplitude *propagates* the system from the initial to final state in the following way:

$$|q_f t_f\rangle = \int_{-\infty}^{\infty} dq_i \, K(q_f, q_i; t_f - t_i) \; |q_i t_i\rangle \tag{1-21}$$

Let us now slice the time interval into a large number N of infinitesimal intervals of size Δt, i.e., $t_f - t_i = N\,\Delta t$. The system evolves then through positions $q_k = q(t_k)$ in time steps $t_k = t_i + k\,\Delta t$, where $k = 0, \dots, N$, and hence $t_0 = t_i$, $t_N = t_f$.

Since,

$$|q_{k+1} t_{k+1}\rangle = \int_{-\infty}^{\infty} dq_k \, K(q_{k+1}, q_k; \Delta t) \; |q_k t_k\rangle \tag{1-22}$$

we can write

$$|q_N t_N\rangle = \int_{-\infty}^{\infty} dq_{N-1} \, K(q_f, q_{N-1}; \Delta t) \dots \int_{-\infty}^{\infty} dq_0 \, K(q_1, q_0; \Delta t) \; |q_0 t_0\rangle \,. \tag{1-23}$$

Looking back at Eq. (1-21) we can see that the full transition amplitude is a superposition of infinitesimal ones:

$$K(q_f, q_i; t_f - t_i) = \int_{-\infty}^{\infty} \prod_{k=1}^{N-1} dq_k \prod_{k=0}^{N-1} K(q_{k+1}, q_k; \Delta t). \tag{1-24}$$

The evolution of the system in an infinitesimal amount of time Δt from position q_k to position q_{k+1}, according to the time-dependent Schrödinger equation, proceeds as follows:

$$K(q_{k+1}, q_k; \Delta t) = \left\langle q_{k+1} | e^{-(i/\hbar)\hat{H}\,\Delta t} | q_k \right\rangle \tag{1-25}$$

where $\hat{H} = H(\hat{p}, \hat{q})$ is the Hamiltonian, and the initial and final states are the eigenstates of the position operator:

$$\hat{q}|q_k\rangle = q_k |q_k\rangle \,, \tag{1-26}$$

that satisfy the orthogonality and completeness conditions:

$$\langle q'|q\rangle = \delta(q - q'), \qquad \int_{-\infty}^{\infty} dq \, |q\rangle \, \langle q| = 1. \tag{1-27}$$

In the momentum representation:

$$|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq \, e^{(i/\hbar)pq} |q\rangle \tag{1-28}$$

these states are the eigenstates of the momentum operators,

$$\hat{p}|p_k\rangle = p_k|p_k\rangle \,, \tag{1-29}$$

and are also orthogonal and complete:

$$\langle p'|p\rangle = \delta(p-p'), \quad \int_{-\infty}^{\infty} dp\,|p\rangle\,\langle p| = 1. \tag{1-30}$$

as can easily be shown by using the integral representation of the delta-function:

$$\delta(q'-q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{ip(q'-q)}\,. \tag{1-31}$$

Let us insert a complete set of momentum p_k eigenstates into Eq. (1-25) so that it becomes

$$K(q_{k+1}, q_k; \Delta t) = \int_{-\infty}^{\infty} dp_k\ \langle q_{k+1}|p_k\rangle \left\langle p_k|e^{-(i/\hbar)\hat{H}\,\Delta t}|q_k\right\rangle . \tag{1-32}$$

Since the Hamiltonian is function of p and q operators only we can write[1]

$$\left\langle p_k|e^{-(i/\hbar)\hat{H}\,\Delta t}|q_k\right\rangle = \langle p_k|q_k\rangle\ e^{-(i/\hbar)H(p_k,q_k)\,\Delta t} \tag{1-33}$$

where H is a function, not an operator. It is the classical Hamiltonian of the system.

Using Eq. (1-28) and the orthogonality condition one sees that

$$\langle p_k|q_k\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-(i/\hbar)p_k q_k}, \quad \langle q_{k+1}|p_k\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{(i/\hbar)p_k q_{k+1}} \tag{1-34}$$

and substituting these in the previous two equations we find

$$K(q_{k+1}, q_k; \Delta t) = \int_{-\infty}^{\infty} \frac{dp_k}{2\pi\hbar}\ \exp\{\tfrac{i}{\hbar}[(q_{k+1}-q_k)p_k - H(p_k,q_k)\,\Delta t]\}. \tag{1-35}$$

This is the expression we substitute in eq. (1-24) to find that in the limit $N \to \infty$, or, equivalently, $\Delta t \to 0$, the transition amplitude is given by

[1] In doing this step we have ignored the operator-ordering problem, see P. Ramond, *Field Theory*, Chapter 2.

$$K(q_f, q_i; t_f - t_i) = \lim_{N\to\infty} \int\limits_{-\infty}^{\infty} \prod_{k=1}^{N-1} dq_k \prod_{k=0}^{N-1} \frac{dp_k}{2\pi\hbar}$$
$$\times \quad \exp\left\{\frac{i}{\hbar}\sum_{k=0}^{N-1}\left[\frac{q_{k+1}-q_k}{\Delta t}p_k - H(p_k, q_k)\right]\Delta t\right\}$$
$$= \int \mathscr{D}q\,\mathscr{D}p\,\exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,[p\dot{q} - H(p,q)]\right\}, \tag{1-36}$$

where in the last step we have used the usual definitions of a derivative and an integral:

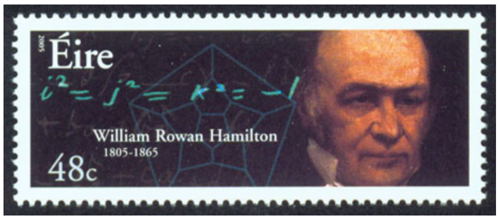
$$\lim_{\Delta t\to 0}\frac{q_{k+1}-q_k}{\Delta t} = \dot{q}_k \tag{1-37}$$
$$\lim_{\Delta t\to 0}\sum_{k=0}^{N-1} f(t_k)\,\Delta t = \int_{t_0}^{t_N} dt\, f(t)\,, \tag{1-38}$$

as well as introduced the following definition for continual measures:

$$\mathscr{D}q(t) = \lim_{N\to\infty}\prod_{k=1}^{N-1} dq(t_k), \quad \mathscr{D}p(t) = \lim_{N\to\infty}\prod_{k=0}^{N-1}\frac{dp(t_k)}{2\pi\hbar}\,. \tag{1-39}$$

Note that in the exponent of Eq. (1-36) we have the classical action of the system, written in terms of the Hamiltonian, — the Hamiltonian action:

$$S = \int_{t_i}^{t_f} dt\,[p\dot{q} - H(p,q)]. \tag{1-40}$$



Thus, we have derived the following simple formula for the probability amplitude of the quantum-mechanical transition from position q_i to q_f during the time $t_f - t_i$:

$$K(q_f, q_i; t_f - t_i) = \int \mathscr{D}q\,\mathscr{D}p\; e^{(i/\hbar)S} \tag{1-41}$$

It is a functional (continual) integral over all possible trajectories in the *phase space* of the system weighed with an exponent of the classical Hamiltonian action.

1.4.1 Path-integral in terms of the Lagrangian action

In many cases the Hamiltonian can be written as

$$H(p,q) = \frac{p^2}{2m} + V(q), \tag{1-42}$$

such that

$$L(q, \dot{q}) = \frac{m\dot{q}^2}{2} - V(q) \tag{1-43}$$

is the Lagrangian. Substituting this Hamiltonian into the path-intagral formula, we can perform the Gaussian integral (cf. Appendix A) over p_k, as follows:

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp_k \, e^{\frac{i}{\hbar}\left(-\frac{p_k^2}{2m} + p_k \dot{q}_k\right)\Delta t} = \left(\frac{m}{2\pi i\hbar\,\Delta t}\right)^{1/2} e^{\frac{i}{\hbar}\frac{m}{2}\dot{q}_k^2 \Delta t} \tag{1-44}$$

and thus obtain

$$\begin{aligned} K(q_f, q_i; t_f - t_i) &= \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\,\Delta t}\right)^{N/2} \int_{-\infty}^{\infty} \prod_{k=1}^{N-1} dq_k \, \exp\left\{\frac{i}{\hbar} \sum_{k=0}^{N-1} L(q_k, \dot{q}_k)\,\Delta t\right\} \\ &= \int \tilde{\mathscr{D}}q \, \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(q, \dot{q})\right\} = \int \tilde{\mathscr{D}}q \; e^{(i/\hbar)S} \end{aligned} \tag{1-45}$$

where the continual measure is defined now as

$$\tilde{\mathscr{D}}q = \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\,\Delta t}\right)^{N/2} \prod_{k=1}^{N-1} dq_k \,, \tag{1-46}$$

and the action S is written in terms of the Lagrangian.

In the absence of interactions and external forces, $V(q) = 0$, we have the free-particle amplitude:

$$K_0(q_f, q_i; t_f - t_i) = \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\,\Delta t}\right)^{N/2} \int_{-\infty}^{\infty} \prod_{k=1}^{N-1} dq_k \, \exp\left\{\frac{i}{\hbar} \sum_{k=0}^{N-1} \frac{m(q_{k+1} - q_k)^2}{2\Delta t}\right\}. \tag{1-47}$$

The integrals here are, again, of the Gaussian type. One integral after another we encounter

$$\begin{aligned} &\int_{-\infty}^{\infty} dq_k \exp\left\{\frac{im}{2\hbar\Delta t}\left[(q_{k+1} - q_k)^2 + \frac{1}{k}(q_k - q_0)^2\right]\right\} \\ &\qquad = \left[\frac{2i\pi\hbar\Delta t}{(1 + 1/k)\, m}\right]^{1/2} \exp\left\{\frac{im}{2\hbar\Delta t}\frac{(q_{k+1} - q_0)^2}{k+1}\right\}, \end{aligned} \tag{1-48}$$

until finally (after $N - 1$ integrations) obtain,

$$K_0(q_f, q_i; t_f - t_i) = \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\,\Delta t}\right)^{1/2} \prod_{k=1}^{N-1} \frac{1}{\sqrt{1 + 1/k}} \exp\left\{\frac{im}{2\hbar}\frac{(q_f - q_i)^2}{N\,\Delta t}\right\}. \tag{1-49}$$

It is not difficult to prove by induction that

$$\prod_{k=1}^{N-1} \frac{1}{\sqrt{1+1/k}} = \frac{1}{\sqrt{N}} \tag{1-50}$$

Therefore, recalling that $N\Delta t = t_f - t_i$, we arrive at the following result:

$$K_0(q_f, q_i; t_f - t_i) = \left(\frac{m}{2\pi i\hbar (t_f - t_i)} \right)^{1/2} \exp\left\{ \frac{im}{2\hbar} \frac{(q_f - q_i)^2}{t_f - t_i} \right\} \tag{1-51}$$

Let's examine this expression. First of all, let us observe that this is exactly the Dirac's formula, Eq. (1-12), for the free-particle case. Indeed the classical trajectory of a free particle is:

$$q_{i\to f}^{(\text{class.})}(t) = q_i + \frac{q_f - q_i}{t_f - t_i}\, t, \tag{1-52}$$

and hence the action is

$$S[q_{i\to f}^{(\text{class.})}(t)] = \frac{m}{2} \frac{(q_f - q_i)^2}{t_f - t_i}\,. \tag{1-53}$$

The prefactor of the exponent can be obtained from the superposition principle

$$\int_{-\infty}^{\infty} dq\, K_0(q_f, q; \tfrac{1}{2}T)\, K_0(q, q_i; \tfrac{1}{2}T) = K(q_f, q_i; T). \tag{1-54}$$

Despite the close connection to the trajectory of a classical particle, the transition amplitude tells us that the quantum-mechanical particle behaves in a qualitatively different way. The amplitude is oscillating, as if the particle is a wave. The wavelength λ of these oscilations can be computed from the periodicity condition,

$$2\pi = \frac{m}{2\hbar} \left[\frac{(X+\lambda)^2}{T} - \frac{X^2}{T} \right], \tag{1-55}$$

where $X = q_f - q_i$, $T = t_f - t_i$. The result is

$$\lambda = \sqrt{X^2 + 4\pi\hbar T/m} - X. \tag{1-56}$$

This looks strange, but for sufficiently large distance X, we have

$$\lambda = \frac{2\pi\hbar T}{mX} = \frac{2\pi\hbar}{p}\,, \tag{1-57}$$

where $p = mX/T$ is the momentum. We have thus obtained the De Broglie relation.

- R. P. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals*, chapptes 1, 2 , and 3.

2

Lecture 2

2.1 Quantization of a scalar field

In this case, we can quite straight-forwardly replace trajectory $q(t)$ with the field $\phi(x)$. The Lagrangian action is replaced by (in $c = 1$ units):

$$S = \int d^4x \left[\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - V(\phi)\right]. \tag{2-1}$$

Note that in 0+1 dimension this is the action of a one-dimensional harmonic oscillator. The conjugate momentum can be defined as

$$\pi(x) \equiv \frac{\delta\mathscr{L}}{\delta\dot{\phi}(x)} = \dot{\phi}(x), \tag{2-2}$$

with the fundamental Poisson bracket is defined for equal time as:

$$\{\phi(x), \pi(x')\}_P|_{x_0'=x_0} = -\{\pi(x'), \phi(x)\}_P|_{x_0'=x_0} = \delta(\vec{x}' - \vec{x}) \tag{2-3}$$

The Hamiltonian density is given by

$$\mathscr{H}(\pi, \phi) = \pi(x)\dot{\phi}(x) - \mathscr{L} = \tfrac{1}{2}\left[\pi^2(x) + (\vec{\nabla}\phi)^2 + m^2\phi^2\right], \tag{2-4}$$

and the Hamiltonian is:

$$H(\pi, \phi) = \int d^3x \mathscr{H}(\pi, \phi). \tag{2-5}$$

To define the path-integral we discretize not only the time $t = x^0$ but also the space $\vec{x}$, and thus obtain a 4-dimensional *lattice* of size L and spacing a. In each of the 4 dimensions the number of sites is $N = L/a$. The lattice sites are labeled by a set $(x^0_\tau, x^1_i, x^2_j, x^3_k)$, with indices τ, i, j, k taking values from 1 to N. At each site the field is given by (a real number)

$$\phi(x^0_\tau, x^1_i, x^2_j, x^3_k) = \phi_{ijk}(t_\tau) = \phi_K(t_\tau), \tag{2-6}$$

where by K we will denote a composite index $K = (i, j, k)$, which runs through the N^3 combinations.

At each value of $t = x^0$ (or τ) we have a three-dimensional box with a given set of field values, called *field configuration*, which we denote as

$$\bar{q}_\tau = (\phi_1(t_\tau), \ldots, \phi_K(t_\tau)). \tag{2-7}$$

In a similar way we introduce the configuration of conjugate momenta:

$$\bar{p}_\tau = (\pi_1(t_\tau), \ldots, \pi_K(t_\tau)). \tag{2-8}$$

Now, the transition amplitude of from a field configuration at t_i to a field configuration at t_f can, in full analogy with quantum mechanics, be written as

$$\begin{aligned}
\langle \bar{q}_f, t_f | \, \bar{q}_i, t_i \rangle &= \langle \bar{q}_f | \, e^{-(i/\hbar)\hat{H}(t_f - t_i)} \, |\bar{q}_i \rangle \\
&= \lim_{N\to\infty} \int_{-\infty}^{\infty} \prod_{K=1}^{N^3} \left(\prod_{\tau=i+1}^{f-1} d\phi_K(t_\tau) \prod_{\tau=i}^{f-1} \frac{d\pi_K(t_\tau)}{2\pi\hbar} \right) \qquad (2\text{-}9) \\
&\times \exp\left\{ \frac{i}{\hbar} \sum_{\tau=i}^{f-1} \sum_{K=1}^{N^3} \Big[\frac{\phi_K(t_{\tau+1}) - \phi_K(t_\tau)}{a} \, \pi_K(t_\tau) - \mathscr{H}\Big(\pi_K(t_\tau), \phi_K(t_\tau)\Big) \Big] a^4 \right\} \\
&\equiv \int \mathscr{D}\phi(x) \mathscr{D}\pi(x) \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \int d^3x \, [\dot{\phi}\pi - \mathscr{H}(\pi, \phi)] \right\} \qquad (2\text{-}10)
\end{aligned}$$

Unfortunately, since the time is treated here in such a special way, the Lorentz invariance is lost. For most of the applications, it suffices to consider

$$Z = \lim_{\substack{t_i\to-\infty \\ t_f\to+\infty}} \langle \bar{q}_f, t_f | \, \bar{q}_i, t_i \rangle = \int \mathscr{D}\phi(x) \mathscr{D}\pi(x) \exp\left\{ \frac{i}{\hbar} \int d^4x \, [\dot{\phi}\pi - \mathscr{H}(\pi, \phi)] \right\}, \tag{2-11}$$

which has a chance to be a Lorentz-invariant quantity.

Substituting the Hamiltonian, we can perform the Gaussian integral over the momenta, to find the expression in terms of the Lagrangian action:

$$Z = \mathcal{N} \int \mathscr{D}\phi(x) \exp\left\{ \frac{i}{\hbar} \int d^4x \, \mathscr{L}(\phi, \partial_\mu \phi) \right\}, \tag{2-12}$$

where the normalization factor is:

$$\mathcal{N} = \int \mathscr{D}\pi \, e^{-(i/\hbar)\frac{1}{2}\int d^4x \, \pi^2(x)}. \tag{2-13}$$

The field configuration at infinite times is called *ground state* or *vacuum*. Therefore Z is the vacuum-to-vacuum transition amplitude, often denoted as

$$\langle 0, +\infty | 0, -\infty \rangle, \quad {}_{\text{out}}\langle 0|0\rangle_{\text{in}}, \quad \ldots \tag{2-14}$$

It has a straightforward analogy with the partition function in statistical mechanics, and therefore sometimes referred to as the *partition function*.

2.2 Green's functions and their generating functional

In the formalism of canonical quantization, the fields are created an annihilated by field operators acting on the vacuum state. The dynamics of quantum fields can fully be described by Green's functions, defined as time-ordered products of field operators between the vacuum states:

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0|\, T(\,\hat{\phi}(x_n) \ldots \hat{\phi}(x_1)\,)\, |0\rangle \tag{2-15}$$

It is convenient to introduce a *generating functional*:

$$\begin{aligned} G[j] &= \langle 0|\, T\, e^{(i/\hbar)\int d^4x j(x)\hat{\phi}(x)} |0\rangle \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!\,\hbar^n} \int d^4x_1 \ldots d^4x_n\, G^{(n)}(x_1, \ldots, x_n)\, j(x_1) \ldots j(x_n) \end{aligned} \tag{2-16}$$

such that

$$G^{(n)}(x_1, \ldots, x_n) = (-i\hbar)^n \frac{\delta^n G[j]}{\delta j(x_n) \ldots \delta j(x_1)}\Big|_{j=0} \tag{2-17}$$

In the path-integral formulation the generating functional for Green's functions is given by

$$G[j] = \frac{Z[j]}{Z[0]} \tag{2-18}$$

where Z is the vacuum-to-vacuum transition amplitude in the presence of classical sources.

For instance, introducing a source field $j(x)$ for the scalar field, we have:

$$Z[j] = \mathcal{N} \int \mathscr{D}\phi(x) \exp\left\{ \frac{i}{\hbar} \int d^4x \left[\mathscr{L}(\phi, \partial_\mu \phi) + \phi(x) j(x) \right] \right\}. \tag{2-19}$$

The use of sources is already apparent from classical field theory. For the scalar field, the Euler-Lagrange equation yields the following field equation:

$$(\partial^2 + m^2)\phi(x) + V'(\phi) = 0, \tag{2-20}$$

where $V'(\phi) = \partial V(\phi)/\partial\phi$. In the presence of the source it becomes:

$$(\partial^2 + m^2)\phi(x) + V'(\phi) = j(x). \tag{2-21}$$

For the free field, $V = 0$, the equation is easily solved by substituting the Fourier transform:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4}\phi(p)\, e^{ip\cdot x}, \tag{2-22}$$

$$j(x) = \int \frac{d^4p}{(2\pi)^4} j(p)\, e^{ip\cdot x}. \tag{2-23}$$

(2-24)

The solution is

$$\phi(p) = -\frac{1}{p^2 - m^2} j(p) \tag{2-25}$$

$$\phi(x) = -\int d^4y\,\Delta(x-y)j(y), \quad \text{with } \Delta(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \tag{2-26}$$

The function Δ is the Green's function, or in this particular case, a *propagator* of the field. Indeed, it describes the propagation of the field from its source at y to an arbitrary point x. The meaning of the "$i\epsilon$" in the expression for Δ will be discussed later. Let us just note that it is important to add to the action a term $\frac{1}{2}i\varepsilon\phi^2$, that leads to a factor

$$e^{-\epsilon/2\hbar \int d^4x\, \phi^2} \tag{2-27}$$

in the path-integral expression for Z, which for $\epsilon > 0$ will ensure the convergence of the integral.

Let us now consider the partition function $Z[j]$ for the same situation: $V = 0$, the free field. The integral over ϕ is Gaussian, and we find

$$G_0[j] = Z_0[j]/Z_0[0] = \exp\Big\{ -\frac{i}{2\hbar}\int d^4x \int d^4y\, j(x)\,\Delta(x-y)\,j(y)\Big\} \tag{2-28}$$

Expanding the exponent,

$$\begin{aligned} G_0[j] &= 1 - \frac{i}{2\hbar}\int d^4x \int d^4y\, j(x)\,\Delta(x-y)\,j(y) \\ &\quad -\frac{1}{8\hbar^2}\Big[\int d^4x \int d^4y\, j(x)\,\Delta(x-y)\,j(y)\Big]^2 + \dots \end{aligned} \tag{2-29}$$

it is easy to see that

$$\begin{aligned} G_0^{(2)}(x_2, x_1) &\equiv (-i\hbar)^2 \frac{\delta^2 G_0[j]}{\delta j(x_2)\delta j(x_1)}\Big|_{j=0} = i\hbar\,\Delta(x_2 - x_1) \\ G_0^{(4)}(x_4, x_3, x_2, x_1) &\equiv (-i\hbar)^4 \frac{\delta^4 G_0[j]}{\delta j(x_4)\delta j(x_3)\delta j(x_2)\delta j(x_1)}\Big|_{j=0} \\ &= \frac{\hbar^2}{2}\Big\{\Delta(x_2 - x_1)\Delta(x_4 - x_3) + \Delta(x_3 - x_1)\Delta(x_4 - x_2) \\ &\qquad + \Delta(x_4 - x_1)\Delta(x_3 - x_2)\Big\}, \end{aligned} \tag{2-30}$$

and so on, while for odd n, $G_0^{(n)} = 0$.

2.3 Interactions. Perturbative expansion

Now, we switch on the interaction, $V(\phi) \neq 0$. A generic example would be:

$$V(\phi) = \frac{\lambda}{n!} m^{4-n} \phi^n(x), \tag{2-31}$$

where n is an integer, and λ is a real constant, called the coupling constant. The mass factor is introduced to make λ dimensionless. The cases $n = 3$ and $n = 4$ are the most well-known. They are called respectively, "the ϕ cube" and "the ϕ to the 4th" theory.

Let us observe that

$$-i\hbar \frac{\delta}{\delta j(y)} e^{(i/\hbar)\int d^4x\, \phi(x) j(x)}\Big|_{j=0} = \phi(y), \tag{2-32}$$

or, more generally,

$$\Big[-i\hbar \frac{\delta}{\delta j(y)} \Big]^n e^{(i/\hbar)\int d^4x\, \phi(x) j(x)}\Big|_{j=0} = \phi^n(y) \tag{2-33}$$

and hence

$$V\left(\frac{\hbar}{i} \frac{\delta}{\delta j(x)} \right) e^{(i/\hbar)\int d^4x\, \phi(x) j(x)}\Big|_{j=0} = V(\phi). \tag{2-34}$$

Using this, the partition of the interacting field in the presence of sources can be written as:

$$\begin{aligned} Z[j] &= \mathscr{N} \int \mathscr{D}\phi(x) \exp\left\{ \frac{i}{\hbar} \int d^4x \left[\mathscr{L}_0(\phi, \partial_\mu \phi) - V(\phi) + \phi(x) j(x) \right] \right\} \\ &= \mathscr{N} \exp\left[-(i/\hbar) \int d^4x\, V\left(\tfrac{\hbar}{i} \tfrac{\delta}{\delta j(x)}\right) \right] \int \mathscr{D}\phi(x) \exp\left\{ \frac{i}{\hbar} \int d^4x \left[\mathscr{L}_0(\phi, \partial_\mu \phi) + \phi(x) j(x) \right] \right\} \\ &= \exp\left[-(i/\hbar) \int d^4x\, V\left(\tfrac{\hbar}{i} \tfrac{\delta}{\delta j(x)}\right) \right] Z_0[j] , \end{aligned} \tag{2-35}$$

where Z_0 is the functional integral of the free field. Recall that

$$Z_0[j] = Z_0[0] \exp\left\{ -\frac{i}{2\hbar} \int d^4x \int d^4y\, j(x)\, \Delta(x-y)\, j(y) \right\} \equiv Z_0[0]\, G_0[j]. \tag{2-36}$$

The complete calculation of $Z[j]$ is still a formidable task, but we can make some progress in perturbation theory, i.e. assuming V is small (e.g., assuming the dimensionless coupling constant $\lambda \ll 1$). In this case, we can make a series expansion:

$$\begin{aligned} Z[j] &= Z_0[0] \sum_{n=0} \frac{1}{n!} \left(-\frac{i}{\hbar} \right)^n \left[\int d^4z\, V\left(\tfrac{\hbar}{i} \tfrac{\delta}{\delta j(z)} \right) \right]^n G_0[j] \\ &= Z_0[0] \left\{ 1 - \frac{i}{\hbar} \int d^4z\, V\left(\tfrac{\hbar}{i} \tfrac{\delta}{\delta j(z)} \right) - \frac{1}{2\hbar^2} \left[\int d^4z\, V\left(\tfrac{\hbar}{i} \tfrac{\delta}{\delta j(z)} \right) \right]^2 + \ldots \right\} G_0[j]. \end{aligned} \tag{2-37}$$

All we need now is to compute the functional derivatives of the type

$$\left[\frac{\hbar}{i}\frac{\delta}{\delta j(z)}\right]^n \exp\Big\{-\frac{i}{2\hbar}\int d^4x\int d^4y\, j(x)\,\Delta(x-y)\,j(y)\Big\} \qquad (2\text{-}38)$$

For example,

$$\begin{aligned}
\frac{\hbar}{i}\frac{\delta}{\delta j(z)}G_0[j] &= -\int d^4x\,\Delta(z-x)\,j(x)\,G_0[j]\\
\left(\frac{\hbar}{i}\right)^2\frac{\delta^2}{\delta j(z)^2}G_0[j] &= \left[i\hbar\,\Delta(0)+\left(\int d^4x\,\Delta(z-x)\,j(x)\right)^2\right]G_0[j]\\
\left(\frac{\hbar}{i}\right)^3\frac{\delta^3}{\delta j(z)^3}G_0[j] &= -\left[3i\hbar\Delta(0)+\left(\int d^4x\,\Delta(z-x)\,j(x)\right)^2\right]\\
&\quad\times\int d^4x\,\Delta(z-x)\,j(x)\,G_0[j]\\
\left(\frac{\hbar}{i}\right)^4\frac{\delta^4}{\delta j(z)^4}G_0[j] &= \left[-3\hbar^2\Delta^2(0)+6i\hbar\Delta(0)\left(\int d^4x\,\Delta(z-x)\,j(x)\right)^2\right.\\
&\quad\left.+\left(\int d^4x\,\Delta(z-x)\,j(x)\right)^4\right]G_0[j]
\end{aligned} \qquad (2\text{-}39)$$

Introducing a "sink" of the source j as

$$s(z)=\int d^4x\,\Delta(z-x)\,j(x), \qquad (2\text{-}40)$$

and denoting the functional derivative as

$$G^{[n]}(z)=\delta^n/\delta j(z)^n, \qquad (2\text{-}41)$$

we observe the following identities:

$$G_0[j] = \exp\Big\{-\frac{i}{2\hbar}\int d^4y\, s(y)\,j(y)\Big\}, \qquad (2\text{-}42)$$

$$\frac{\delta}{\delta j(y)}s(z) = \Delta(z-y)\equiv\frac{1}{i\hbar}\tilde{\Delta}(z-y) \qquad (2\text{-}43)$$

$$G_0^{[1]}(z) = -\frac{i}{\hbar}s(z)\,G_0[j]. \qquad (2\text{-}44)$$

We can thus rewrite the above functional derivatives more economically:

$$\begin{aligned}
-i\hbar\,G_0^{[1]}(z) &= -\,s(z)\,G_0[j],\\
(-i\hbar)^2\,G_0^{[2]}(z) &= \left[\tilde{\Delta}(0)+s^2(z)\right]G_0[j]
\end{aligned}$$

$$\begin{aligned}
(-i\hbar)^3\, G_0^{[3]}(z) &= -\left[3\tilde{\Delta}(0) + s^2(z)\right]\, s(z)\, G_0[j] \\
(-i\hbar)^4\, G_0^{[4]}(z) &= \left[3\tilde{\Delta}^2(0) + 6\tilde{\Delta}(0)\, s^2(z) + s^4(z)\right]\, G_0[j] \\
(-i\hbar)^5\, G_0^{[5]}(z) &= -\left[15\tilde{\Delta}^2(0) + 10\tilde{\Delta}(0)\, s^2(z) + s^4(z)\right] s(z)\, G_0[j] \\
(-i\hbar)^6\, G_0^{[6]}(z) &= \left[15\tilde{\Delta}^3(0) + 45\tilde{\Delta}^2(0)\, s^2(z) + 15\tilde{\Delta}(0)\, s^4(z) + s^6(z)\right]\, G_0[j]
\end{aligned} \tag{2-45}$$

In general, we have the following recurrent relation:

$$(-i\hbar)^n\; G_0^{[n]}(z) = (n-1)\tilde{\Delta}(0)\,(-i\hbar)^{n-2} G_0^{[n-2]}(z) - s(z)\,(-i\hbar)^{n-1} G_0^{[n-1]}(z), \tag{2-46}$$

with $G_0^{[0]}(z) = G_0[j]$.

Thus, for example, the partition function of the ϕ^3 theory (to the first order in V) is given by:

$$Z_{\phi^3}[j] = Z_0[j]\left(1 + \frac{i}{\hbar}\frac{\lambda m}{3!}\int d^4z\left[3i\hbar\Delta(0) + \left(\int d^4x\,\Delta(z-x)\,j(x)\right)^2\right]\right), \tag{2-47}$$

while the partition function of ϕ^4-theory is given by:

$$\begin{aligned}
Z_{\phi^4}[j] &= Z_0[j] - \frac{i}{\hbar}\frac{\lambda}{4!} Z_0[j]\int d^4z\left[-3\hbar^2\Delta(0)^2 + 6i\hbar\Delta(0)\left(\int d^4x\,\Delta(z-x)\,j(x)\right)^2\right. \\
&\qquad \left. + \left(\int d^4x\,\Delta(z-x)\,j(x)\right)^4\right].
\end{aligned} \tag{2-48}$$

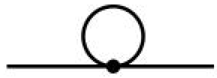
It can be seen that the contributions which do not contain sources (i.e., pure vacuum contributions) cancel out of the generating functional for the Green's functions, e.g.

$$\begin{aligned}
G_{\phi^4}[j] &= \frac{Z_{\phi^4}[j]}{Z_{\phi^4}[0]} = G_0[j]\left\{1 - \frac{i}{\hbar}\frac{\lambda}{4!}\int d^4z\left[6i\hbar\Delta(0) + \left(\int d^4x\,\Delta(z-x)\,j(x)\right)^2\right]\right. \\
&\qquad \left. \times\left(\int d^4x\,\Delta(z-x)\,j(x)\right)^2\right\}.
\end{aligned} \tag{2-49}$$

From here we can easily obtain the Green's functions, e.g.

$$\begin{aligned}
G_{\phi^4}^{(2)}(x_1,x_2) &= (-i\hbar)^2 \frac{\delta^2 G_{\phi^4}[j]}{\delta j(x_1)\delta j(x_2)}\bigg|_{j=0} = (-i\hbar)^2 \frac{\delta^2}{\delta j(x_1)\delta j(x_2)} \\
&\qquad \times\left\{G_0[j] + \frac{\lambda}{4}\Delta(0)\int d^4z\left(\int d^4x\,\Delta(z-x)\,j(x)\right)^2\right\}\bigg|_{j=0} \\
&= i\hbar\,\Delta(x_1-x_2) + (i\hbar)^2\frac{\lambda}{2}\Delta(0)\int d^4z\,\Delta(x_2-z)\,\Delta(z-x_1).
\end{aligned} \tag{2-50}$$

The latter expression can be interpreted in terms of Feynman diagrams:

$G^{(2)}_{\phi^4}(x_1, x_2) = i\hbar \Big($ $+$  $\Big)$ (2-51)

where each line denotes a propagator Δ, the vertex denotes the interaction at a point (here z), while the factor 1/2 (the so-called "symmetry factor") and the factor $i\hbar$ are inferred by the rules, see below.

It is convenient to go to the momentum space by means of

$$\Delta(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}}{p^2 - m^2 + i\epsilon} \equiv \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \Delta(p). \tag{2-52}$$

where p is the four-momentum.

For the two-point Green's function we, for instance, have:

$$\begin{aligned} G^{(2)}_{\phi^4}(x_1, x_2) &= i\hbar \left[\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x_2 - x_1)}}{p^2 - m^2 + i\epsilon} + \tfrac{1}{2}\lambda i\hbar \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \right. \\ &\quad \left. \times \int d^4z \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{e^{ip_1\cdot(x_1 - z)}}{p_1^2 - m^2 + i\epsilon} \frac{e^{ip_2\cdot(x_2 - z)}}{p_2^2 - m^2 + i\epsilon} \right] \\ &= i\hbar \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x_2 - x_1)}}{p^2 - m^2 + i\epsilon} \left[1 + \tfrac{1}{2}\lambda \frac{1}{p^2 - m^2 + i\epsilon} i\hbar \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \right] \end{aligned} \tag{2-53}$$

This means that in the momentum space the Green's function takes the following form:

$$G^{(2)}_{\phi^4}(p) = i\hbar\, \Delta(p) \left[1 + \tfrac{1}{2}\lambda\, \Delta(p)\, i\hbar \int \frac{d^4k}{(2\pi)^4} \Delta(k) \right] \tag{2-54}$$

$G^{(2)}_{\phi^4}(p) = i\hbar \Big($ $+$  $\Big)$ (2-55)

Feynman rules for ϕ^4 theory:

- Propagator: $\Delta(x_2 - x_1)$ or $\Delta(p)$
- Vertex: λ
- Loop: $i\hbar \int \frac{d^4p}{(2\pi)^4}$
- Symmetry factors.

Using these rules we can easily write down any contribution to the Green's functions. The rule for the loop in particular shows that the expansion in $\hbar$ is equivalent to the expansion in the number of loops.

2.3.1 Ultraviolet divergence, regularization and renormalization

Unfortunately, the typical loop contribution:

$$\Delta(0) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \tag{2-56}$$

is a divergent integral. This is a very common problem in QFT, the problem of ultraviolet (UV) divergencies.[1]

It turns out that in many cases the UV divergencies can be *renormalized* away, such that the physical quantities are not affected by them. Theories where this cannot be done are called *non-renormalizable*. Gravity is the most famous example of a non-renormalizable theory. The ϕ^4 theory is renormalizable. Let us see how one get rids of the UV divergence in our example.

The first step is to *regularize* the divergent integral,

$$J_1(m^2) \equiv i \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \tag{2-57}$$

There many ways to do that, here are the two most common ones:

1. *Sharp cutoff:* the integration is cut off at some large but finite number Λ:

$$\int \frac{d^4p}{(2\pi)^4} = \lim_{\Lambda\to\infty} \frac{1}{(2\pi)^4} \int_{-\Lambda}^{\Lambda} dp_0 dp_1 dp_2 dp_3 \tag{2-58}$$

2. *Dimensional regularization:* the integral is evaluated in D dimensions and the result is expanded around $d-4$ equal to 0. Introducing $\varepsilon = \frac{1}{2}(4-D)$, and an arbitrary mass scale μ:

$$\int \frac{d^4p}{(2\pi)^4} = \lim_{\varepsilon\to 0} \mu^{4-D} \int \frac{d^Dp}{(2\pi)^D} \tag{2-59}$$

In the case of dimreg the evaluation of the integral goes as follows:

$$\begin{aligned} J_1(m^2) &= i\mu^{4-D} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 - m^2 + i\epsilon} = i\mu^{4-D} \int \frac{d^Dp}{(2\pi)^D} \int_0^\infty dz\, e^{iz(p^2 - m^2 + i\epsilon)} \\ &= -\frac{\mu^{4-D}}{(4\pi i)^{D/2}} \int_0^\infty dz\, z^{-D/2}\, e^{iz(-m^2+i\epsilon)} = \frac{\mu^{4-D}}{(4\pi)^{D/2}} (m^2)^{D/2-1}\, \Gamma(1 - D/2) \end{aligned}$$

[1] These divergencies are called ultraviolet because they appear at large momenta, $p \to \infty$. There also sometimes divergencies at small momenta $p \to 0$, called infrared divergencies.

where

$$\Gamma(\alpha) = \int_0^\infty dz\, z^{\alpha-1}\, e^{-z}\,. \tag{2-60}$$

is the Gamma-function.

For small $\varepsilon = \frac{1}{2}(4-D)$, we obtain:

$$J_1(m^2) = \frac{m^2}{(4\pi)^2}\left[-\frac{1}{\varepsilon} + \gamma_E - \log 4\pi - 1 + \log\frac{m^2}{\mu^2} + O(\varepsilon)\right] \tag{2-61}$$

with $\gamma_E = -\Gamma'(1) \simeq 0.5772$, the Euler constant.

Now, the renormalization. Observe that

$$G^{(2)}_{\phi^4}(p) = i\hbar\,\Delta(p)\left[1 + \tfrac{1}{2}\lambda\,\Delta(p)\,i\hbar\int\frac{d^4k}{(2\pi)^4}\,\Delta(k)\right] = i\hbar\,\frac{1}{\Delta^{-1} - \Sigma} + O(\lambda^2) \tag{2-62}$$

where

$$\Sigma = \tfrac{1}{2}\hbar\lambda\, J_1(m^2) \tag{2-63}$$

is called the self-energy. The two-point Green's function described the propagation of the scalar field in the presence of quantum fluctuations. The role of the latter is to renormalize the mass of the scalar field:

$$m_R^2 = m^2 + \Sigma \tag{2-64}$$

such that

$$G^{(2)}_{\phi^4}(p) = i\hbar\frac{1}{p^2 - m_R^2 + i\epsilon} + O(\lambda^2) \tag{2-65}$$

we can choose the renormalized mass to be the physical mass of field (the on-shell renormalization scheme).

- C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, Singapore, 1988), chapters 9-1, 9-2, 9-3 and 6-1-1.

3

Lecture 3

3.1 Path integral in the presence of constraints

The path integral derived in the previous section can easily be generalized to the n-dimensional case:

$$K(\bar{q}_f, \bar{q}_i; t_f - t_i) = \int \prod_{i=1}^{n} \mathscr{D}q_i \, \mathscr{D}p_i \; e^{(i/\hbar) \int_{t_i}^{t_f} dt \, [\bar{p}\,\dot{\bar{q}} - H(\bar{p},\bar{q})]}\,, \tag{3-1}$$

where $\bar{q} = (q_1, \ldots, q_n)$, $\bar{p} = (p_1, \ldots, p_n)$. These vectors span a $2n$-dimensional phase space Γ, and the path-integral is performed over all trajectories in Γ. There is an important class of systems, called *singular* or *constrained* systems, where this integration procedure is not correct.

> The system is called *constrained* if
>
> $$\det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = 0.$$

In this case, the defining equation for the momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3-2}$$

does not give a relation between the momentum and velocity, $p_i \sim \dot{q}_i$, as it normally does, but, at least for some i, it gives a relation between the momentum and coordinates: $p_i = f(\bar{q})$. The latter relation is a *constraint*, since it constrains the allowed (or, physical) phase-space of the system. In general, *constraints* are relations of the type $\varphi(\bar{q}, \bar{p}) = 0$.

Now, if the phase-space of the system Γ is constrained by $2m$ conditions

$$\varphi_a(\bar{p}, \bar{q}) = 0, \quad a = 1, \ldots, 2m \tag{3-3}$$

which define the *constraint surface* $\Phi \in \Gamma$, all the trajectories will lie on the *physical* phase space $\Gamma^* = \Gamma \bigcap \Phi$ of dimension $2(n-m)$. Clearly, in the path integral we need to integrate over the trajectories in Γ^* only. The most straightforward way to do this would be to resolve all the constraints in order to find the *physical* $\bar{Q}$ and $\bar{P}$ which span Γ^*, and then write the path integral as

$$K(\bar{q}_f, \bar{q}_i; t_f - t_i) = \int \prod_{l=1}^{n-m} \mathscr{D}Q_l \, \mathscr{D}P_l \; e^{(i/\hbar)\int_{t_i}^{t_f} dt\,[\bar{P}\dot{\bar{Q}} - H(\bar{P},\bar{Q})]} \,. \tag{3-4}$$

However, it is often easier and more desirable to work in terms of original variables q and p. To be able to do this we need to understand how to transit from one set of variables to the other.

A simple change of variables $(Q, P) \to (q, p)$ would not work, since the number of independent variables in these sets is not equal, unless we are on the constraint surface. Let us therefore extend the physical set by introducing auxiliary variables Ω_a, which vanish on the constraint surface. The latter requirement means that the auxiliary variables can be presented as linear combinations of constraints:

$$\Omega_a = u_{ab}\, \varphi_b \,, \tag{3-5}$$

where (u_{ab}) is a constant non-singular matrix, and summation over repeating indices is understood. We may choose u's such that Ω's to be *canonical* variables, that is, to assume that their *Poisson bracket* is normalized to one:

$$\{\Omega_a(t), \Omega_b(t)\}_P = \delta_{ab} \,. \tag{3-6}$$

So we have at hand a transformation from one set of canonical variables (Q, P, Ω) to another set of canonical variables (q, p). According to the *Leouville theorem*, such a *canonical transformation* leaves the volume of the phase space invariant. This of course is true as well for an element of the phase-space volume, and therefore,

$$\prod_{l=1}^{n-m} \mathscr{D}Q_l \, \mathscr{D}P_l \prod_{a=1}^{2m} \mathscr{D}\Omega_a = \prod_{i=1}^{n} \mathscr{D}q_i \, \mathscr{D}p_i \tag{3-7}$$

Next we multiply both sides with $\prod_{a=1}^{2m} \delta(\Omega_a)$, to eliminate the integration over the auxiliary variables on the left-hand-side, leaving us with

$$\prod_{l=1}^{n-m} \mathscr{D}Q_l \, \mathscr{D}P_l = \prod_{i=1}^{n} \mathscr{D}q_i \, \mathscr{D}p_i \prod_{a=1}^{2m} \delta(\Omega_a) \,. \tag{3-8}$$

The last step is to express the delta-function of auxiliary variables in terms of the constraints by substituting Eq. (3-5):

$$\delta(\Omega_a) = \delta(u_{ab}\,\varphi_b) = \frac{1}{\det u}\,\delta(\varphi_a)\,, \tag{3-9}$$

and use that the Poisson bracket of constraints is

$$\{\varphi_a, \varphi_b\}_P = (u^{-1})_{aa'}\,(u^{-1})_{bb'}\,\{\Omega_{a'}, \Omega_{b'}\}_P = (u^{-2})_{ab}, \tag{3-10}$$

to rewrite $\det^{-1} u$ as $\det(u^{-1}) = \det^{1/2} u^{-2} = \det^{1/2}(\{\varphi_a, \varphi_b\}_P)$.

Thus, we finally arrive at

$$\prod_{l=1}^{n-m} \mathscr{D}Q_l\,\mathscr{D}P_l = \prod_{i=1}^{n} \mathscr{D}q_i\,\mathscr{D}p_i\,\sqrt{\det(\{\varphi_a, \varphi_b\}_P)}\prod_{a=1}^{2m}\delta(\varphi_a)\,. \tag{3-11}$$

This is something we wanted: we have expressed the measure of the reduced phase space in terms of the measure of the original phase space with the constraints explicitly taken into account.

The resulting path integral written in terms of original variables, in the presence of constraints, takes form

$$K(\bar{q}_f, \bar{q}_i; t_f - t_i) = \int \prod_{i=1}^{n} \mathscr{D}q_i\,\mathscr{D}p_i\,\sqrt{\det(\{\varphi_a, \varphi_b\}_P)}\prod_{a=1}^{2m}\delta(\varphi_a)\;e^{(i/\hbar)\int_{t_i}^{t_f} dt\,[\bar{p}\,\dot{\bar{q}} - H(\bar{p},\bar{q})]}\,. \tag{3-12}$$

This is clearly somewhat different from what one would naively write down, see Eq. (3-1). The measure acquires an additional factor which depends solely on constraints and which purpose is to ensure that the continual integration is constrained to the physical domain of the phase space.

A classic example of constraint system in field theory is electromagnetism:

$$\mathscr{L}_{\text{Maxwell}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3-13}$$

Indeed, an explicit calculation gives

$$\frac{\partial^2 \mathscr{L}_{\text{Maxwell}}}{\partial \dot{A}_\lambda\,\partial \dot{A}_\sigma} = -(g^{\lambda\sigma}g^{00} - g^{\lambda 0}g^{\sigma 0}) = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{3-14}$$

The determinant of this diagonal matrix is, obviously, 0.

It is even simpler to demonstrate that Dirac's theory is singular. In fact, all fields with spin are constrained.

3.2 Dirac-Bergmann algorithm

Suppose we have a singular system completely specified by a Lagrangian $L(\bar{q},\dot{\bar{q}})$. As we already discussed, at the stage of introducing the conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \ldots, n \tag{3-15}$$

we might discover a few (let, for certainty, say m) relations of the type

$$\varphi_a^{(1)} \equiv p_a - f_a(q) = 0, \quad a = 1, \ldots, m. \tag{3-16}$$

These are the relations among coordinates and momenta — constraints in the phase-space of the system. Now, are these the only constraints in the system? Not necessarily. To find out whether there are further constraints and determine all them we shall employ the Dirac-Bergmann algorithm, which let us state in the following form:

1. the constraints $\varphi_a^{(1)}$ — the *primary constraints* — are added to the Hamiltonian H via the Lagrange multipliers $\lambda_a^{(1)}$ to form the *primary Hamiltonian*:

$$H^{(1)} = H + \sum_{a=1}^{m} \lambda_a^{(1)} \varphi_a^{(1)} \,. \tag{3-17}$$

2. require that constraint do not change in time — the time-constancy requirement (TCR), or in other words, the Poisson bracket of the constraint with the Hamiltonian must vanish

$$\text{TCR:} \quad \{\varphi_a^{(1)}, H^{(1)}\} \stackrel{!}{=} 0 \tag{3-18}$$

At this stage, when computing the Poisson bracket explicitly, for a fixed index a, we can obtain three different answers:
1) if $\{\varphi_a^{(1)}, H^{(1)}\} = 0$, then *stop* the iteration for this a;
2) if $\{\varphi_a^{(1)}, H\} \neq 0$ and $\{\varphi_a^{(1)}, \varphi_{a'}^{(1)}\} = 0$, for all a', then $\varphi^{(2)} = \{\varphi_a^{(1)}, H\}$ is a *secondary constraint*;
3) if $\{\varphi_a^{(1)}, H\} \neq 0$ and $\{\varphi_a^{(1)}, \varphi_{a'}^{(1)}\} \neq 0$, for some a', then the TCR is an equation for the Lagrange multiplier $\lambda_{a'}^{(1)}$.

3. Case 2) yields further constraints, $\varphi_b^{(2)}$, which need to be added to the Hamiltonian via the Lagrange multipliers $\lambda_b^{(2)}$, to form the *secondary Hamiltonian*:

$$H^{(2)} = H^{(1)} + \sum_{b} \lambda_b^{(2)} \varphi_b^{(2)} \,. \tag{3-19}$$

Then we go back to step 2, thus imposing the TSR on the secondary constraints, and so on.

We repeat this iterative procedure until the case 2) in step 2 does not arise anymore. This means there are no further constraints, we have determined all of them.

Finally, after all constraints are determined, it is important to verify that they fulfill the TCR with the *total Hamiltonian*, the one that includes all the constraints. This last step may yield more equations for the Lagrange multipliers.

3.3 Classification of constraints into first and second class

Once we determined all the constraints we do not need to pay attention to whether they are primary, secondary, tertiary, or whatever. However, it is important to distinguish the two classes of constraints, which can, for instance, be done according to the following

> *Definition*:
> constraints which Lagrangian multipliers can be determined are *second class*; constraints which Lagrangian multipliers cannot be determined are *first class*.

Any constraint will belong to one of the two classes. Let us label the first-class constraints as $\varphi_a^{(I)}$ with $a = 1, \ldots, m_I$, and the second-class ones as $\varphi_b^{(II)}$ with $b = 1, \ldots, m_{II}$. Then, from the description of the Dirac-Bergmann algorithm and the definition of classes, it is clear that the Poisson bracket of a first-class constraint with any constraint should vanish (at least on the constraint surface):

$$\{\varphi_a^{(I)}, \varphi_{a'}^{(I)}\} = 0 = \{\varphi_a^{(I)}, \varphi_b^{(II)}\}, \tag{3-20}$$

for any a, a', and b. In contrast, the Poisson bracket of second-class constraints, among themselves, is not trivial:

$$\{\varphi_b^{(II)}, \varphi_{b'}^{(II)}\} = C_{bb'}, \tag{3-21}$$

where C is a not singular matrix: $\det C \neq 0$, at the constraint surface.

The presence of first-class constraints is closely related to the *gauge symmetries* of the system. We shall discuss this relation later in the context of field theories. One can demote the first-class constraints into second class by introducing *gauge-fixing*

3.4 Degrees-of-freedom counting

We began describing the system by n coordinates and n momenta, so $2n$ degrees of freedom to start with. Obviously, if there are constraints, the number of degrees of freedom is reduced. By how much it is reduced is dependent on the class of the constraints: a second-class constraint removes one degree of freedom, while a first-class constraint removes two degrees of freedom in the phase space of the system. Thus the reduced (or, physical) phase space has

$$\#d.o.f. = 2n - 2m_I - m_{II}\,. \tag{3-22}$$

3.5 Electromagnetism

We will now start applying the constraint considerations of the previous two lectures to fields. Everything we said above will be true as well if we transit from classical paths $q(t)$ to classical fields which depend on the space-time coordinates $x = (t, \vec{x})$. We thus immerse into the 4-dimensional Minkowski space-time with the metric: $\mathrm{diag}(g^{\mu\nu}) = (1, -1, -1, -1)$.

The Lagrangian density of the electromagnetic (EM) field $A_\mu(x)$ is given by

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{3-23}$$

To find the conjugate momenta, defined as

$$E^\mu(x) = \frac{\partial \mathscr{L}}{\partial \dot{A}_\mu(x)}, \quad \dot{A}_\mu(x) \equiv \frac{\partial}{\partial x_0} A_\mu(x), \tag{3-24}$$

let us write out the Lagrangian density in terms of the time and space components of the EM field, i.e., A_0 and A_i $(i = 1, 2, 3)$:

$$\mathscr{L} = \tfrac{1}{2}F_{0i}F_{0i} - \tfrac{1}{4}F_{ij}F_{ij}\,. \tag{3-25}$$

In writing this the metric is taken into account, so hereafter no distinction between lower and upper indices is made. We will use only lower indices.

We can now easily see that

$$E_0 = 0 \tag{3-26}$$

$$E_i = F_{0i} = \dot{A}_i - \partial_i A_0, \tag{3-27}$$

where $\dot{A}_\mu \equiv \frac{\partial}{\partial x_0} A_\mu$. The second equation here gives us the relation between momenta[1] E_i and "velocities" $\dot{A}_i$, while the first equation is the (primary) constraint:

$$\varphi^{(1)}(x) = E_0(x)\,. \tag{3-28}$$

At this point we postulate the fundamental Poisson brackets (defined at equal times, $x_0 = y_0$):

$$\begin{aligned} \{A_0(x), E_0(y)\} &= \delta(\vec{x} - \vec{y}) = -\{E_0(x), A_0(y)\} \\ \{A_i(x), E_j(y)\} &= \delta_{ij}\, \delta(\vec{x} - \vec{y}) = -\{E_i(x), A_j(y)\} \\ \{A_\mu(x), A_\nu(y)\} &= \{E_\mu(x), E_\nu(y)\} = 0. \end{aligned} \tag{3-29}$$

[1] In this case E^i is equal to the electric field strength. This should explain our choice of notation for conjugate momenta.

Next, we invoke the Dirac-Bergmann algorithm to find further constraints.

The primary Hamiltonian is

$$\begin{aligned} H^{(1)} &= \int d\vec{x}\, \left[E_0\dot{A}_0 + E_i\dot{A}_i - \mathscr{L} + \lambda^{(1)}E_0\right] \\ &= \int d\vec{x}\, \left[E_i\dot{A}_i - \tfrac{1}{2}(\dot{A}_i - \partial_i A_0)E_i + \tfrac{1}{2}B_iB_i + \lambda^{(1)}E_0\right] \\ &= \int d\vec{x}\, \left[\tfrac{1}{2}E_i(E_i + \partial_i A_0) + \tfrac{1}{2}(\partial_i A_0)E_i + \tfrac{1}{2}B_iB_i + \lambda^{(1)}E_0\right] \qquad (3\text{-}30) \\ &= \int d\vec{x}\, \left[\tfrac{1}{2}(E_i^2 + B_i^2) + (\partial_i A_0)E_i + \lambda^{(1)}E_0\right] \qquad (3\text{-}31) \end{aligned}$$

where in the second line of this equation we have redefined the Lagrange multiplier: $\lambda^{(1)} \to \lambda^{(1)} - \dot{A}_0$ and introduced the vector of the magnetic field: $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$. We also would like to use the partial integration to write

$$\int d\vec{x}\,(\partial_i A_0)E_i = \int d\vec{x}\,[\partial_i(A_0E_i) - A_0\partial_i E_i]. \qquad (3\text{-}32)$$

and use the Gauss theorem to argue that the total-derivative term vanishes. Thus,

$$H^{(1)} = \int d\vec{x}\, \left[\tfrac{1}{2}(E_i^2 + B_i^2) - A_0\partial_i E_i + \lambda^{(1)}E_0\right] \qquad (3\text{-}33)$$

Using the fundamental Poisson brackets we can compute the Poisson bracket of the primary constraint with the Hamiltonian:

$$\{E_0(x), H^{(1)}\} = \int d\vec{y}\,[\partial_i^y E_i(y)]\,\delta(\vec{x} - \vec{y}) = \partial_i^x E_i(x)\,, \qquad (3\text{-}34)$$

where we have written out $\partial_i^y \equiv \frac{\partial}{\partial y_i}$.

Therefore, this Poisson bracket does not vanish and we do not have an equation for the Lagrange multiplier. We thus have obtained a secondary constraint:

$$\varphi^{(2)} = \partial_i E_i\,. \qquad (3\text{-}35)$$

The secondary Hamiltonian is

$$H^{(2)} = H^{(1)} + \lambda^{(2)}\,\partial_i E_i\,. \qquad (3\text{-}36)$$

In computing the Poisson bracket, $\{\partial_i E_i(x), H^{(2)}\}$, we face the follwing calculation,

$$\begin{aligned} &\int d\vec{y}\,\{\partial_i E_i(x), \tfrac{1}{2}B_j^2(y)\} = \int d\vec{y}\,B_j(y)\,\partial_i^x\,\{E_i(x), B_j(y)\} \\ &= \int d\vec{y}\,B_j(y)\,\partial_i^x\,\partial_k^y\,\varepsilon_{jkm}\{E_i(x), A_m(y)\} = -\int d\vec{y}\,\varepsilon_{ijk}\,B_j(y)\,\partial_i^x\,\partial_k^y\delta(\vec{x} - \vec{y}) \\ &= \int d\vec{y}\,\varepsilon_{ijk}\,B_j(y)\,\partial_i^x\,\partial_k^x\delta(\vec{x} - \vec{y}) = 0\,, \qquad (3\text{-}37) \end{aligned}$$

because $\varepsilon_{ijk}\partial_i\,\partial_k = 0$. We thus find

$$\{\partial_i E_i(x), H^{(2)}\} = 0\,. \tag{3-38}$$

Hence there are no further constraints.

The two constraints are first class, since their Lagrange multipliers cannot be determined. Also, it is easy to verify that, all the Poisson brackets of these constraints vanish:

$$\{E_0(x), E_0(y)\} = 0 = \{\partial_i E_i(x), \partial_j E_j(y)\} = \{E_0(x), \partial_j E_j(y)\}\,. \tag{3-39}$$

The degrees-of-freedom counting in this case refers to the *spin* degrees of freedom. The vector field has 4 components and so does its conjugate momentum. Therefore, 8 degrees of freedom to start with. The 2 first-class constraints reduce this number to $8 - 2\cdot 2 = 4$. These degrees of freedom are shared equally between the field and its conjugate momentum, two independent components for each. This is precisely what is needed to describe the two possible polarizations a massless particle with spin.

3.6 Free massive vector field (Proca model)

The electromagnetic Lagrangian, considered above, describes a massless spin-1 particle — the photon. Let us briefly see what changes if this particle is given a mass M. The Lagrangian density will then be complemented by a mass term:

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}M^2 A_\mu A^\mu. \tag{3-40}$$

The conjugate momenta and the primary constraint remain to be the same as for EM theory. The Hamiltonian acquires the mass term, hence

$$H^{(1)} = H^{(1)}_{\mathrm{EM}} - \tfrac{1}{2}M^2\left(A_0^2 - A_i^2\right)\,, \tag{3-41}$$

where $H^{(1)}_{\mathrm{EM}}$ is given above in Eq. (3-33). The secondary constraint now obviously changes:

$$\{E_0(x), H^{(1)}\} = \partial_i E_i(x) + M^2 A_0(x) \equiv \varphi^{(2)}(x)\,. \tag{3-42}$$

We note that, unlike in EM case, $\varphi^{(1)}$ and $\varphi^{(2)}$ do not commute:

$$\{\varphi^{(1)}(x), \varphi^{(2)}(y)\} = \{E_0(x), [\partial_i E_i(y) + M^2 A_0(y)]\} = -M^2\,\delta(\vec{x}-\vec{y})\,. \tag{3-43}$$

The secondary Hamiltonian is as usual, $H^{(2)} = H^{(1)} + \lambda^{(2)}\varphi^{(2)}$. Applying TCR to the secondary constraint, we obtain an equation for the Lagrange multiplier $\lambda^{(1)}$:

$$\{\varphi^{(2)}(x), H^{(2)}\} = -M^2\partial_i A_i(x) + M^2\lambda^{(1)}(x) \stackrel{!}{=} 0\,, \tag{3-44}$$

which is easily solved: $\lambda^{(1)} = \partial_i A_i$.

The second Lagrange multiplier is determined form the TCR for the primary constraint:

$$\{\varphi^{(1)}(x), H^{(2)}\} = \varphi^{(2)}(x) + \lambda^{(2)}(x) \int d\vec{y}\, \{\varphi^{(1)}(x), \varphi^{(2)}(y)\} \stackrel{!}{=} 0\,. \tag{3-45}$$

We obtain $\lambda^{(2)} = \frac{1}{M^2}\partial_i E_i + A_0$.

The degrees-of-freedom counting now goes as follows. The two constraints are second class, so they remove 2 phase-space degrees of freedom. We have then $8-2 = 3+3$ degrees of freedom, 3 for the field and 3 for its conjugate momentum. This is precisely what's needed to describe the $2s+1 = 3$ spin polarizations of a massive particle with spin 1.

4

Lecture 4

4.1 Yang-Mills theory

We next perform a similar analysis of the Yang-Mills theory, specified by the Lagrangian density:

$$\mathscr{L}_{\rm YM} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}, \tag{4-1}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\,, \tag{4-2}$$

and where a, b, c are the color indices, which run from 1 to $N_c^2 - 1$, with N_c the number of colors; f^{abc} are the SU(N_c) structure constants.

In the way similar to EM we find the conjugate momenta:

$$E^a_i = F^a_{0i}, \tag{4-3}$$

and $N_c^2 - 1$ primary constraints:

$$\varphi^{(1)a} = E^a_0\,. \tag{4-4}$$

The only non-vanishing fundamental Poisson brackets are:

$$\begin{aligned} \{A^a_0(x), E^b_0(y)\} &= \delta^{ab}\,\delta(\vec{x}-\vec{y}) = -\{E^a_0(x), A^b_0(y)\} \\ \{A^a_i(x), E^b_j(y)\} &= \delta^{ab}\,\delta_{ij}\,\delta(\vec{x}-\vec{y}) = -\{E^a_i(x), A^b_j(y)\} \end{aligned} \tag{4-5}$$

and the Hamiltonian is

$$H^{(1)} = \int d\vec{x}\ \left[\tfrac{1}{2}(E^a_i E^a_i + B^a_i B^a_i) - A^a_0 \nabla^{ab}_i E^b_i + \lambda^{(1)a} E^a_0\right] \tag{4-6}$$

where $B^a_i = \frac{1}{2}\varepsilon_{ijk} F^a_{jk}$; $\nabla^{ab}_i = \delta^{ab}\partial_i - g f^{abc} A^c_i$.

The secondary constraints are

$$\varphi^{(2)a}(x) = \{E_0^a(x), H^{(1)}\} = \nabla_i^{ab} E_i^b(x)\,. \tag{4-7}$$

This constraint certainly commutes with the primary constraint, $\{E_0^a, \nabla_i^{ab} E_i^b(y)\} = 0$. To find out whether it commutes with the Hamiltonian,

$$H^{(2)} = H^{(1)} + \lambda^{(2)a}\, \nabla_i^{ab} E_i^b(x)\,. \tag{4-8}$$

we compute first the following Poisson bracket (at $x_0 = y_0$)

$$\begin{aligned}
\{\varphi^{(2)a}(x), \varphi^{(2)b}(y)\} &= \{\nabla_i^{ac} E_i^c(x), \nabla_j^{bd} E_j^d(y)\} \\
&= -g f^{bde} E_j^d(y)\, \partial_i^x \{E_i^a(x),\, A_j^e(y)\} - g f^{ace} E_i^c(x)\, \partial_j^y \{A_i^e(x),\, E_j^b(y)\} \\
&\quad + g^2 f^{ace} f^{bdf} \{E_i^c(x)\, A_i^e(x),\, E_j^d(y)\, A_j^f(y)\} \\
&= g f^{bca} E_i^c(y)\, \partial_i^x\, \delta(\vec{x}-\vec{y}) - g f^{acb} E_i^c(x)\, \partial_i^y\, \delta(\vec{x}-\vec{y}) \\
&\quad + g^2 \left[f^{acd} f^{bdf}\, E_i^c(x)\, A_i^f(y) - f^{ace} f^{bdc}\, A_i^e(x)\, E_i^d(y) \right] \delta(\vec{x}-\vec{y}) \\
&= g f^{abc} \left[\partial_i^y E_i^c(y) - g f^{cde} E_i^c(y)\, A_i^e(y) \right]\, \delta(\vec{x}-\vec{y}) \\
&= g f^{abc} (\nabla_i^{cd} E_i^d)\, \delta(\vec{x}-\vec{y})\,,
\end{aligned} \tag{4-9}$$

where in the last step we neglected the total-derivative terms, used the properties of the δ-function:

$$\partial_i^y \delta(\vec{x}-\vec{y}) = -\partial_i^x \delta(\vec{x}-\vec{y}), \quad f(\vec{x})\, \delta(\vec{x}-\vec{y}) = f(\vec{y})\, \delta(\vec{x}-\vec{y})\,, \tag{4-10}$$

as well as the Jacobi identity for the SU(N_c) structure constants:

$$f^{acd} f^{bde} - f^{ade} f^{bcd} = f^{abd} f^{dec}\,. \tag{4-11}$$

Thus we observe that the Poisson brackets of secondary constraints obeys the SU(N_c) algebra:

$$\{\varphi^{(2)a}(x), \varphi^{(2)b}(y)\} = g f^{abc} \varphi^{(2)c}(x)\, \delta(\vec{x}-\vec{y})\,. \tag{4-12}$$

Another bracket we must compute here is

$$\begin{aligned}
&\int d\vec{y}\, \{\nabla_i^{ac} E_i^c(x), \tfrac{1}{2}(B_j^b)^2(y)\} = \int d\vec{y}\, \{\nabla_i^{ac} E_i^c(x), B_j^b(y)\} B_j^b \\
&= \int d\vec{y}\, \{(\partial_i^x E_i^a - g f^{acf} E_i^c A_i^f),\, \varepsilon_{jkl} (\partial_k^y A_l^b - \tfrac{1}{2} g f^{bde} A_l^d A_k^e)\} \\
&= \int d\vec{y}\, B_j^b(y)\, \varepsilon_{jkl} [-\delta^{ab} \delta_{il} \partial_i^x \partial_k^y + g f^{abf} \delta_{il} A_i^f\, \partial_k^y \\
&+ \tfrac{1}{2} g f^{bde} (\delta^{ad} \delta_{il} A_k^e + \delta^{ae} \delta_{ik} A_l^d) \partial_i^x - \tfrac{1}{2} g^2 f^{acf} f^{bde} A_i^f (\delta^{cd} \delta_{il} A_k^e + \delta^{ce} \delta_{ik} A_l^d)]\, \delta(\vec{x}-\vec{y}) \\
&= \int d\vec{y}\, B_j^b(y)\, \varepsilon_{ijk} [g f^{abc} (A_i^c(x)\, \partial_k^y - A_k^c(y) \partial_i^x) - g^2 f^{acf} f^{bce} A_i^f A_k^e]\, \delta(\vec{x}-\vec{y}) \\
&= -g^2 f^{acf} f^{bce}\, \varepsilon_{ijk}\, B_j^b(x)\, A_i^f(x)\, A_k^e(x) = 0\,.
\end{aligned} \tag{4-13}$$

Now it easy to see that $\{\varphi^{(2)a}, H^{(2)}\} \sim g^{abc}\varphi^{(2)c}(x)$, so it vanishes at the constraint surface — there are no further constraints, and the Lagrange multipliers cannot be determined. This theory, therefore, has two $\times(N_c^2 - 1)$ first-class constraints.

4.2 Faddeev-Popov ghosts in Yang-Mills theory

We would like now to quantize the YM theory, bearing in mind that constraints must be taken into account. The path-integral in the presence of constraints that was considered in the first lecture should be applicable to our situation, provided the usual changes in transiting from paths to fields are made. We then obtain the *functional integral* of the theory in the form,

$$Z = \int \prod_{\mu,\, a} \mathscr{D}A^a_\mu \mathscr{D}E^a_\mu \prod_a \delta(E^a_0)\delta(\nabla^{ab}_i E^b_i)\, \det{}^{1/2}(\{\varphi^{(r)a}, \varphi^{(s)b}\})\, e^{i\int d^4x[E^a_i \dot{A}^a_i - \frac{1}{2}(E^a_i)^2 - \frac{1}{2}(B^a_i)^2]} \tag{4-14}$$

The problem is that in deriving this functional integral we assumed that the matrix of Poisson brackets of constraint is non-singular. In other, words we assumed that all the constraints are second class. In the YM system the constraints $\varphi^{(r)a}$ (where $r = 1, 2$) are first class.

We can deal with the first-class constraints by introducing the gauge-fixing conditions $\chi^{(r)a}$ such that

$$\det(\{\chi^{(r)a}, \varphi^{(s)b}\}) \neq 0\,, \tag{4-15}$$

and

$$\{\chi^{(r)a}, \chi^{(s)b}\} = 0 \tag{4-16}$$

for any values of the indices.

The gauge-fixing conditions are constraints themselves. After introducing them the first-class constraints become second class.

Coming back to our analysis of the YM theory we supplement the first-class constraints by the following gauge-fixing conditions:

$$\chi^{(1)a} = A^a_0, \quad \chi^{(2)a} = \partial_i A^a_i \tag{4-17}$$

These conditions define the *Coulomb gauge*. The functional integral, in this gauge, takes the form,

$$\begin{aligned} Z &= \int \prod_{\mu,\, a} \mathscr{D}A^a_\mu \mathscr{D}E^a_\mu \prod_a \delta(E^a_0)\,\delta(A^a_0)\,\delta(\nabla^{ab}_i E^b_i)\,\delta(\partial_i A^a_i)\, \det{}^{1/2} M \\ &\times \exp i \int d^4x[E^a_i \dot{A}^a_i - \tfrac{1}{2}(E^a_i)^2 - \tfrac{1}{2}(B^a_i)^2]\,, \end{aligned} \tag{4-18}$$

where the matrix M of Poisson brackets of constraints is now composed of four square matrices:

$$M = \begin{pmatrix} \{(\varphi^{(r)a}(x),\, \varphi^{(s)b}(y)\}) & (\{\varphi^{(r)a}(x),\, \chi^{(s)b}(y)\}) \\ (\{\chi^{(r)a}(x),\, \varphi^{(s)b}(y)\}) & (\{\chi^{(r)a}(x),\, \chi^{(s)b}(y)\}) \end{pmatrix} \tag{4-19}$$

Since the only non-vanishing elements are the brackets of the constraints with the gauge-fixing conditions we have

$$\begin{aligned} \det{}^{1/2} M &= \det(\{\chi^{(r)a}(x),\, \varphi^{(s)b}(y)\}) = \det\left[\begin{pmatrix} \delta^{ab} & 0 \\ 0 & \partial_i^x\, \nabla_i^{y\,ba} \end{pmatrix} \delta(\vec{x}-\vec{y})\right] \\ &= -\det[\nabla_i^{x\,ab}\, \partial_i^x\, \delta(\vec{x}-\vec{y})]\,. \end{aligned} \tag{4-20}$$

We can perform the functional integral over the conjugate momenta E, using the rules of Gaussian integration. The result is

$$Z = \int \prod_{\mu,\,a} \mathscr{D}A_\mu^a \prod_a \delta(A_0^a)\, \delta(\partial_i A_i^a)\, \det[\nabla_i^{x\,ab}\, \partial_i^x\, \delta(\vec{x}-\vec{y})]\, e^{i\int d^4x\, \mathscr{L}_{\rm YM}}\,, \tag{4-21}$$

For perturbative calculations of this functional integral, it is convenient to exponentiate the determinant and the δ-functions, such that instead of appearing in the measure they appear as new terms in the Lagrangian.

The determinant can be exponentiated by using the Grassmann (anti-commuting) fields, $c^a(x)$. It can be shown in general that

$$\det M^{ab}(x-y) = \int \prod_a \mathscr{D}\bar{c}^a \mathscr{D}c^a \exp\left[-\int dx \int dy\, \bar{c}^a(x)\, M^{ab}(x-y)\, c^b(y)\right]\,. \tag{4-22}$$

In our case, up to an overall factor,

$$\begin{aligned} \det[\nabla_i^{x\,ab}\, \partial_i^x\, \delta(\vec{x}-\vec{y})] &= \int \prod_a \mathscr{D}\bar{c}^a \mathscr{D}c^a\, \exp\left[i \int d\vec{x} \int d\vec{y}\, \bar{c}^a(x)\, \nabla_i^{x\,ab}\, \partial_i^x\, \delta(\vec{x}-\vec{y})\, c^b(y)\right] \\ &= \int \prod_a \mathscr{D}\bar{c}^a \mathscr{D}c^a\, \exp i \int d\vec{x}\, \left[-(\partial_i \bar{c}^a)(\partial_i c^a) + g f^{abc} (\partial_i \bar{c}^a)\, c^b\, A_i^c\right] \end{aligned}$$

The functional integral can thus be written as

$$Z = \int \prod_a \prod_\mu \mathscr{D}A_\mu^a \mathscr{D}\bar{c}^a \mathscr{D}c^a\, \delta(A_0^a)\, \delta(\partial_i A_i^a)\, e^{i\int d^4x\, \mathscr{L}_{\rm YM} + \mathscr{L}_{\rm ghost}}\,, \tag{4-23}$$

where the new term in the Lagrangian is

$$\mathscr{L}_{\rm ghost} = -(\partial_i \bar{c}^a)(\partial_i c^a) + g f^{abc} (\partial_i \bar{c}^a)\, c^b\, A_i^c\,. \tag{4-24}$$

The new fields, $c^a(x)$, were introduced in this way by Faddeev and Popov, and are called *Faddeev-Popov ghosts*.

The expression for the ghost Lagrangian depends on the gauge. There is a gauge, the so-called *unitary gauge*, where the ghosts disappear. In any other gauge, however, ghosts are present and give contributions at the loop level.

We will finally write down here the functional integral of the YM theory in a relativistically invariant gauge, called the *Lorentz gauge*:

$$Z = \int \prod_{\mu,\, a} \mathscr{D} A^a_\mu \prod_a \delta(\partial_\mu A^{\mu\, a}) \; \det[\nabla^{ab}_\mu \, \partial^\mu \, \delta^4(x-y)] \, e^{i \int d^4x \mathscr{L}_{\rm YM}} \,, \tag{4-25}$$

Or, writing the determinant in terms of ghosts,

$$Z = \int \prod_a \left[\prod_\mu \mathscr{D} A^a_\mu \right] \mathscr{D}\bar{c}^a \mathscr{D} c^a \; \delta(\partial_\mu A^{\mu\, a}) \, e^{i \int d^4x \mathscr{L}_{\rm YM} + \mathscr{L}_{\rm ghost}} \,, \tag{4-26}$$

with

$$\mathscr{L}_{\rm ghost} = \partial_\mu \bar{c}^a \, \partial^\mu c^a - g f^{abc} (\partial_\mu \bar{c}^a) \, c^b \, A^{\mu\, c} \,. \tag{4-27}$$

Ghost contributions are necessary to maintain gauge invariance in YM theory at the quantum level.

4.3 Literature

- P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964), chapters 1 and 2.

A

Gaussian integrals

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad a > 0 \tag{A-1}$$

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+2bx} = \sqrt{\frac{\pi}{a}}\, e^{b^2/a}, \quad a > 0 \tag{A-2}$$

Similar formulae exists for complex coefficients, e.g.,

$$\int_{-\infty}^{\infty} dx\, e^{iax^2} = \sqrt{\frac{i\pi}{a}}, \quad a \in \mathbb{R}. \tag{A-3}$$

This result can be proved by changing the variable, $ax^2 = y$, for positive a, or $ax^2 = -y$, for negative a. Let's consider $a > 0$:

$$\int_{-\infty}^{\infty} dx\, e^{iax^2} = 2\int_{0}^{\infty} dx\, e^{iax^2} = \frac{1}{\sqrt{a}} \int_{0}^{\infty} dy\, y^{-1/2}\, e^{iy}\,.$$

Next, choose the closed contour in the first quadrant of the complex y plane, over which, according to the Cauchi theorem for analytic functions, the integral vanishes:

$$\left(\int_0^\infty + \int_{i\infty}^0\right) dy\, y^{-1/2}\, e^{iy} + \lim_{R\to\infty} iR^{1/2} \int_0^{\pi/2} d\varphi\, e^{i\varphi/2} e^{iRe^{i\varphi}} = 0$$

The last term can, by partial integration (choosing $u = -iR^{-1/2}\exp(-i\varphi/2)$, $v = \exp[iRe^{i\varphi}]$), be written as

$$i\sqrt{R}\int_0^{\pi/2} d\varphi\, e^{i\varphi/2} e^{iRe^{i\varphi}} = \frac{1}{\sqrt{R}}\left[-i\big(e^{-R}e^{-i\pi/4} - e^{iR}\big) + \tfrac{1}{2}\int_0^{\pi/2} d\varphi\, e^{-i\varphi/2} e^{iRe^{i\varphi}}\right],$$

which vanishes in the limit of $R \to \infty$. We thus have,

$$\int_0^\infty dy\, y^{-1/2}\, e^{iy} = \int_0^{i\infty} dy\, y^{-1/2}\, e^{iy} \overset{y=iz}{=} i^{1/2}\int_0^\infty dz\, z^{-1/2}\, e^{-z} = \sqrt{i\pi}, \quad \text{qed.}$$