Path integral method for fermions

Before : spin 0 , $\varphi(x)$ SCALAR , mass m

Universe : spin 1/2 , $\psi(x)$ Dirac field, mass m

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \,, \quad \psi_a \; \left(\begin{array}{c}\text{four components} \\ 4\times 1\end{array}\right)$$

Anti - COMMUTATION RELATIONS (CONSEQUENCE OF PAULI PRINCIPLE)

$$[\psi(x), \psi(x')]_+ = 0 \qquad [A,B]_+ = AB + BA$$

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu} \mathbb{1}_{4\times 4}$$

x, x' SPACE SEPERATED POINTS

ONE NEEDS A METHOD TO PERFORM PATH INTEGRAL FOR THE ANTI-COMMUTING FIELDS

IMPORTANT TOOL FOR QUANTIZATION OF THE NON - ABELIAN FIELD THEORIES

GOAL : CALCULATE

$$\langle 0 | T(\psi(x)\bar{\psi}(y)) | 0 \rangle = G^{(2)}(x,y)$$

GRASSMANN VARIABLES

GRASSMANN ALGEBRA - ALGEBRA GENERATED BY OBJECTS THAT ANTI-COMMUTE

G_N

CONSIDER ALGEBRA OVER $\mathbb{C}$ $\Rightarrow$ GENERATED BY $2N$ ANTI-COMMUTING GENERATORS

$$\theta_i, \bar{\theta}_i, \quad i = 1, \dots, N, \qquad \theta_i \theta_j + \theta_j \theta_i = [\theta_i, \theta_j]_+ = 0$$

$$[\theta_i, \bar{\theta}_j]_+ = [\bar{\theta}_i, \bar{\theta}_j] = 0$$

$$\Downarrow$$

$$\theta_i^2 = \bar{\theta}_j^2 = 0$$

GENERAL ELEMENT OF G_N (SUM OF THE ALL POSSIBLE TERMS)

$$g = \underbrace{c}_{\text{COMPLEX NUMBER}} + c_i \theta_i + \tilde{c}_i \bar{\theta}_i + c_{ij} \underbrace{\theta_i \bar{\theta}_j}_{\text{CONVENTION: PUT } \theta_i \text{ TO LEFT, } \bar{\theta}_i \text{ TO RIGHT}} + \dots$$

$$\dots + c_{\dots} \theta_{i_1} \dots \theta_{i_k} \bar{\theta}_{j_1} \dots \bar{\theta}_{j_n}$$

MULTIPLICATION

$g_1 \cdot g_2$

$$g_1 = a_1 + b_1 \theta + c_1 \bar{\theta} + d_1 \theta \bar{\theta}$$
$$g_2 = a_2 + b_2 \theta + c_2 \bar{\theta} + d_2 \theta \bar{\theta}$$

$N = 1$ EXAMPLE

$$g_1 \cdot g_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1)\theta + (a_1 c_2 + a_2 c_1)\bar{\theta} + (a_2 d_1 + a_1 d_2 + b_1 c_2 \underbrace{-}_{\text{BECAUSE OF ANTICOM.}} b_2 c_1)\theta\bar{\theta}$$

IN GENERAL EACH ELEMENT HAS EVEN NUMBER OF $\theta, \bar\theta$ and ODD NUMBER OF $\theta, \bar\theta$:

ONE CAN SPLIT (WRITE) $\Rightarrow$ $G = G_{even} \oplus G_{odd}$ ($\mathbb{Z}_2$ graded algebra)

ALGEBRA OVER $\mathbb{C}$ $\rightsquigarrow$ NEED TO DEFINE THE CONJUGATION

(ANALOG OF CONJUGATION OF COMPLEX NUMBERS, MATRICES)

$$g \to g^*$$

FOR COMPLEX NUMBER, $c^* = \bar c$

FOR GENERATORS $\theta_i^* = \bar\theta_i$, $\bar\theta_i^* = \theta_i$

$$(g_1 \cdot g_2)^* = g_2^* g_1^*$$ LIKE FOR MATRICES $M \to M^+$

EXAMPLE

$$g^* = (a + b\theta + c\bar\theta + d\theta\bar\theta)^* =$$

$$= \bar a + \bar c\theta + \bar b\bar\theta + \bar d\theta\bar\theta$$

$$(\theta\bar\theta)^* = \bar\theta^*\theta^* = \theta\bar\theta$$

IN GENERAL: SIGNS

$$(\theta_{i_1}\dots\theta_{i_k}\bar\theta_{j_1}\dots\bar\theta_{j_h})^* = \theta_{j_1}\dots\theta_{j_h}\bar\theta_{i_1}\dots\bar\theta_{i_k}(-1)^{[k/2]+[h/2]}$$

INTEGER

Differentiation & INTEGRATION

$$\frac{\partial \theta_j}{\partial \theta_i} = \delta_{ij} \,, \quad \frac{\partial}{\partial \theta_i} 1 = 0$$

Operator $\frac{\partial}{\partial \theta_i}$ DEFINES A **LEFT DERIVATIVE**:

1) COMMUTE θ_i TO THE LEFT
2) ACT $\frac{\partial \theta_i}{\partial \theta_i} = 1$: (SUPRESS IT)

GENERAL CASE

$$\frac{\partial}{\partial \theta_i} (\underbrace{\theta_{i_1} \dots \theta_{i_k}}_{\text{no } \theta_i}) = 0$$

$$\frac{\partial}{\partial \theta_i} (\theta_{i_1} \dots \underline{\theta_i} \dots \theta_{i_k}) = \underbrace{\theta_{i_1} \dots \theta_{i_k}}_{\text{no } \theta_i} (-1)^{i-i_1}$$

EXAMPLE: $g = a + b\theta + c\bar{\theta} + d\theta\bar{\theta}$

$$\begin{cases} \frac{\partial g}{\partial \theta} = b + d\bar{\theta} \\ \frac{\partial g}{\partial \bar{\theta}} = c - d\theta \end{cases}$$

ONE CAN CHECK THAT

DEREVATIVES ANTICOMMUTE

$$\left[\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right]_+ = 0$$

$$\left[\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \bar{\theta}_j}\right]_+ = 0$$

$$\left[\frac{\partial}{\partial \bar{\theta}_i}, \frac{\partial}{\partial \bar{\theta}_j}\right]_+ = 0$$

INTEGRATION

DEFINED TO BE IDENTICAL TO DIFFERETIATION

$$\boxed{\int d\Theta_i \, \ldots = \frac{\partial}{\partial \Theta_i} \, \ldots}$$

THERE IS NO BOUNDARY TERM (GRASSMANN VAR IS AN ABSTRACT THING)

ANTI COMMUTING OBJE

$$\Rightarrow \int d\Theta_i = 0$$

STILL HAS FORMAL PROPERTIES OF INTEGRATION

- LINEAR
- AFTER INTEGRATION OVER A VARIABLE, AN EXPRESSION DOES NOT DEPEND ON THIS VARIABLE ANY MORE

g has max one Θ_i due to $\Theta_i^2 = 0$.

GAUSSIAN INTEGRAL

(Berezin integrals over Gaussian)

G_N algebra $\Theta_1, \ldots, \Theta_N$
$\bar{\Theta}_1, \ldots, \bar{\Theta}_N$

CONSIDER FOLLOWING ELEMENT OF THE ALGEBRA

$$e = \exp\left(-\bar{\Theta}_i A_{ij} \Theta_j\right), \quad A_{ij} \text{ is } N \times N \text{ matrix}, \quad A^{+} = A$$

DEFINE

$$\exp(g) = 1 + g + \frac{1}{2} g^2 + \ldots$$

THIS SUM ENDS AT ORDER N

(SUM over $\sum_{i,j}^{N}$)

EXAMPLE: $N=1$ $\exp(-\bar{\theta} A \theta) = 1 + A\,\theta\bar{\theta}$

EXERCISE $\Rightarrow$ $N=2$ $\exp(-\bar{\theta}_i A_{ij} \theta_j) = 1 + \dots + \underbrace{(A_{12}A_{21} - A_{11}A_{22})\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2}_{\text{HIGHEST TERM}}$

$N=3$ etc.

INTEGRAL

$$\int \prod_{i=1}^{N} d\bar{\theta}_i\, d\theta_i = \int d\bar{\theta}_1 \int d\theta_1 \int d\bar{\theta}_2 \int d\theta_2 \int \dots$$

ONE HAS TO BE CAREFUL ABOUT THE ORDER OF INTEGRATION

$$\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2 = -\theta_2\bar{\theta}_2\theta_1\bar{\theta}_1$$

$$\exp(-\bar{\theta}_i A_{ij}\theta_j)_{N=2} = 1 + \dots + \underbrace{(A_{11}A_{22} - A_{12}A_{21})}_{\det(A)}\,\underline{\theta_2\bar{\theta}_2\theta_1\bar{\theta}_1}$$

PICK HIGHEST TERM, OTHER WILL GIVE "0"
$\int d\theta\, 1 = 0.$

$$\Rightarrow \boxed{\int \prod_{i=1}^{N} d\bar{\theta}_i\, d\theta_i\, \exp(-\bar{\theta}_i A_{ij}\theta_j) = \det A}$$

Fermionic expectation value

$$\langle g \rangle = \frac{\int \prod_{i=1}^{N} d\bar{\theta}_i \, d\theta_i \, \exp\left(-\bar{\theta}_i A_{ij} \theta_j\right) g}{\int \prod_{i=1}^{N} d\bar{\theta}_i \, d\theta_i \, \exp\left(-\bar{\theta}_i A_{ij} \theta_j\right)}$$

$$\langle \underbrace{\theta_i \dots}_{\text{odd}} \rangle = 0$$ because here even powers of θ

$$\langle \theta_i \bar{\theta}_j \rangle \overset{N=1}{=} \frac{\int d\bar{\theta}\, d\theta \, \exp(-\bar{\theta} A \theta)\, \theta \bar{\theta}}{\int d\bar{\theta}\, d\theta \, \underbrace{\exp(-\bar{\theta} A \theta)}_{1 + A\theta\bar{\theta}}} = \frac{1}{A}$$

$$\boxed{\langle \theta_i \bar{\theta}_j \rangle \overset{\text{GENERAL}}{=} \frac{\text{Minor}(A)_{ij}}{\det A} = \left(A^{-1}\right)_{ij}}$$

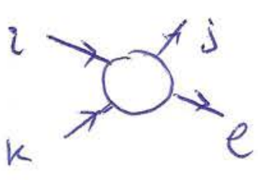
EXERCISE

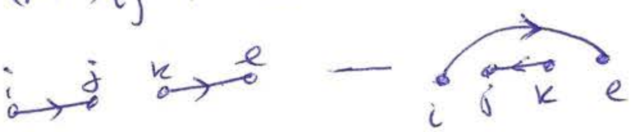
$$\langle \bar{\theta}_i \theta_j \rangle = -\left(A^{-1}\right)_{ij}$$

i.e. expect. values also anticommute.

Wick theorem (expectation value of 4 point function)

$$\langle \theta_i \bar{\theta}_j \theta_k \bar{\theta}_e \rangle = \left(A^{-1}\right)_{ij} \left(A^{-1}\right)_{ke} - \left(A^{-1}\right)_{ie} \left(A^{-1}\right)_{kj}$$





$\Rightarrow$ THESE ARE RECEIPTS FOR PERFORMING FERMIONIC PATH INTEGRALS

Dirac field

$\psi = \{ \psi^a(x),\ a = 1, \dots, 4 \}$

↑ COMPONENT OF DIRAC FIELD

$\bar{\psi} = \{ \bar{\psi}^a(x), \dots$

↑

CONSIDER THEM AS INDEPENDENT GENERATORS OF GRASSMANN ALGEBRA (BIG, INF DIM)

EACH $\psi^a(x)$ PLAYS THE ROLE OF THE θ. & $\bar{\psi}^a \to \bar{\theta}$.

$\boxed{\text{PATH INTEGRAL IS DEFINED AS GRASSMANN GAUSSIAN INTEGRAL}}$

$$\underbrace{\int D[\bar{\psi}, \psi]}_{\text{MEASURE}} \exp\left(i \int d^4x\, \mathcal{L} \right) = \det(i\partial\!\!\!/ - m)$$

$$\prod_{a=1}^{4} \prod_{x} d\bar{\psi}^a(x)\, d\psi^a(x)$$

$$\mathcal{L} = \bar{\psi}^a(x) \left(i(\gamma^\mu)_{ab} \partial_\mu - m\, \delta_{ab} \right) \psi^b(x)$$

$$\boxed{\langle 0 | T(\psi(x) \bar{\psi}(y)) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p\!\!\!/ - m} e^{-ip(x-y)} = i\Delta(x-y)}$$

$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \psi(p)$ ⟵ USE HERE DIFFERENT CONVENTION FOR FOURIER TRASFORM. ⚠

$\bar{\psi}(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \bar{\psi}(k)$

$$i \int d^4x\, \mathcal{L} = i \int \cancel{d^4x} \cdot \int \frac{d^4p}{(2\pi)^4} \cancel{\frac{d^4k}{(2\pi)^4}} \cancel{e^{ikx}} \bar{\psi}(k) \left(\gamma^\mu p_\mu - m \right) \cancel{e^{-ipx}} \psi(p)$$

$$= - \int \frac{d^4p}{(2\pi)^4} \bar{\psi}(p) \underbrace{(-i)(\gamma^\mu p_\mu - m)}_{A} \psi(p)$$