Symmetries in Physics: Introduction and Overview

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- Gauge principle $\Rightarrow$ generation of interactions/dynamics
- Symmetry and asymmetry/symmetry breaking as distinguishing features of dynamics

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3. (existence of inverse) For all $a \in G$ there is an element, denoted by a^{-1}, such that
$$aa^{-1} = a^{-1}a = e$$
4. (Abelian group) If $ab = ba$ for all $a, b \in G$ the group is called Abelian

Examples of groups 1

Example 1: Permutations of n objects

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5. countably infinite group

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Examples for uncountably infinite continuous groups

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$$\mathrm{U}(1) = \{z \in \mathbb{C} \,|\, |z| = 1\} = \{\exp(i\varphi) \,|\, 0 \leq \varphi < 2\pi\}$$

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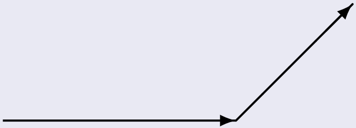

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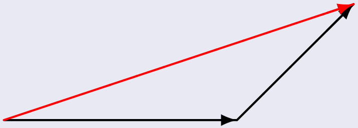

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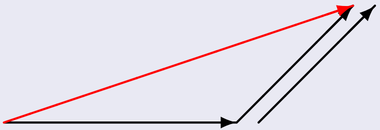

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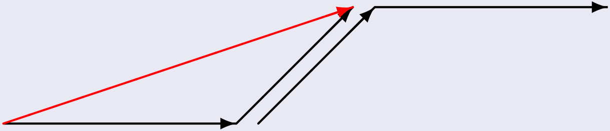

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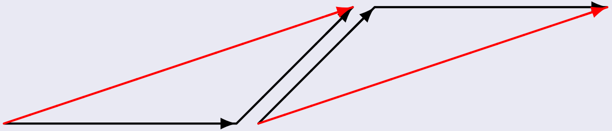

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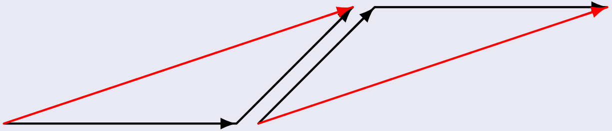

- Abelian (order of composition does not matter)
- Not compact (parameters of a translation may be arbitrarily large)

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Examples of groups 4

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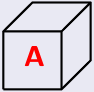

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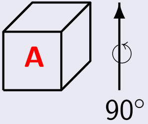

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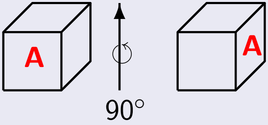

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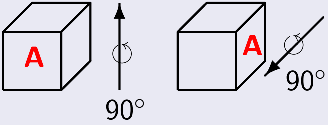

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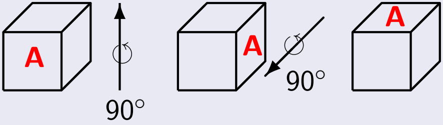

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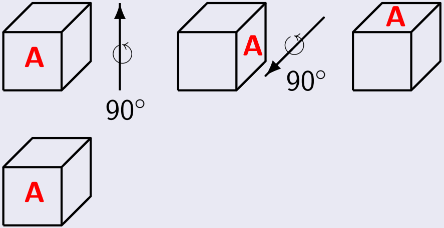

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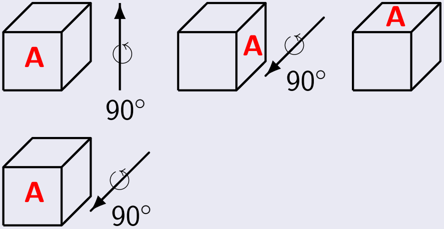

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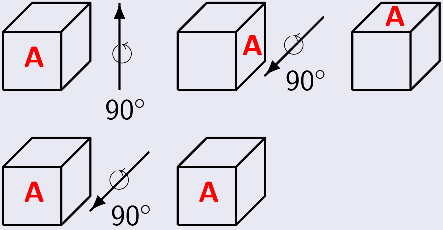

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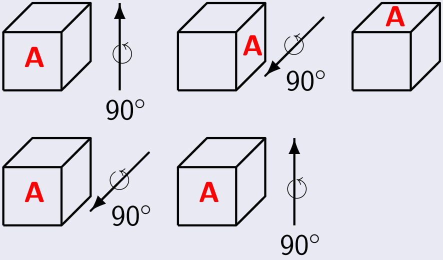

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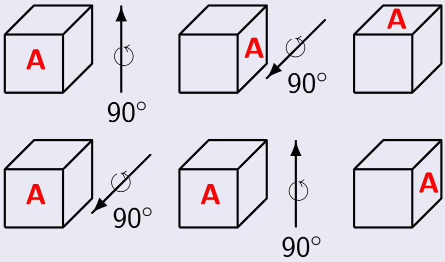

- Not Abelian (order of composition matters)
- Compact

Examples of groups 5

4. Unitary group U(2):

$$\mathrm{U}(2) := \{U|\text{complex } 2\times 2 \text{ matrix}, U^\dagger U = UU^\dagger = \mathbb{1}\}$$

with matrix multiplication as composition

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6. Analogous U(n) and SU(n): Quark model, QCD, etc.

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3. Nonlinear realization (spontaneous symmetry breaking)

Action of a group 1

Definition

Let $M = \{m\}$ and G be a non-empty set and a group, respectively. A mapping A, which associates with each pair $(g, m) \in G \times M$ a unique element $A(g, m) \in M$, defines an action of the group G on M, if the following conditions are satisfied:

1. $A(e, m) = m \ \forall \ m \in M$,
2. $A(g_1, A(g_2, m)) = A(g_1 g_2, m) \ \forall \ g_1, g_2 \in G, \ \forall \ m \in M$.

Applications in Physics

1. M: states, dynamical variables, fields, ...
2. M vector space $\Rightarrow$ representation of a group
3. Nonlinear realization (spontaneous symmetry breaking)
4. Symmetry $\leftrightarrow$ group invariants

Action of a group 2

Example from classical physics

The Hamiltonian of a particle in a central potential,

$$H(\vec{p},\vec{x}) = \frac{\vec{p}^{\,2}}{2m} + V(|\vec{x}\,|),$$

is invariant under

$$\begin{aligned} x_i &\mapsto \sum_{j=1}^{3} R_{ij} x_j, \\ p_i &\mapsto \sum_{j=1}^{3} R_{ij} p_j, \end{aligned}$$

where $R \in \mathrm{O}(3)$. $\Rightarrow$

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The angular momentum is a conserved quantity.

Action of a group 3

Example from quantum mechanics

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}: \quad \text{electron polarized in positive } z \text{ direction}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \frac{\hbar}{2}\sigma_z, \quad \sigma_z|\uparrow\rangle = \vec{\sigma}\cdot\hat{e}_z|\uparrow\rangle = |\uparrow\rangle.$$

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$$\hat{n} = \sin(\theta)\cos(\phi)\hat{e}_x + \sin(\theta)\sin(\phi)\hat{e}_y + \cos(\theta)\hat{e}_z$$

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$$\begin{aligned} U(\phi,\hat{e}_z)U(\theta,\hat{e}_y)|\uparrow\rangle &= \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right)\\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ &= \begin{pmatrix}\cos\left(\frac{\theta}{2}\right)e^{-i\frac{\phi}{2}}\\ \sin\left(\frac{\theta}{2}\right)e^{i\frac{\phi}{2}}\end{pmatrix}\end{aligned}$$

Action of a group 4

Example from field theory

$$G = \mathrm{O}(2) = \mathrm{SO}(2) \,\dot\cup\, S_1\mathrm{SO}(2),$$

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}: \quad \text{reflection over the 1-axis.}$$

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Each $g \in G$ can be written either as

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where $0 \leq \varphi < 2\pi$.

Action of a group 5

Lagrange density

$$\mathcal{L}(\Phi_1, \Phi_2, \partial_\mu \Phi_1, \partial_\mu \Phi_2) = \frac{1}{2} \sum_{i=1}^{2} \left(\partial_\mu \Phi_i \partial^\mu \Phi_i - m_i^2 \Phi_i^2 \right) - \mathcal{V}(\Phi_1, \Phi_2)$$

of two real scalar fields $\Phi_i(t, \vec{x})$, $\Phi_i \in C^2(M^4)$, $i = 1, 2$,
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Define the action of the group G on $M = \{(\Phi_1, \Phi_2)\}$,

$$\begin{pmatrix}\Phi_1' \\ \Phi_2'\end{pmatrix} := A(R(\varphi), (\Phi_1, \Phi_2)) := \begin{pmatrix}\cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi)\end{pmatrix}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} \in M,$$

for $R(\varphi) \in \mathrm{SO}(2)$ and analogously for $S_1 R(\varphi) \in S_1\mathrm{SO}(2)$.

Action of a group 6

Note

$$A(R(0),(\Phi_1,\Phi_2)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix},$$

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- $\mathcal{V}$ is a function of $\Phi_1^2 + \Phi_2^2$.

Remarks

1. Since $\mathrm{U}(1) \cong \mathrm{SO}(2)$, the invariant Lagrange density may be used to describe a pair of oppositely charged (pseudo-)scalar particles. The coupling to the electromagnetic field is generated in terms of the gauge principle.
2. S_1 may be regarded as the charge conjugation transformation.
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Outlook

The Lagrangian of the Standard Model of Particle Physics is a group invariant with $G =$SU(3)$\times$SU(2)$\times$U(1). The construction requires the (local) operation of the group G on the set of the quarks, leptons (matter fields) and the gauge bosons and the Higgs fields.

SU(N) and quarks 1

Aim: Classification of composite states

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SU(N) and quarks 1

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A simpler and more elegant scheme can be constructed if we allow non-integral values for the charges. We can dispense entirely with the basic baryon b if we assign to the triplet t the following properties: spin $\frac{1}{2}$, $z = -\frac{1}{3}$, and baryon number $\frac{1}{3}$. We then refer to the members $u^{\frac{2}{3}}$, $d^{-\frac{1}{3}}$, and $s^{-\frac{1}{3}}$ of the triplet as "quarks" 6) q and the members of the anti-triplet as anti-quarks $\bar{q}$. Baryons can now be constructed from quarks by using the combinations $(q\,q\,q)$, $(q\,q\,q\,q\,\bar{q})$, etc., while mesons are made out of $(q\,\bar{q})$, $(q\,q\,\bar{q}\,\bar{q})$, etc. It is assuming that the lowest baryon configuration $(q\,q\,q)$ gives just the representations **1**, **8**, and **10** that have been observed, while the lowest meson configuration $(q\,\bar{q})$ similarly gives just **1** and **8**.

6) James Joyce, Finnegan's Wake: „three quarks for Muster Mark"

Physical motivation

SU(N) and quarks 2

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1. Evidence for substructure of hadrons

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SU(N) and quarks 2

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SU(N) and quarks 2

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4. Matter fields of QCD (quarks) are fermions with spin 1/2, which show up in six different *flavors*

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SU(N) and quarks 3

Light quarks

flavor	u	d	s
masse [MeV]	$2.2^{+0.6}_{-0.4}$	$4.7^{+0.5}_{-0.4}$	96^{+8}_{-4}
charge [$e > 0$]	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
I_3	$+\frac{1}{2}$	$-\frac{1}{2}$	0
			strangeness: -1

SU(N) and quarks 4

Heavy quarks

flavor	c	b	t
mass [GeV]	1.28 ± 0.03	$4.18^{+0.04}_{-0.03}$	173.1 ± 0.6
charge [$e > 0$]	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
I_3	0	0	0
	charm: $+1$	bottom: -1	top: $+1$

See `http://pdg.lbl.gov`

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SU(N) and quarks 5

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SU(N) and quarks 5

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   - Solution: Slater determinant

$$\frac{1}{\sqrt{6}}\begin{vmatrix} r_1 & g_1 & b_1 \\ r_2 & g_2 & b_2 \\ r_3 & g_3 & b_3 \end{vmatrix}$$

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   - General N_c:
$$\frac{1}{\sqrt{N_c!}}\epsilon_{i_1\ldots i_{N_c}}\chi^{i_1}\otimes\ldots\otimes\chi^{i_{N_c}}$$

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SU(N) and quarks 6

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11. Mesons: $q\bar{q}$ (quark-antiquark) states; color neutral via $\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$

Quark model

SU(N) and quarks 7

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SU(N) and quarks 7

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SU(N) and quarks 7

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$$|\uparrow\rangle, \quad |\downarrow\rangle$$

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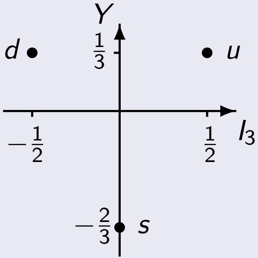

SU(N) und quarks 8

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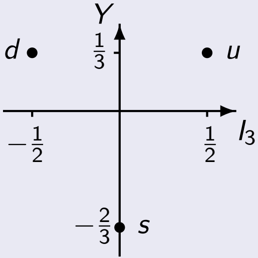

3. Antiquark triplet

   $|\bar{u}\rangle$

   $|\bar{d}\rangle$

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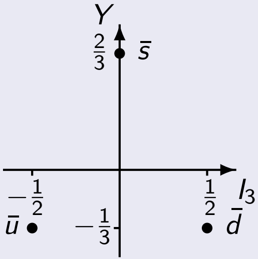

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2. Antiquarks transform under the complex conjugate representation:

$$\bar{q} = \begin{pmatrix} \psi_{\bar{u}} \\ \psi_{\bar{d}} \\ \psi_{\bar{s}} \end{pmatrix}, \quad \bar{q} \mapsto U^* \bar{q}.$$

SU(N) and quarks 10

SU(6)

Assume a Hamiltonian, where the interaction between quarks does not depend on spin and flavor.

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Composite states

Description in terms of tensor product

$$X \otimes X \otimes X \qquad \text{for baryons}$$

$$X \otimes X^* \qquad \text{for mesons}$$

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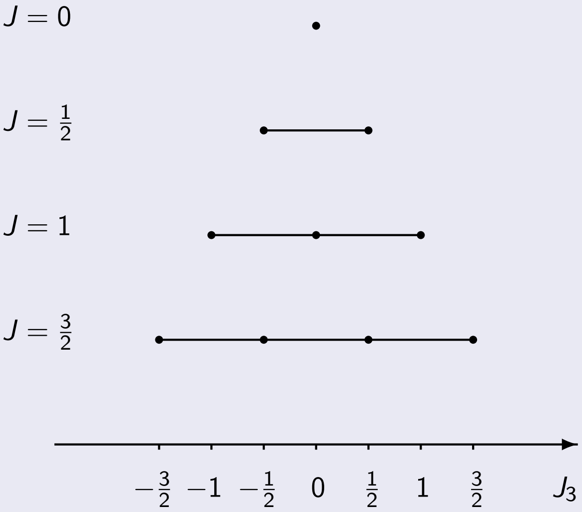
Graphical method, spin



Sequential coupling

Eigenvalues of $J_3 = J_3(1) + J_3(2)$ are additive

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$$\frac{1}{2} \otimes \frac{1}{2} =$$

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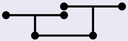

$$=$$

$$= 1 \oplus 0$$

Symmetry properties

$$\mathcal{H} = \mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}} = \mathcal{H}_1 \oplus \mathcal{H}_0$$

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$$\begin{aligned} |1,1\rangle &:= |\uparrow,\uparrow\rangle, \\ |1,0\rangle &:= \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle), \\ |1,-1\rangle &:= |\downarrow,\downarrow\rangle \end{aligned}$$

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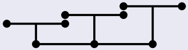

$$=$$

$$= \frac{3}{2} \oplus \frac{1}{2}$$

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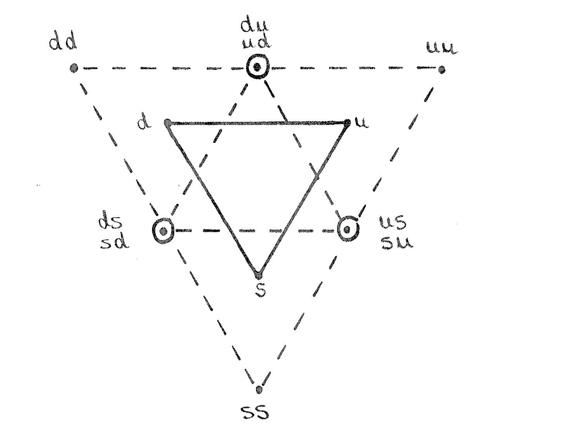
Number of states

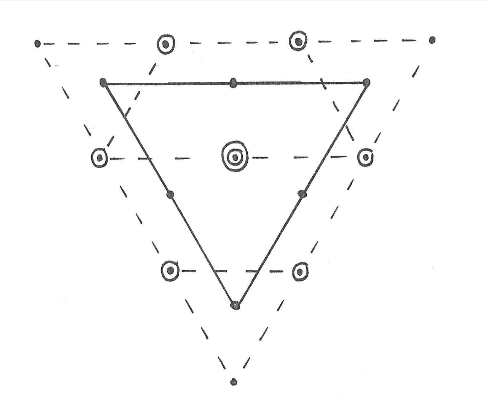
$$2 \cdot 2 \cdot 2 = 4 + 2 + 2$$

Coupling in SU(3) analogous

$$3 \otimes 3 = 6 \oplus \bar{3}$$



$$6 \otimes 3 = 10 \oplus 8$$



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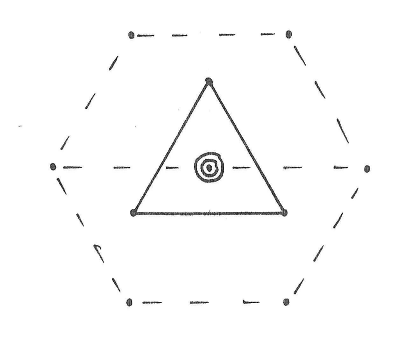
$$\bar{3} \otimes 3 = 8 \oplus 1$$



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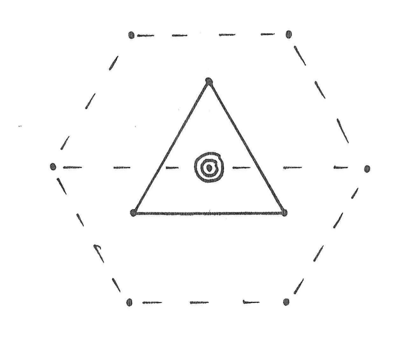
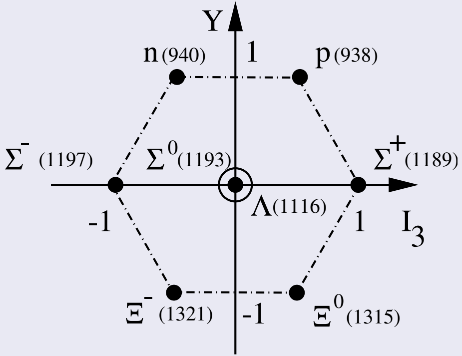


Dimensions of the vector spaces

$$3 \cdot 3 \cdot 3 = 10 + 8 + 8 + 1$$

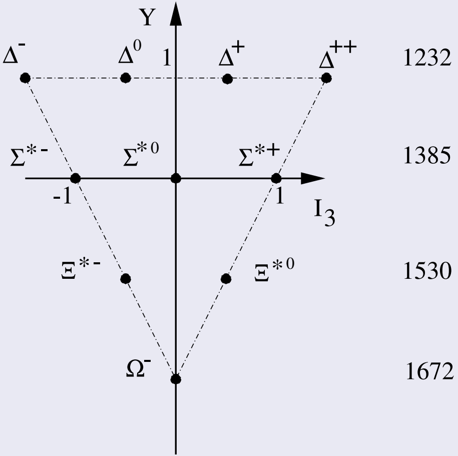
Baryon octet



Y=B+S, B=1

baryon octet with $J = \frac{1}{2}$

Baryon decuplet



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baryon decuplet with $J = \frac{3}{2}$

Mathematical procedure

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1. Decompose the tensor product into a direct sum

$$Z = X \otimes X \otimes X = \bigoplus_{j=1}^{M} P_j(Z)$$

Each of the linear subspaces $P_j(Z)$ is irreducible with respect to the product representation (states of different subspaces do not mix under transformations)

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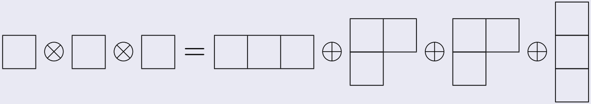

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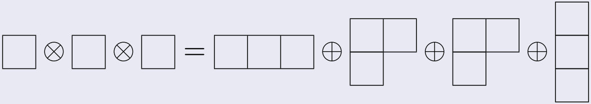

$$6 \otimes 6 \otimes 6 \;=\; \underbrace{56}_{\mathrm{S}} \oplus \underbrace{70}_{\mathrm{M,S}} \oplus \underbrace{70}_{\mathrm{M,A}} \oplus \underbrace{20}_{\mathrm{A}}$$

4. Physical interpretation

$$56 = \underbrace{10}_{\text{SU(3) decuplet}} \cdot \underbrace{4}_{\text{spin } 3/2} + \underbrace{8}_{\text{SU(3) octet}} \cdot \underbrace{2}_{\text{spin } 1/2}$$

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- Realization for $3q$ baryons in terms of color Slater determinant

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Gauge theories 1

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5. Introduce additional, so-called gauge fields, in order to guarantee invariance under local transformations
6. $\Rightarrow$ interaction between gauge fields and elementary particles

Example: Quantum electrodynamics (QED, U(1), Abelian)

Lagrange density of a free electron:

$$\mathcal{L}_0(\Psi, \partial_\mu \Psi) = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi,$$

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix},$$

$$\partial\!\!\!/ = \gamma^\mu \partial_\mu,$$

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix},$$

$$\bar{\Psi} = \Psi^\dagger \gamma^0.$$

Gauge theories 3

$\mathcal{L}_0$ is invariant under a *global* U(1) transformation:

$$\Psi(x) \mapsto \Psi'(x) = e^{-i\alpha}\Psi(x),$$
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Remark

All components Ψ_i are multiplied by the same phase.

Transformation property

Convention: Electron has negative electric charge ($q_e = -1$)

$$\mathrm{U}(1) \ni e^{-i\alpha} \mapsto e^{-i\alpha q_e} = e^{i\alpha}$$

We make use of the following **local** transformation:

$$\Psi(x) \mapsto e^{i\alpha(\mathbf{x})}\Psi(x).$$

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Gauge theories 4

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Gauge potential $\mathcal{A}_\mu(x)$

$$\mathcal{A}_\mu(x) \mapsto \mathcal{A}'_\mu(x) = \mathcal{A}_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \quad e > 0,$$

$$\begin{aligned} D_\mu \Psi(x) &:= [\partial_\mu - ie\mathcal{A}_\mu(x)]\Psi(x) \mapsto D'_\mu \Psi'(x) \\ &= [\partial_\mu - ie\mathcal{A}_\mu(x) - i\partial_\mu \alpha(x)] \left[e^{i\alpha(x)} \Psi(x) \right] \\ &= e^{i\alpha(x)} [\partial_\mu + i\partial_\mu \alpha(x) - ie\mathcal{A}_\mu(x) - i\partial_\mu \alpha(x)]\Psi(x) \\ &= e^{i\alpha(x)} [\partial_\mu - ie\mathcal{A}_\mu(x)]\Psi(x). \end{aligned}$$

Gauge theories 5

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\end{aligned}
$$

New Lagrange density

$$
\mathcal{L}_0(\Psi, D_\mu\Psi) = \bar{\Psi}(i\not{D} - m)\Psi = \mathcal{L}_0(\Psi, \partial_\mu\Psi) + e\bar{\Psi}\gamma^\mu\Psi\mathcal{A}_\mu.
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Gauge theories 5

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Invariant under so-called gauge transformation of the second kind:

$$\Psi(x) \mapsto e^{i\alpha(x)}\Psi(x),$$

$$\mathcal{A}_\mu(x) \mapsto \mathcal{A}_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x).$$

Lagrange density of QED

Interpret $\mathcal{A}_\mu$ as a dynamical variable. Define field-strength tensor

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$$

and introduce a "kinetic" term:

$$\mathcal{L}_{\mathrm{QED}} = \bar{\Psi} i \gamma^\mu (\partial_\mu - ie\mathcal{A}_\mu)\Psi - m\bar{\Psi}\Psi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}.$$

Lagrange density of QED

Interpret $\mathcal{A}_\mu$ as a dynamical variable. Define field-strength tensor

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Gauge theories 6

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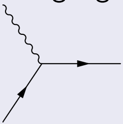
- After quantization, the dynamical gauge field is identified with the photon.
- Interaction between the matter field and the gauge field

$$\mathcal{L}_{\mathrm{int}} = -(-e)\bar{\Psi}\gamma^\mu\Psi\mathcal{A}_\mu = -J^\mu_{\mathrm{em}}\mathcal{A}_\mu$$



Remarks

1. A mass term

$$\frac{1}{2}M^2\mathcal{A}_\mu\mathcal{A}^\mu \mapsto \frac{1}{2}M^2(\mathcal{A}_\mu\mathcal{A}^\mu + \frac{2}{e}\partial_\mu\alpha\mathcal{A}^\mu + \frac{1}{e^2}\partial_\mu\alpha\partial^\mu\alpha)$$
$$\neq \frac{1}{2}M^2\mathcal{A}_\mu\mathcal{A}^\mu$$

would destroy gauge invariance.
Gauge bosons are massless! (without spontaneous symmetry breaking)

2. The coupling of the photon to the matter field is dictated by the transformation property of the matter field under U(1). Consider matter field Ψ_q for a particle with charge q

$$\Psi_q(x) \mapsto e^{-iq\alpha}\Psi_q(x),$$

$\Rightarrow$ so-called minimal substitution $(\partial_\mu \mapsto \partial_\mu + ieq\mathcal{A}_\mu)$

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- Electron: $q = -1$
- Proton: $q = +1$
- Neutron: $q = 0$
- up quark: $q = 2/3$
- etc.

3. The requirement of renormalizability of QED excludes further gauge-invariant couplings such as the coupling to an anomalous magnetic moment,

$$-\frac{e\kappa}{4m}\mathcal{F}_{\mu\nu}\bar{\Psi}\sigma^{\mu\nu}\Psi, \quad \sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu].$$

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   This is not a group-theoretical argument!

4. Due to the Abelian nature of U(1), photons do not directly interact with each other.

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Non-Abelian case

Non-Abelian case

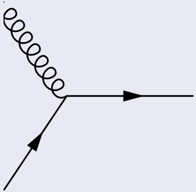
Quantum chromodynamics (QCD, SU(3))

Gauge theories 9

Non-Abelian case

Quantum chromodynamics (QCD, SU(3))
Matter fields: Quark fields u_A, d_A, ..., $A = 1, 2, 3$
Gauge fields: Gluons $\mathcal{A}_a$, $a = 1, \ldots, 8$

Gluon-quark interaction



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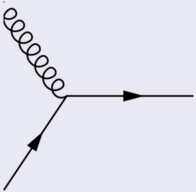

New: Gluons interact with each other (because SU(3) is non-Abelian)

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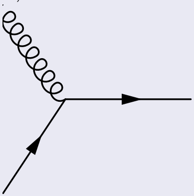

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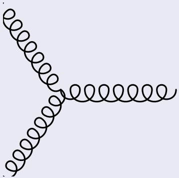
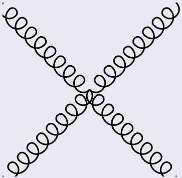
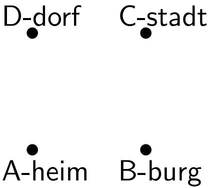




Spontaneous symmetry breaking 1

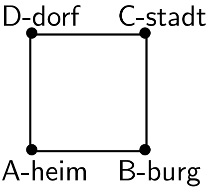
1. Illustration



Goal: Find the shortest routes network connecting the four cities

Spontaneous symmetry breaking 2

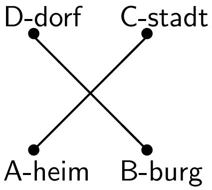
1. Illustration



Total length: $4a$ (a side length of the square)

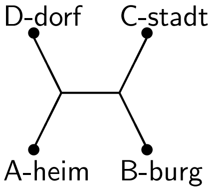
Spontaneous symmetry breaking 3

1. Illustration



Total length: $2\sqrt{2}\,a < 4a$

1. Illustration



Total length: $(1+\sqrt{3})a < 2\sqrt{2}\,a < 4a$

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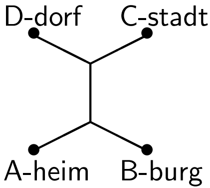
1. Illustration



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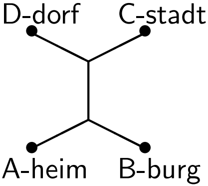

object	cities	Hamilton operator
symmetry	D_4	G
criterion	shortes routes network	ground state
symmetry of solution	D_2	subgroup H of G

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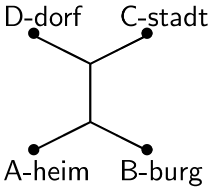
Spontaneous symmetry breaking 5

1. Illustration

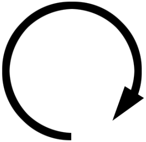


object	cities	Hamilton operator
symmetry	D_4	G
criterion	shortes routes network	ground state
symmetry of solution	D_2	subgroup H of G

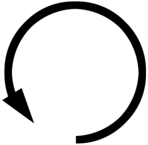
2. Goldstone-Theorem (1961, 1962): For each generator of the Lie group G which does not annihilate the ground state, one obtains a massless Goldstone boson.

$<$ Greek *cheir* $\gg$hand$\ll$

right-handed (clockwise) screw



left-handed (counterclock-wise) screw



mirror

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Spontaneous symmetry breaking in QCD 1

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- Theoretical limit: $m_u = m_d = m_s = 0$

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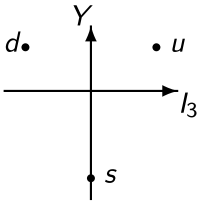

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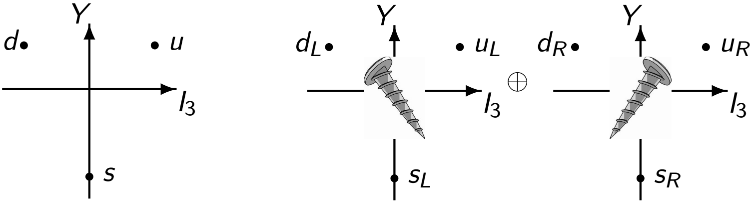

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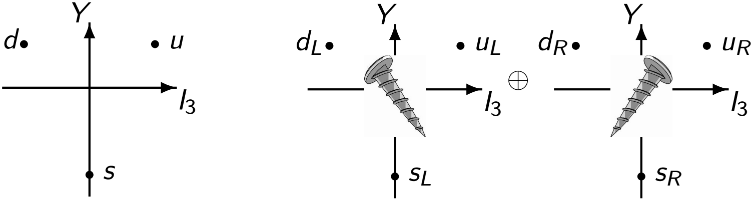

- H_0 is invariant under $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$

Spontaneous symmetry breaking in QCD 1

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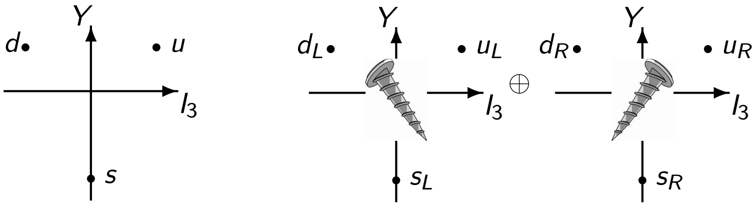
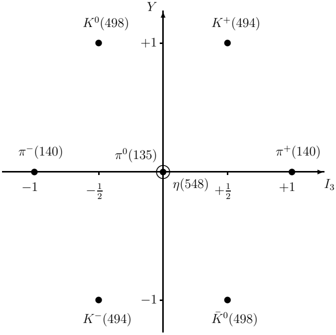


- H_0 is invariant under $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$
- Ground state is invariant under $H = \mathrm{SU}(3)_V$ only

Spontaneous symmetry breaking in QCD 2

- 8 (almost) massless Goldstone bosons: π, K, η



- Physical masses result from explicit symmetry breaking:

$$m_u = 2.2\,\text{MeV}, \quad m_d = 4.7\,\text{MeV}, \quad m_s = 96\,\text{MeV}.$$

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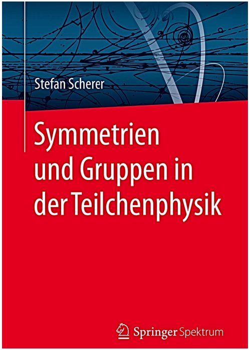
1. H. F. Jones, *Groups, Representations and Physics* (Adam Hilger, Bristol, 1990)
2. S. Scherer, *Symmetrien und Gruppen in der Teilchenphysik* (Springer Spektrum, Berlin, 2016)



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