Handout 3 (read by Nov. 6)

Homogeneous Lorentz group L or O(1,3)

Let $V = \mathbb{R}^4$. Define the Minkowski metric as

$$M(x,x) := x_0 x_0 - \sum_{i=1}^{3} x_i x_i = x^T G x,$$

with

$$x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad G = (G_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad i,j = 0,1,2,3.$$

The enumeration matches the convention used in Physics.

Consider the linear map

$$x \mapsto x' = \Lambda x, \quad x'_i = \Lambda_{ij} x_j,$$

with

$$M(x',x') = M(x,x) \Leftrightarrow x^T \Lambda^T G \Lambda x = x^T G x,$$

i.e.

$$\Lambda^T G \Lambda = G. \qquad (*)$$

The abstract group is defined as

$$\mathrm{L} = \mathrm{O}(1,3) := \{\Lambda \in \mathrm{GL}(4,\mathbb{R}) | \Lambda^T G \Lambda = G\}.$$

Properties of the 4×4 matrices Λ

1. $\det(\Lambda) = \pm 1$, because

   $$\det(G) = -1 = \det(\Lambda^T G \Lambda) = \underbrace{\det(\Lambda^T)}_{= \det(\Lambda)} \det(G) \det(\Lambda) = -(\det(\Lambda))^2.$$

   $\det(\Lambda) = +1$: proper Lorentz transformations

2. Either $\Lambda_{00} \geq 1$ or $\Lambda_{00} \leq -1$.

   Explanation: Consider the matrix equation $(*)$ for $i = j = 0$:

   $$1 = G_{00} = \underbrace{\Lambda^T_{0k}}_{= \Lambda_{k0}} G_{kl} \Lambda_{l0} = \Lambda_{00}^2 \underbrace{- \sum_{k=1}^{3} \Lambda_{k0}^2}_{\leq 0} \quad \Rightarrow \quad \Lambda_{00}^2 = 1 + \sum_{k=1}^{3} \Lambda_{k0}^2 \geq 1 \quad \Rightarrow \quad |\Lambda_{00}| \geq 1.$$

   $\Lambda_{00} \geq 1$: orthochronous Lorentz transformations

3. Four so-called branches $\mathrm{L}_{+,-}^{\uparrow,\downarrow}$:

(a) $+:\ \det(\Lambda) = 1$,

(b) $-:\ \det(\Lambda) = -1$,

(c) $\uparrow$: $\Lambda_{00} \geq 1$,

(d) $\downarrow$: $\Lambda_{00} \leq -1$.

4. Subgroup of **proper, orthochronous Lorentz transformations**:

$$\mathrm{L}_+^\uparrow := \{\Lambda \in \mathrm{GL}(4, \mathbb{R}) | \Lambda^T G \Lambda = G, \det(\Lambda) = +1, \Lambda_{00} \geq 1\}.$$

5. Special transformations

(a) identity

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{L}_+^\uparrow$$

(b) parity

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathrm{L}_-^\uparrow$$

(c) time reversal

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{L}_-^\downarrow$$

(d) product PT

$$PT = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathrm{L}_+^\downarrow.$$

$\{E, P, T, PT\}$: Klein four-group (Felix Klein, 1849 - 1925), Abelian (diagonal matrices commute)

6. $\mathrm{L}_+^\downarrow$, $\mathrm{L}_-^\uparrow$, $\mathrm{L}_-^\downarrow$ are not subgroups of L, because they do not contain the identity.

7. $\mathrm{L} = \mathrm{L}_+^\uparrow \cup P\mathrm{L}_+^\uparrow \cup T\mathrm{L}_+^\uparrow \cup PT\mathrm{L}_+^\uparrow$ (explanation analogous to O(3))

Discussion of $\mathbf{L}_+^\uparrow$

1. Proper rotations (strictly speaking isomorphic to SO(3)):

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \quad \text{with} \quad R \in \mathrm{SO}(3).$$

   For a discussion of R (in terms of Euler angles), see examples 1.2.15.

2. Lorentz boosts $L\left(\frac{\vec{p}}{E_p}\right)$

   (a) Interpretation as a (passive) transformation in a primed coordinate frame which moves relative to the rest frame of a particle of mass m with $-\frac{\vec{p}}{E_p}$

   (b) Interpretation as an active transformation with

$$(m, \vec{0}\,) \mapsto (E_p, \vec{p}\,), \quad E_p = \sqrt{m^2 + \vec{p}^{\,2}}.$$

   Make use of the bra-ket notation

$$|\vec{p}\rangle = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}, \quad \langle \vec{p}| = \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}.$$

   A Lorentz boost may then be written as

$$L\left(\frac{\vec{p}}{E_p}\right) = \frac{1}{m} \left(\begin{array}{c|c} E_p & \langle \vec{p}| \\ \hline |\vec{p}\rangle & m\mathbb{1}_{3\times 3} + \frac{|\vec{p}\rangle\langle\vec{p}|}{E_p + m} \end{array} \right),$$

   where

$$\begin{aligned} 1 \leq \frac{E_p}{m} &=: \gamma, \\ \vec{v} = \frac{\vec{p}}{\gamma m} = \frac{\vec{p}}{E_p} &=: \beta\hat{n}, \quad 0 \leq \beta < 1, \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}}. \end{aligned}$$

   Example: $\vec{p} = |\vec{p}|\hat{e}_z$, i.e., the primed system moves with $-\beta\hat{e}_z$ relative to the unprimed system, where $\beta = |\vec{p}|/E_p$:

$$L(\beta\hat{e}_z) = L\left(\frac{|\vec{p}|}{E_p}\hat{e}_z\right) = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}.$$

We made use of:

$$
\begin{aligned}
\frac{E_p}{m} &= \gamma, \\
\frac{p_z}{m} = \frac{|\vec{p}|}{m} &= \beta\gamma, \\
|\vec{p}\rangle\langle\vec{p}| &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |\vec{p}|^2 \end{pmatrix}, \\
\frac{|\vec{p}|^2}{m(E_p+m)} = \frac{(E_p+m)(E_p-m)}{m(E_p+m)} &= \frac{E_p-m}{m} = \gamma - 1.
\end{aligned}
$$

Check:

$$
L(\beta\hat{e}_z)\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma m \\ 0 \\ 0 \\ \beta\gamma m \end{pmatrix} = \begin{pmatrix} E_p \\ 0 \\ 0 \\ |\vec{p}| \end{pmatrix}.
$$

Any Lorentz boost may be expressed in terms of three parameters, e.g., the direction $\hat{n}$ and the rapidity λ with $0 \leq \lambda < \infty$, where

$$
\sinh(\lambda) = \beta\gamma, \quad \cosh(\lambda) = \gamma, \quad \beta = \tanh(\lambda) = \frac{|\vec{p}|}{E_p}.
$$

3. Any proper, orthochronous Lorentz transformation $\Lambda \in \mathrm{L}_+^\uparrow$ may be written as

$$
\Lambda = L\left(\frac{\vec{p}}{E_p}\right)\mathcal{R}(\alpha, \beta, \gamma).
$$

Note that there are different ways to parametrize $\Lambda \in \mathrm{L}_+^\uparrow$. However, in any case you need 6 real parameters to fully specify a $\Lambda \in \mathrm{L}_+^\uparrow$.