Handout 2 (read by Oct 30)

Example 1.3.6 from field theory

$$G = \mathrm{O}(2) = \mathrm{SO}(2) \cup S_1\mathrm{SO}(2),$$
$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \quad \text{reflection over the 1-axis.}$$

Each $g \in G$ can be written either as

$$R(\varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}, \quad \det(R(\varphi)) = 1,$$

or as

$$S_1R(\varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ -\sin(\varphi) & -\cos(\varphi) \end{pmatrix}, \quad \det(S_1R(\varphi)) = -1,$$

where $0 \le \varphi < 2\pi$.

Model: Consider two real scalar fields $\Phi_i(t, \vec{x})$, $\Phi_i \in C^2(M^4)$, $i = 1, 2$,
M^4: Minkowski space.

Lagrange density (in natural units $\hbar = c = 1$),

$$\mathcal{L}(\Phi_1, \Phi_2, \partial_\mu\Phi_1, \partial_\mu\Phi_2) = \frac{1}{2}\sum_{i=1}^{2}\left(\partial_\mu\Phi_i\partial^\mu\Phi_i - m_i^2\Phi_i^2\right) - \mathcal{V}(\Phi_1, \Phi_2).$$

Define the action of the group G on $M = \{(\Phi_1, \Phi_2)\}$,

$$\begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} := A(R(\varphi), (\Phi_1, \Phi_2)) := \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \in M,$$

for $R(\varphi) \in \mathrm{SO}(2)$ and analogously for $S_1R(\varphi) \in S_1\mathrm{SO}(2)$,

$$\begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} := A(S_1R(\varphi), (\Phi_1, \Phi_2)) := \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ -\sin(\varphi) & -\cos(\varphi) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \in M.$$

(The (real) linear combination of two real fields is a real field.)
Note that

$$A(R(0), (\Phi_1, \Phi_2)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix},$$
$$A(g_1, A(g_2, \underbrace{(\Phi_1, \Phi_2)}_{=:\Phi})) = A(g_1, \tilde{R}_2\Phi) = \tilde{R}_1(\tilde{R}_2\Phi)$$
$$= (\tilde{R}_1\tilde{R}_2)\Phi = A(g_1g_2, (\Phi_1, \Phi_2)).$$

(The product of two O(2) matrices is an O(2) matrix.)

The Lagrange density $\mathcal{L}$ is a so-called group invariant, i.e.

$$\mathcal{L}(\Phi_1, \Phi_2, \partial_\mu\Phi_1, \partial_\mu\Phi_2) = \mathcal{L}(\Phi_1', \Phi_2', \partial_\mu\Phi_1', \partial_\mu\Phi_2'),$$

iff

- $m_1 = m_2$

and

- $\mathcal{V}$ is a function of $\Phi_1^2 + \Phi_2^2$.

Remarks:

1. Since U(1) $\cong$ SO(2), the invariant Lagrange density may be used to describe a pair of oppositely charged (pseudo-)scalar particles. The coupling to the electromagnetic field is generated in terms of the gauge principle.

2. S_1 may be regarded as the charge conjugation transformation.

3. Historical remark: In 1934 Pauli and Weisskopf discussed the quantization of the complex Klein-Gordon field with positive and negative charge,

$$\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2),$$
$$\Phi^\dagger = \frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2).$$

4. Noether-Theorem $\Rightarrow$ conservation laws

Outlook 1.3.7 The Lagrangian of the Standard Model of Particle Physics is a group invariant with G =SU(3)$\times$SU(2)$\times$U(1). The construction requires the (local) operation of the group G on the set of the quarks, leptons (matter fields) and the gauge bosons and the Higgs fields.