Handout 1 (read by Oct 23)

Definition 1.1.1 A group G is a non-empty set of elements $\{a, b, \ldots\}$ with a law of composition (multiplication) $(a, b) \mapsto c = ab \in G$ satisfying the following conditions:

1. (associative law) $a(bc) = (ab)c \; \forall \; a, b, c \in G$
2. (unit element) G contains an element, the identity element, denoted by e, such that for all $a \in G$
$$ea = ae = a$$
3. (existence of inverse) For all $a \in G$ there is an element, denoted by a^{-1}, such that
$$aa^{-1} = a^{-1}a = e$$
4. (Abelian group) If $ab = ba$ for all $a, b \in G$ the group is called Abelian

Convention: Interpret order of product ab such that b is applied "before" a.

Terminology 1.1.2

- order $|G|$ = number of group elements (finite, countably infinite, uncountably infinite)
- Suppose that n is the smallest positive integer such that $g^n = e$: n is the order of the element g
- structure of a group = specification of outcome of all possible compositions ab
- A finite group $G = \{g_1, \ldots, g_n\}$ may be represented by a group table $T = (t_{ij})$, where $t_{ij} = g_i g_j \in G$.
- Infinite groups: Specify structure in terms of a composition rule
- Two groups G and G' are isomorphic, $G \cong G'$, if there exists a unique correspondence $g \leftrightarrow g'$ between their elements, which preserves the group structure, i.e.,
$$\underbrace{a'b'}_{\text{product in } G'} = (\underbrace{ab}_{\text{product in } G})'.$$
Isomorphic groups have the same group structure.
- faithful realization of an abstract group = one-to-one mapping of the abstract group onto a group of concrete elements with a concrete specification of the group multiplication which preserves the structure

  All faithful realizations of an abstract group are isomorphic to the group and to each other.

  Non-faithful realizations preserve the structure of the abstract group, but the map is not injective.