

Exercise sheet 7
Theoretical Physics 3 : QM SS2018
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Exercise 0.

How much time did it take to complete the task?

Exercise 1. (20 points)

- a) (15 p.) Show that the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in three dimensions in spherical coordinates takes the form

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

- b) (5 p.) Show that the radial term can also be written as

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r.$$

Exercise 2. (45 points)

- a) (15 p.) Show that $L_{\pm} Y_l^m = \hbar \sqrt{l(l+1) - m(m \pm 1)} Y_l^{m \pm 1}$.

Hint: Consider the norm of $L_{\pm} Y_l^m$.

- b) (15 p.) Show that for eigenfunctions of \hat{L}_z , we have

$$\langle \hat{L}_x \rangle = \langle \hat{L}_y \rangle = 0, \quad \langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle \quad \text{and} \quad \langle \hat{L}_x \hat{L}_y + \hat{L}_y \hat{L}_x \rangle = 0.$$

Hint: Consider $\langle \hat{L}_{\pm} \rangle$ and $\langle \hat{L}_{\pm}^2 \rangle$.

- c) (15 p.) In the state ψ_{lm} with definite angular momentum l and its z -component m , find the mean values $\langle \hat{L}_x^2 \rangle$, $\langle \hat{L}_y^2 \rangle$ as well as the mean values $\langle \hat{L}_{\tilde{z}} \rangle$ and $\langle \hat{L}_{\tilde{z}}^2 \rangle$ of the angular momentum projection along the \tilde{z} -axis making an angle α with the z -axis.

Hint: Use $\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \hat{L}^2$.

Exercise 3. (35 points)

- a) (20 p.) Prove that for a particle in a potential $V(\vec{r})$ the rate of change of the expectation value of the orbital angular momentum \vec{L} is equal to the expectation value of the torque (rotational analog to Ehrenfest's theorem)

$$\frac{d}{dt} \langle \vec{L} \rangle = \langle \vec{N} \rangle \quad \text{where} \quad \vec{N} = \vec{r} \times (-\vec{\nabla} V).$$

Show that $\langle \vec{L} \rangle$ is constant for any spherically symmetric potential. (This is one form of the quantum statement of conservation of angular momentum.)

- b) (15 p.) Show that the mean values of the vectors \vec{L} , \vec{r} , \vec{p} for the particle state with wave function $\psi = \exp(i\vec{p}_0 \cdot \vec{r}/\hbar)\phi(\vec{r})$ are connected by the classical relation $\vec{L} = \vec{r} \times \vec{p}$. Here, p_0 is a real vector and $\phi(\vec{r})$ is a real function.

Exercise 4. (*Bonus* 15 points)

Confirm or invalidate the following assertions:

- a) (10 p.) If $[\hat{H}, \hat{L}] = \vec{0}$, the energy levels do not depend on m (i.e., on the eigenvalues of the projection of one of the components of the angular momentum \hat{L}).
- b) (5 p.) If $[\hat{H}, \hat{L}^2] = 0$, the energy levels do not depend on l .