Exercise sheet 2
Theoretical Physics 3 : QM SS2018
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The goal of this exercise sheet is to investigate the time evolution of particles which were initially "living" in an infinite square well potential of width a,

$$V(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le a \\ +\infty & \text{otherwise,} \end{cases} \tag{1}$$

after an instantaneous shift at the moment $t = 0$ of the left border of the potential by a to the left. So that new potential being

$$\tilde{V}(x) = \begin{cases} 0 & \text{for} \quad -a \le x \le a \\ +\infty & \text{otherwise} \end{cases} \tag{2}$$

and the initial wave functions are no more stationary states for $t > 0$. The "instantaneous shift" implies that it does not affect the state of the particle, which is the same before and immediately after the shift.

The system within the potential $V(x)$ was investigated during the lecture and the stationary states

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n}{a} x\right), \quad n = 1, 2, 3, \ldots$$

with the corresponding energies

$$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{a}\right)^2$$

were found.

Exercise 0.

How much time did it take to complete the task?

Exercise 1. (40 points)

We should start by investigating the stationary states $\tilde{\Psi}_n(x)$ within the potential $\tilde{V}(x)$.

a) Starting from the stationary Schrödinger equation, find the energy spectrum of the system, $\tilde{E}_n$. How could one guess the obtained results using the spectrum E_n?
Apart from the Schrödinger equation itself, which conditions determine the spectrum?

b) Show that the set of the stationary states consists of (spatially) even and odd ones:

$$\tilde{\Psi}_n^{\text{even}}(x) \propto \cos\left(\frac{\pi n x}{2a}\right), \quad n = 1, 3, \ldots$$
$$\tilde{\Psi}_n^{\text{odd}}(x) \propto \sin\left(\frac{\pi n x}{2a}\right), \quad n = 2, 4, \ldots,$$

which can be combined:

$$\tilde{\Psi}_n(x) \propto \sin\left(\frac{\pi n}{2a}(x+a)\right), \quad n = 1, 2, 3, \ldots.$$

Normalize this result.

Exercise 2. (60 points)

The obtained set of stationary states $\tilde{\Psi}_n(x)$ form a complete basis within the linear space spanned by the potential $\tilde{V}(x)$. As result, any function in this space can be written as a linear combination of $\tilde{\Psi}_n(x)$. We are going to apply this fact to the initial wave functions $\Psi_n(x)$.

a) Assuming each initial state is expanded in series

$$\Psi_m(x) = \sum_{n=1}^{\infty} c_n^m \tilde{\Psi}_n(x)$$

Find the expansion coefficients c_n^m.
Hint: Is $\sin(\pi n)/(n^2 - m^2)$ equal to 0 for all $n, m \in \mathbb{Z}$?

b) At times $t > 0$ the corresponding time-dependent wave functions can be found as

$$\Psi_m(x,t) = \sum_{n=1}^{\infty} c_n^m \tilde{\Psi}_n(x,t).$$

Check that normalization holds with time.
What is the probability P_n^m of finding a particle which had initially the energy E_m in the $\tilde{\Psi}_n(x)$ eigenstate of the new system for $t > 0$?
Determine P_2^1.

c) When will the probability of finding any particle in the right half of the well ($0 \leq x \leq a$) become zero?

d) Consider a particle which was initially in the ground state $\Psi_1(x)$.
Calculate the expectation value of the energy at any time $t > 0$.
What is the meaning of this result?
Hint: $\sum_{n=0}^{\infty} \left(\frac{2n+1}{(2n+1)^2-4}\right)^2 = \frac{\pi^2}{16}$

(*Bonus*) Exercise 3. (20 points)

A particle in the infinite square well has as its initial wave function an even mixture of the first and second excited stationary states:

$$\Psi(x,0) = A\,(\,\Psi_2(x) \,+\, \Psi_3(x))\ .$$

Note, that Ψ_2 is the spatially odd wave function of the first excited state of the system ($n = 2$). While Ψ_3 is the spatially even wave function of the second excited state of the system ($n = 3$).

a) Normalize $\Psi(x,0)$.

b) Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. To simplify the result, define $\omega \equiv \pi^2\hbar/2ma^2$.
Check that normalization holds with time.

c) Compute time evolution of $\langle x \rangle$.
What are the frequency and the amplitude of its oscillation?
Compute $\langle p \rangle$ using $\langle p \rangle = m\frac{\mathrm{d}}{\mathrm{d}x}\langle x \rangle$.

d) Find the expectation value of H. How does it compare to E_2 and E_3?