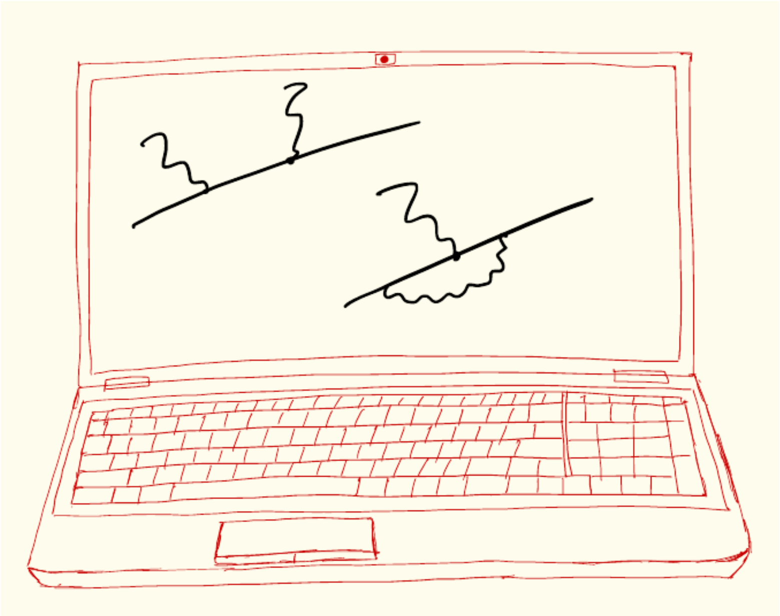
# Computer Algebra for Feynman Graphs



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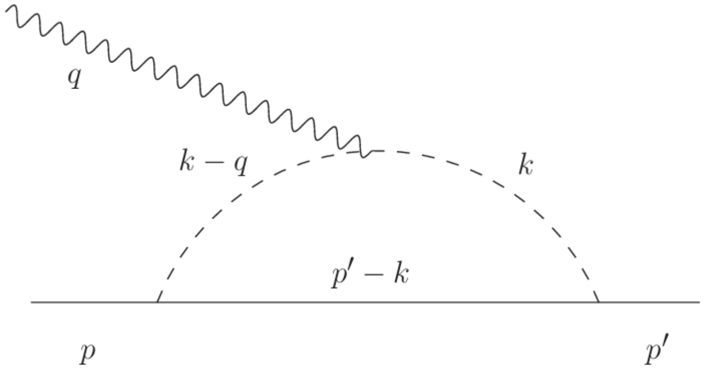
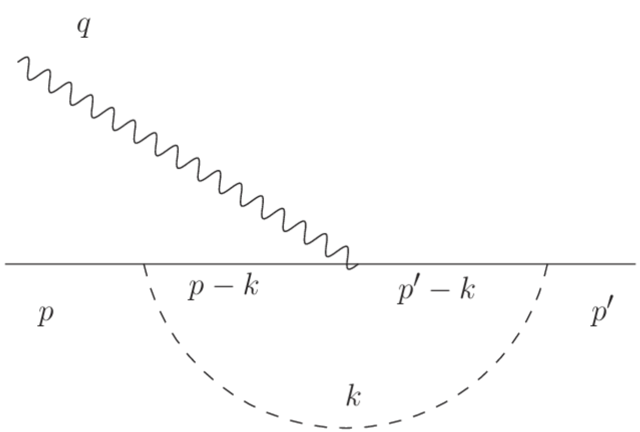
# In this Lecture

- A practical example in LoopTools: renormalization of one-loop electromagnetic vertex and self-energy correction in a theory with nucleons, pions, and photons

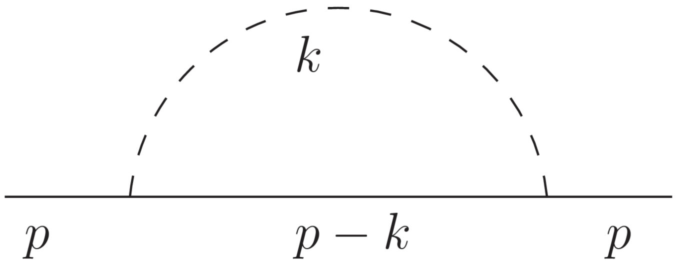
# Example: One-Loop Renormalization

- Let us consider the electromagnetic vertex loop corrections that we looked at earlier, in a theory with interactions of pions, nucleons, and photons:




- Let us also consider the nucleon self-energy correction:



# The Context: Chiral Perturbation Theory

- These graphs are [part of] the leading-order (one-loop) corrections to the electromagnetic interactions of the nucleon due to its interactions with pions (the so-called pion cloud corrections)
- The theory that they originate from is called the Chiral Perturbation Theory (ChPT). It is a low-energy theory of QCD, and an example of what is called an Effective Field Theory (EFT)
- An EFT as a rule has the following three properties:
  - It works in terms of relevant low-energy degrees of freedom (i.e., hadrons – pions and nucleons – rather than quarks and gluons)
  - It inherits the symmetries of the underlying theory (in this case – the chiral symmetry of QCD)
  - It works at low energies, and the amplitudes are expanded in powers of the small parameter, which is the ratio of the low energy to the typical high-energy scale (which is about the mass of the rho meson in ChPT)

# The Lagrangian

- Even though we have the broader context of QCD and ChPT as its effective theory, we can look at these graphs just as coming from a realistic model of pion-nucleon interactions. The Lagrangian of our model is this:

$$\mathcal{L} = \frac{1}{2}[D^\mu \pi]^a [D_\mu \pi]^a - \frac{m_\pi^2}{2}\pi^a \pi^a - \frac{e^2}{32\pi^2 f} F^{\mu\nu}\tilde{F}_{\mu\nu}\pi^3$$

$$+ \bar{N}\left[ i\not{D} - m - i\frac{m g_A}{f}\gamma^5 \pi^a \tau^a + \frac{m g_A^2}{f^2}\pi^a \pi^a \right] N$$

$$+ i\frac{e}{8m}\bar{N}[\gamma^\mu, \gamma^\nu]\left\{ \frac{1}{2}\kappa_p (1+\tau^3) + \frac{1}{2}\kappa_n (1-\tau^3) \right\} N F_{\mu\nu}$$

Pions $\pi^a, \; a = 1, 2, 3$

Nucleons – protons and neutrons

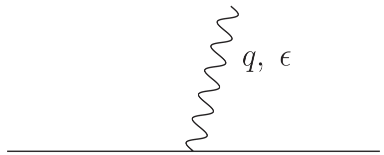
- Interactions with photons are encoded by the covariant derivative,

$$[D_\mu \pi]^a = \partial_\mu \pi^a - e A_\mu \epsilon^{ab3}\pi^b, \;\; \not{D} = \not{\partial} - i\frac{e}{2}(1+\tau^3)\not{A}$$

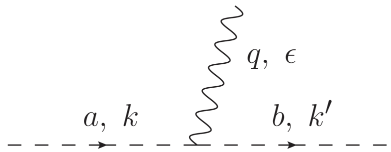
and by the nucleon anomalous magnetic moment coupling (the last line); we also have the chiral anomaly that is responsible for $\pi^0 \to \gamma\gamma$

# The Lagrangian: Feynman Rules

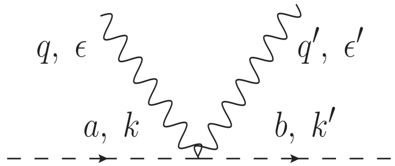
- Relevant Feynman rules that follow from that Lagrangian are



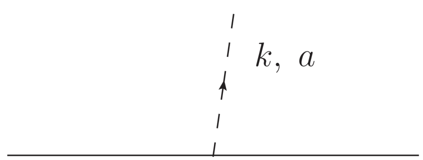

$$ie\left[\frac{1}{2}(1+\tau^3)\gamma^\mu + \frac{\kappa_s + \kappa_v\tau^3}{2m}\gamma^{\mu\nu}q_\nu\right]\epsilon_\mu$$



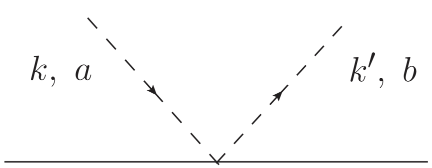

$$e\epsilon^{a3b}(k'^\mu + k^\mu)\epsilon_\mu$$




$$2ie^2(\delta^{ab} - \delta^{a3}\delta^{b3})\epsilon\cdot\epsilon'^*$$




$$-\frac{mg_A}{f_\pi}\gamma^5\tau^a$$




$$\frac{mg_A^2}{2f_\pi^2}\delta^{ab}$$

$q$ outgoing; $\epsilon^\mu$ is the photon polarization vector;

$$\gamma^{\mu\nu} = \tfrac{1}{2}[\gamma^\mu, \gamma^\nu];$$

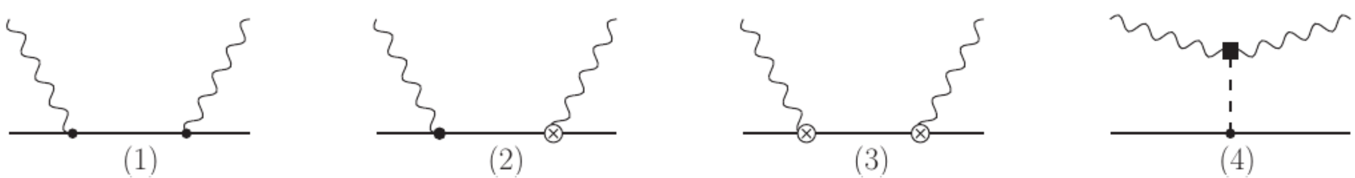
$$\kappa_s = \kappa_p + \kappa_n,\ \ \kappa_v = \kappa_p - \kappa_n$$

Other vertices are not needed or don't contribute in our calculation

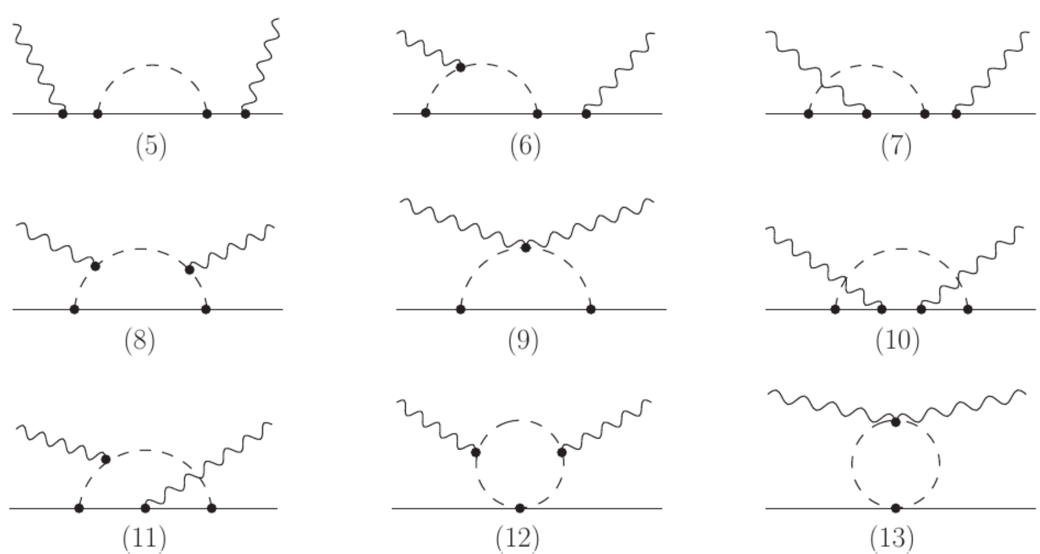
Note that only the leading photon-nucleon coupling (the term with $\gamma^\mu$) is taken into account when we consider the loop diagrams.
The anomalous magnetic moments coupling enters only the tree graphs below.

# The Context: Nucleon Compton Scattering

- As mentioned above, we can consider the interactions as a reasonable model of low-energy interactions of the nucleons with photons and pions
- Furthermore, we will think of the vertex corrections as entering the nucleon Compton scattering amplitude (with both photons real): we have the leading-order tree graphs,

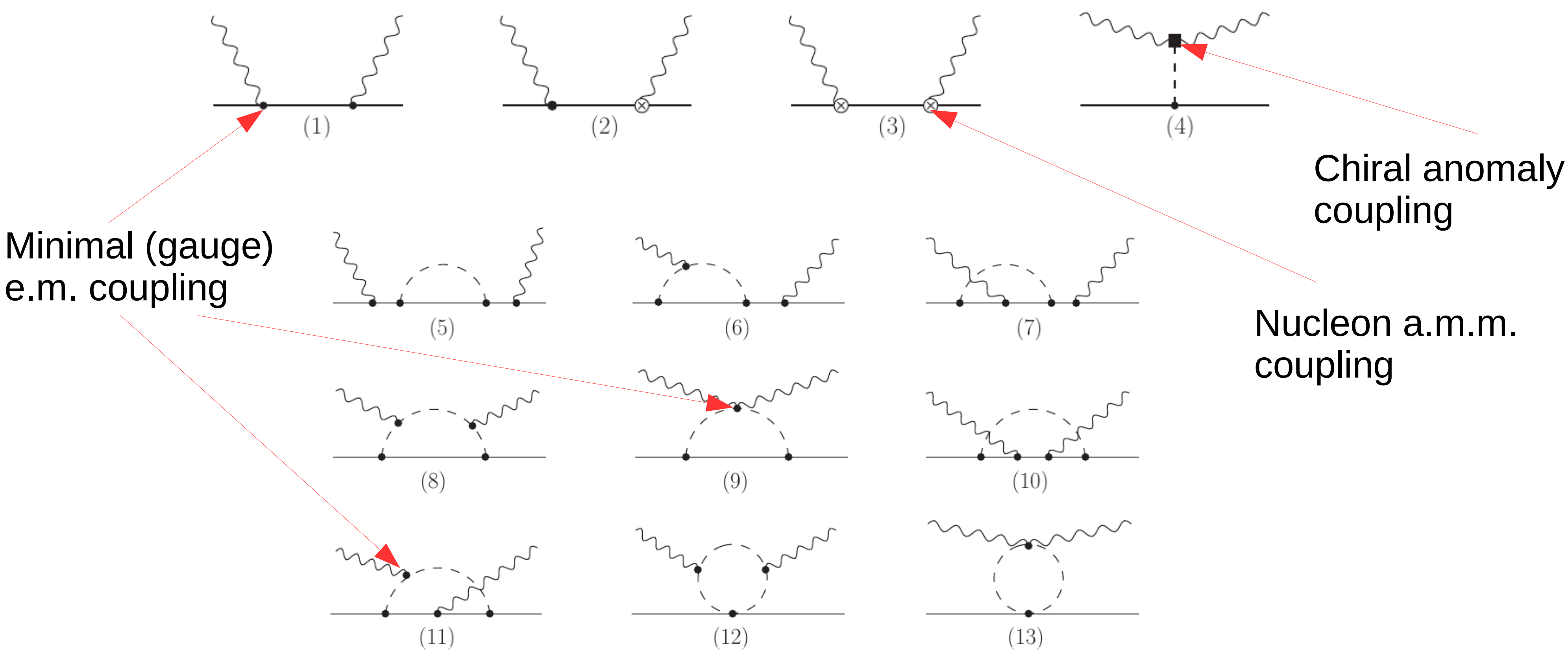



and the one-loop corrections



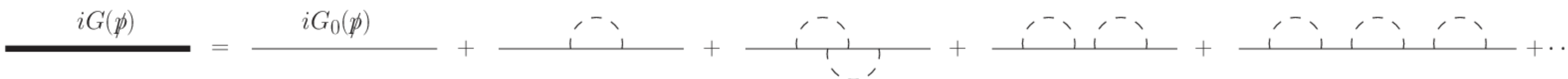
# The Context: Nucleon Compton Scattering

- Loops (5), (6), and (7) have to be properly renormalized, which is what we are going to do; all the other loops do not have divergences and can just be calculated [loops (12) and (13) have divergences separately but they cancel in the sum of these two graphs]



# Self Energy: Sum and Renormalize

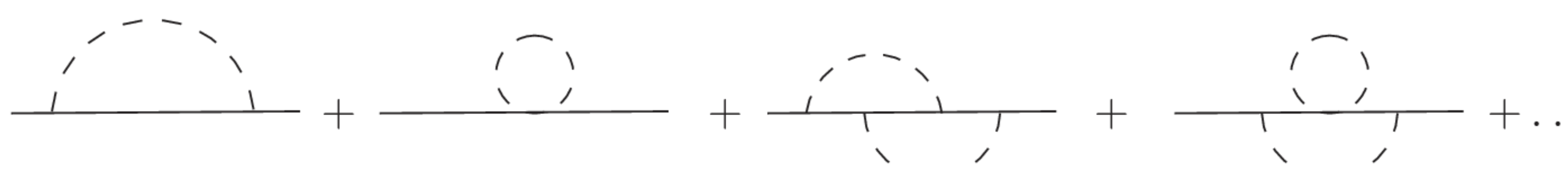
- Let’s consider nucleon (more generally, one-particle) Green’s function and corrections to it (note here the graphs include the external propagators):




$$iG(\not p) = iG_0(\not p) + iG_0(\not p)(\dots)iG_0(\not p) + iG_0(\not p)(\dots)iG_0(\not p)(\dots)iG_0(\not p) + \dots$$

- Recall that $iG_0(\not p) = [\not p - m]^{-1} = \dfrac{\not p + m}{p^2 - m^2 + i0}$ has a pole at $p^2 = m^2$
- We see that there are graphs that have double pole, triple pole, and so on – the whole pole structure of the “corrected” or “interacting” Green’s function seemingly changes a lot
- To deal with that, one defines the (total) self-energy as the sum of all amputated (without external propagators) one-nucleon irreducible graphs (those that don’t fall apart if one internal nucleon line is cut):

$i\Sigma(\not p) =$



# Self Energy: Sum and Renormalize

- The whole Green's function can now be rewritten as

$$
\begin{aligned}
iG(\not p) &= iG_0(\not p) + iG_0(\not p) i\Sigma(\not p) iG_0(\not p) + iG_0(\not p) i\Sigma(\not p) iG_0(\not p) i\Sigma(\not p) iG_0(\not p) + \dots \\
&= iG_0(\not p)[1 - i\Sigma(\not p) iG_0(\not p)]^{-1} = i[G_0^{-1}(\not p) + \Sigma(\not p)]^{-1} \\
&= i[\not p - m + \Sigma(\not p)]^{-1}
\end{aligned}
$$

- This looks as the free Green's function with corrections, and we expect that it has a single pole at some point $\not p = m_R$ ; the self-energy, on the other hand, is expected to be a smooth function of $\not p$, so that the whole denominator can be expanded in a Taylor series around that point:

$$
\begin{aligned}
\not p - m + \Sigma(\not p) &= \not p \underbrace{-m + \Sigma(m_R)}_{-m_R} + (\not p - m_R)\Sigma'(m_R) + O[(\not p - m_R)^2] \\
&= (\not p - m_R)[1 + \Sigma'(m_R)] + \Sigma_R(\not p)
\end{aligned}
$$

- The renormalized self-energy is defined by

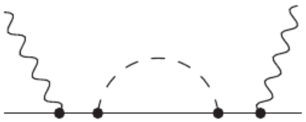
$$
\Sigma_R(\not p) = \Sigma(\not p) - \Sigma(\not p)\big|_{\not p = m_R} - \left[\not p - m_R\right] \Sigma'(\not p)\big|_{\not p = m_R}
$$

# Self Energy: Sum and Renormalize

- The Green's function is now rewritten as

$$iG(\not{p}) = iZ\left[\not{p} - m_R + Z\Sigma_R(\not{p})\right]^{-1} = iZ[\not{p} - m_R] + iZ[\not{p} - m_R]^{-1} i\Sigma_R(\not{p}) iZ[\not{p} - m_R]^{-1} + \ldots,$$

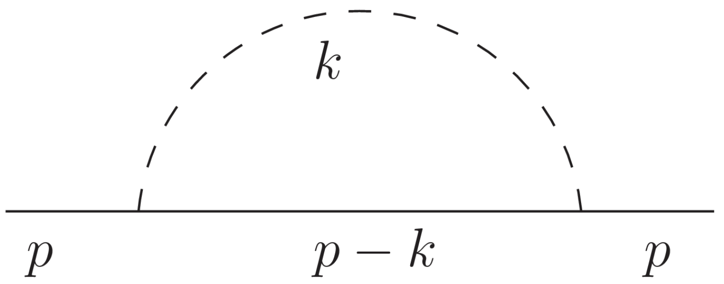
  where we expanded in the end, and $Z^{-1} = 1 + \Sigma'(\not{p})\big|_{\not{p}=m_R}$
- This has a few consequences:
  - The Green's function looks very similar to the free Green's function, except that it is renormalized and the pole is shifted
  - The expansion tells us that it is the renormalized self-energy that has to be used in more complicated Feynman graphs
- Furthermore, we can use $m_R$ equal to our physical mass (hoping that the differences will be small), and, since we work in perturbation theory, we can put $Z = 1$ in the corrections containing the self-energy at the first order (since $Z \simeq 1 - \Sigma'(\not{p})\big|_{\not{p}=m_R} + \ldots$, and the self-energy is a small correction)
- This means, for example, that loop diagram (5) that contains the self-energy is



(5)

$$\propto \gamma^\mu[\not{p} - m]^{-1}\Sigma_R(\not{p})[\not{p} - m]^{-1}\gamma^\nu$$

# Nucleon Self Energy




- The self-energy loop corresponds to

$$i\Sigma(p) = -3\frac{m^2 g_A^2}{f_\pi^2}\int \frac{d^4k}{(2\pi)^4}\frac{\not{k}-\not{p}+m}{[k^2-m_\pi^2][(k-p)^2-m^2]}$$

we choose $p_1 = p,$ getting $k_1 = p,\ m_1 = m_\pi,\ m_2 = m$

- After the sign change we get in the LoopTools notation

$$i\Sigma(p) = 3\frac{m^2 g_A^2}{f_\pi^2}\int \frac{d^4k}{(2\pi)^4}\frac{\not{k}+\not{p}-m}{[k^2-m_\pi^2][(k+k_1)^2-m^2]}$$

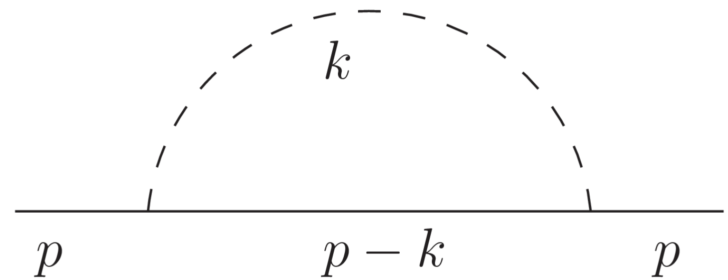
- This becomes (up to a constant factor, recall the different normalization conventions used by us and in LoopTools; we disregard it for the moment)

$$i\Sigma(\not{p}) = 3\frac{m^2 g_A^2}{f_\pi^2}\left[\not{p}B_1(p^2,m_\pi^2,m^2)+[\not{p}-m]B_0(p^2,m_\pi^2,m^2)\right]$$

- The derivative of the self-energy is (the derivatives of B-functions are w.r.t. $p^2$ )

$$i\frac{d\Sigma(\not{p})}{d\not{p}} = 3\frac{m^2 g_A^2}{f_\pi^2}\left[B_1+B_0+2p^2B_1'+2(p^2-m\not{p})B_0'\right]$$

# Nucleon Self Energy




- After the renormalization we have

$$
\begin{aligned}
i\Sigma_R(\not{p}) &= i\left(\Sigma(\not{p}) - \Sigma(\not{p})\Big|_{\not{p}=m} - (\not{p}-m)\frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}=m}\right) \\
&= 3\frac{m^2 g_A^2}{f_\pi^2}\left[(\not{p}-m)\left(B_1 + B_0 - \bar{B}_1 - \bar{B}_0 - 2m^2\bar{B}_1'\right) + m\left(B_1 - \bar{B}_1\right)\right]
\end{aligned}
$$

  Here, $B_1 \equiv B_1(p^2, m_\pi^2, m^2)$, $\bar{B}_1 \equiv B_1(m^2, m_\pi^2, m^2)$, and so on

- Now, we can plot the scalar prefactors in the expression for renormalized self energy and see that they do not depend on the divergence $\Delta$

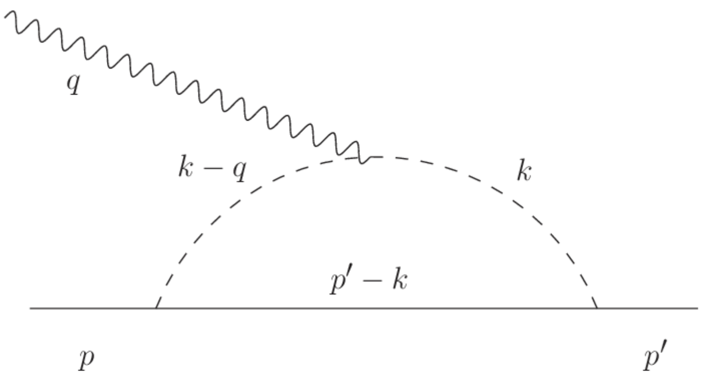
- We also have to note down the renormalization factor that enters the resummed Green's function:

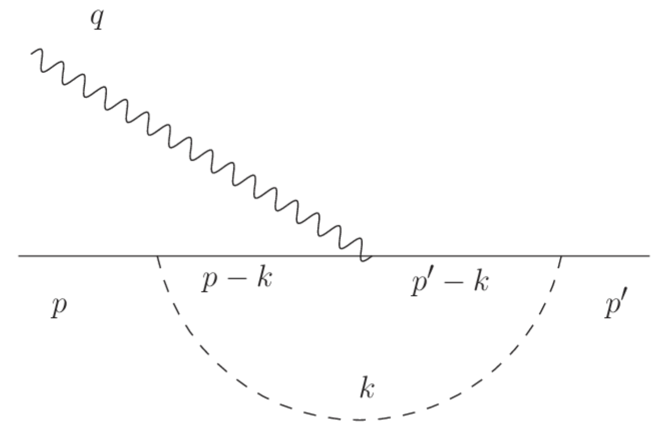
$$
iG(\not{p}) = \frac{iZ}{\not{p}-m}, \quad Z^{-1} = 1 + \frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}=m} \Longrightarrow Z \simeq 1 - \frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}=m} \equiv 1 + \delta Z
$$

- We will need this when we consider the vertices and check the Ward-Takahashi identity

# Vertex Corrections

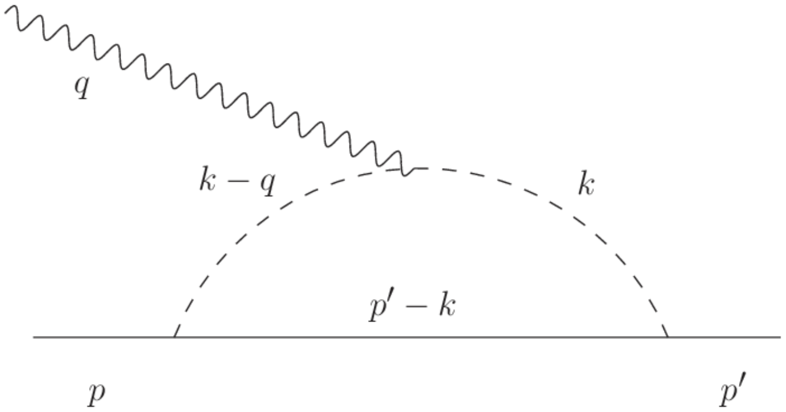
- The vertex corrections that we had previously:




$$i\Gamma^{\nu}(p,p') = \frac{em^2 g_A^2}{f_\pi^2}\Bigg[2\tau^3 \int \frac{d^4k}{(2\pi)^4}\frac{(2k-q)^{\nu}(\not{k}-\not{p}'+m)}{[k^2-m_\pi^2][(k-q)^2-m_\pi^2][(k-p')^2-m^2]}$$
$$-\frac{1}{2}(3-\tau^3)\int \frac{d^4k}{(2\pi)^4}\frac{(\not{k}-\not{p}'+m)\gamma^{\nu}(\not{k}-\not{p}+m)}{[k^2-m_\pi^2][(k-p)^2-m^2][(k-p')^2-m^2]}\Bigg]$$

# Pion-Photon Coupling




- Start with the pion-coupling loop:

$$\int \frac{d^4k}{(2\pi)^4} \frac{(2k-q)^\nu(\not{k}-\not{p}+m)}{[k^2-m_\pi^2][(k-q)^2-m_\pi^2][(k-p')^2-m^2]}$$

- Here, we can choose $p_1 = q,\ p_2 = p,\ p_3 = -p',$ so

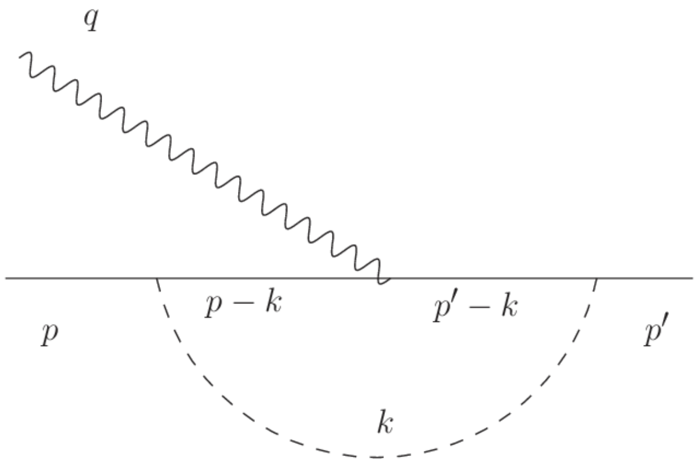
$$k_1 = q,\ k_2 = q+p = p',\ m_1 = m_2 = m_\pi,\ m_3 = m$$

- In order to match the LoopTools notation, however, we have to have $k + k_i$ in the denominators; this can be done by changing the sign of the loop momentum, giving

$$\int \frac{d^4k}{(2\pi)^4} \frac{(2k+q)^\nu(\not{k}+\not{p}-m)}{[k^2-m_\pi^2][(k+q)^2-m_\pi^2][(k+p')^2-m^2]}$$

- This form conforms the LoopTools notation and can be decomposed in the LoopTools functions; we will do it in a FORM script later

# Nucleon-Photon Coupling




- With the nucleon-coupling loop, we use

$$\int \frac{d^4k}{(2\pi)^4} \frac{(\not{k} - \not{p}' + m)\gamma^\nu(\not{k} - \not{p} + m)}{[k^2 - m_\pi^2][(k-p)^2 - m^2][(k-p')^2 - m^2]}$$

  and choose $p_1 = p,\ p_2 = q,\ p_3 = -p',$ getting
  $k_1 = p,\ k_2 = q + p = p',\ m_1 = m_\pi,\ m_2 = m_3 = m$

- After the sign change similarly to what we did before, we get an expression that conforms to the LoopTools notation

$$\int \frac{d^4k}{(2\pi)^4} \frac{(\not{k} + \not{p}' - m)\gamma^\nu(\not{k} + \not{p} - m)}{[k^2 - m_\pi^2][(k+p)^2 - m^2][(k+p')^2 - m^2]}$$

- Note that we still have a different normalization of the integral – this has to be fixed at some stage

# Photon-Nucleon Vertex

- Before calculating the vertex corrections, let us consider the general structure of the photon-nucleon vertex and its connection with the S-matrix element
- First of all, if we define the photon-nucleon vertex $\Gamma^\mu(p,p')$ as the sum of all (or up to a given order) amputated three-point ($\gamma NN$) Feynman graphs, it is related to the S-matrix element between the initial and the final state as

$$eZ\Gamma^\mu(p,p')\epsilon_\mu = \langle f|S|i\rangle$$

- Here, $Z$ is exactly the renormalization factor in the Green's function, and we only take into account renormalization coming from the nucleon's side
- This relation can be rigorously derived (the Lehmann-Symanzik-Zimmermann reduction formula); generaly, the connection between the amplitude $\mathcal{M}$ that comes from a set of amputated Feynman graphs and the S-matrix element is

$$\langle f|S|i\rangle = \prod_i \sqrt{Z_i}\,\mathcal{M},$$

  where there is a $\sqrt{Z}$ factor from each incoming or outgoing particle
- A handwaving argument can be made like that: the factor of $Z$ in the Green's function has to be split between the two external ends, thus giving a $\sqrt{Z}$ for each of the external nucleons, and this is generalized to arbitrary amplitude $\mathcal{M}$

# Nucleon Form Factors

- So, having the relation between $\Gamma^\mu(p,p')$ and the S-matrix element,

$$eZ\Gamma^\mu(p,p')\epsilon_\mu = \langle f|S|i\rangle$$

  let us now consider the most general structure for the latter; it has to be Lorentz-covariant, which for on-shell nucleons gives the general form

$$\langle f|S|i\rangle = e\left[F_1(q^2)\gamma^\mu + \frac{1}{2m}F_2(q^2)\gamma^{\mu\nu}q_\nu + F_3(q^2)q^\mu\right]\epsilon_\mu$$

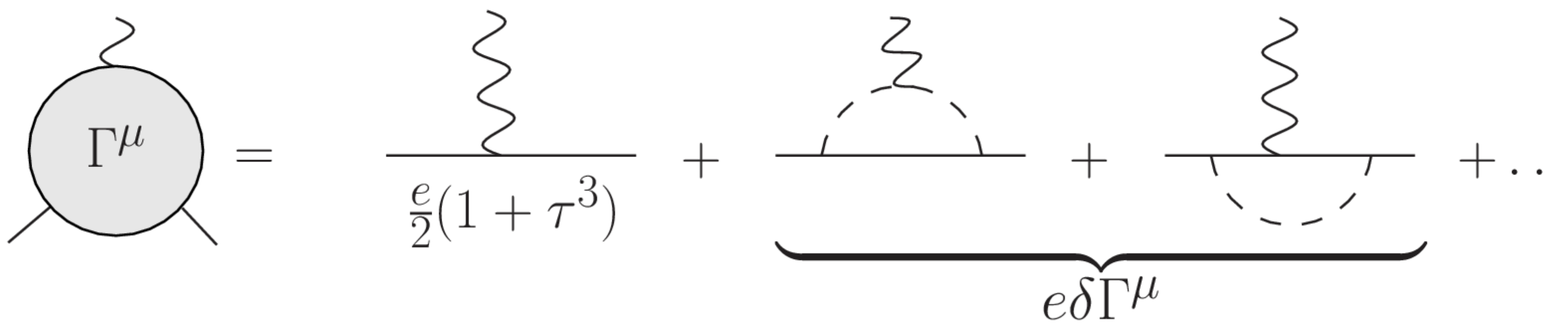
- This also has to be gauge invariant, which means that it has to vanish if $\epsilon_\mu$ is replaced by $q_\mu$; the function $F_3(q^2)$ has therefore to vanish at all values of $q^2$
- Functions $F_1(q^2)$ and $F_2(q^2)$ are called the Dirac and Pauli form factors, and they are normalized such that $F_1(0)$ equals the nucleon charge in units of proton charge (i.e., 1 for the proton and 0 for the neutron), and $F_2(0)$ equals the nucleon's anomalous magnetic moment in nuclear magneton units
- Let us consider what happens with the Dirac form factor and the electric charge when we take into account one-loop vertex corrections

# Dirac FF: Ward-Takahashi Identity

- We write (for on-shell nucleons) the one-loop correction as

$$\delta\Gamma^\mu(p,p') = \delta\Gamma^\mu(q) = \delta F_1(q^2)\gamma^\mu + \frac{1}{2m}\delta F_2(q^2)\gamma^{\mu\nu}q_\nu$$

- Besides that, we also have the leading nucleon charge coupling:




- Writing now the expansion for $eZ\Gamma^\mu(p,p')\epsilon_\mu = \langle f|S|i\rangle$ , we get for the $F_1$ part:

$$Z\Gamma^\mu = (1+\delta Z)\left[\frac{1}{2}(1+\tau^3) + \delta\Gamma^\mu \simeq \frac{1}{2}(1+\tau^3)\gamma^\mu + \frac{1}{2}(1+\tau^3)\delta Z + \delta F_1(q^2)\right]$$

- We see that the nucleon charge does not get renormalized (at this order) if

$$\delta F_1(0) + \frac{1}{2}(1+\tau^3)\delta Z = \delta F_1(0) - \frac{1}{2}(1+\tau^3)\frac{d\Sigma(\not{p})}{d\not{p}}\bigg|_{\not{p}=m} = 0$$

# Renormalized One-Loop Vertex

- This is (a form of) the Ward-Takahashi Identity for our one-loop graphs:

$$\delta F_1(0) + \frac{1}{2}(1+\tau^3)\delta Z = \delta F_1(0) - \frac{1}{2}(1+\tau^3)\frac{d\Sigma(\not{p})}{d\not{p}}\bigg|_{\not{p}=m} = 0$$

- A few further remarks:
  - If this works, we can define the renormalized one-loop vertex as

$$\delta\Gamma^\mu_R(p,p') = \delta\Gamma^\mu(p,p') - \delta F_1(0)\gamma^\mu$$

    and forget about the charge and wave function renormalization
  - But what happens with the correcton to nucleon a.m.m. coming from $\delta Z$ and $\delta F_2(0)$ ? Formally, the expansion of $Z\Gamma^\mu$ gives renormalized a.m.m.

$$\kappa = \kappa_0 + \kappa_0 \delta_Z + \delta F_2(0)$$

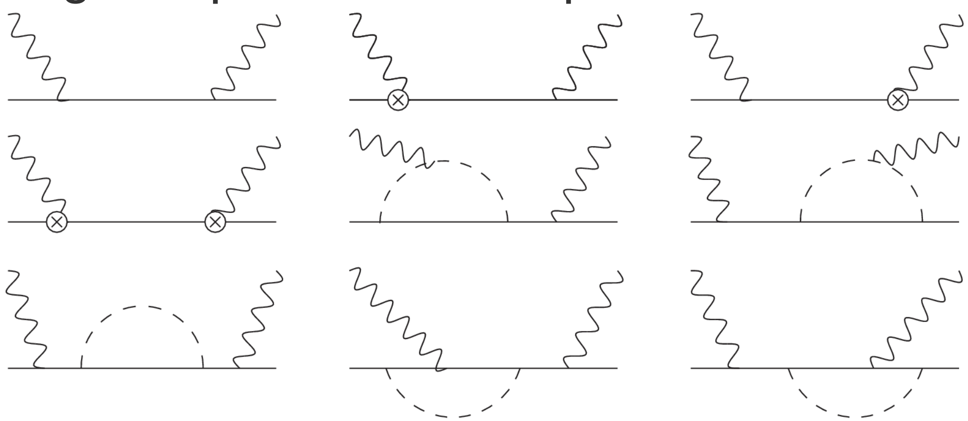
  - Here, we do an additional renormalization, saying that the value of a.m.m. that we use in the leading-order tree vertex already is the physical value of the nucleon a.m.m. This means that we remove these corrections to $\kappa_0$, substitute the physical value of a.m.m., and at the same time we also have to subtract the Pauli form factor part from the renormalized vertex

# Renormalized One-Loop Vertex

- Finally, the renormalized vertex takes the form

$$\delta\Gamma^{\mu}_{R}(p,p') = \delta\Gamma^{\mu}(p,p') - \delta F_1(0)\gamma^{\mu} - \frac{1}{2m}\delta F_2(0)\gamma^{\mu\nu}q_{\nu}$$

- With this and the renormalized self-energy we can finally calculate nucleon Compton scattering; the parts that take part in the renormalization now look



where the loop corrections – the self-energy and the vertices – are renormalized, and the leading order vertex uses the physical values of the charge and the a.m.m. (strictly speaking, using those everywhere in the above graphs means implicitly accounting for higher-order renormalization effects and higher-order corrections to the a.m.m. but this is fine – at least, it does not break the gauge invariance [due to the fact that the Ward-Takahashi identity works at all orders and that the a.m.m. coupling is gauge invariant itself])

# Renormalized One-Loop Vertex

- Let us now use FORM to calculate the one-loop vertices in terms of LoopTools functions (let us do it for two on-shell nucleons for simplicity – even though in the Compton scattering graphs we have only one nucleon on-shell; we will take care of that later)
- After that, we will check, using LoopTools, that
  - The formfactor $F_3(q^2)$ is zero for all values of $q^2$ (since we do it numerically it is really “for a few reasonable [spacelike] values”)
  - The Ward-Takahashi identity

$$\delta F_1(0) + \frac{1}{2}(1+\tau^3)\delta Z = \delta F_1(0) - \frac{1}{2}(1+\tau^3)\frac{d\Sigma(\not{p})}{d\not{p}}\bigg|_{\not{p}=m} = 0$$

    works (numerically) both for the proton and for the neutron (not trivial!!!)
- We will look at a few numerical subtleties related to divergences and how LoopTools treats them, and also to how it calculates things
- We will also plot the renormalized Dirac and Pauli form factors