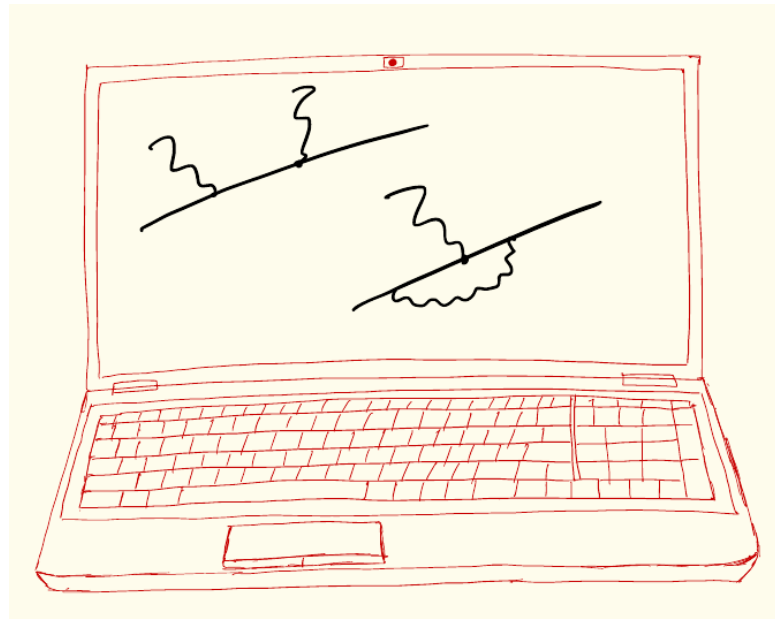


Computer Algebra for Feynman Graphs



10.



In this Lecture

- One-loop Feynman graphs – general information
 - General one-loop Feynman graphs
 - Reduction of tensor integrals to scalar functions
- LoopTools: a package to evaluate general one-loop integrals
 - Introduction and notations
 - Example of loop integral reduction

One-Loop Feynman Graphs

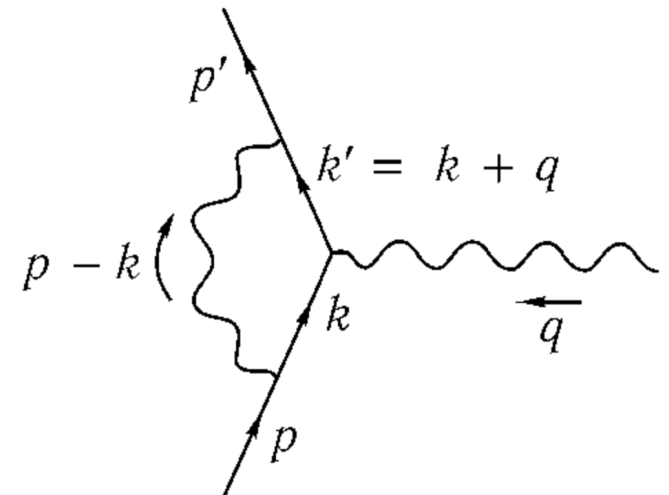
- Amplitudes that correspond to one-loop Feynman graphs have the general form

$$A = \int \frac{d^4 k}{(2\pi)^4} \frac{L(k, p_i, m_i, \gamma, \dots)}{[k^2 - m_1^2][(k - k_1)^2 - m_2^2] \dots [(k - k_{n-1})^2 - m_n^2]}$$

where the numerator is a polynomial of the integration momentum, external momenta, masses, gamma matrices, polarization vectors etc., while the denominator arises from the propagators of internal particles.

- Example: electron vertex correction in QED (a three-point vertex)

$$\Gamma^\mu \propto \int \frac{d^4 k}{(2\pi)^4} \frac{\gamma^\nu (\not{k} + \not{q} + m) \gamma^\mu \not{k} \gamma_\nu}{[k^2 - m^2][(k + q)^2 - m^2][(k - p)^2]}$$



One-Loop Feynman Graphs

- The external quantities in the numerator can be factored out, so instead of the complete expression one only needs to consider tensor loop integrals with simple numerators, for instance, for the one-loop electron vertex in QED,

$$\Gamma^\mu \propto \int \frac{d^4 k}{(2\pi)^4} \frac{\gamma^\nu (\not{k} + \not{q} + m) \gamma^\mu \not{k} \gamma_\nu}{[k^2 - m^2][(k + q)^2 - m^2][(k - p)^2]}$$

it is sufficient to only consider integrals with numerators $1, k^\lambda, k^\lambda k^\sigma$.

- The general amplitude, analogously,

$$A = \int \frac{d^4 k}{(2\pi)^4} \frac{L(k, p_i, m_i, \gamma, \dots)}{[k^2 - m_1^2][(k - k_1)^2 - m_2^2] \dots [(k - k_{n-1})^2 - m_n^2]}$$

can be expressed through integrals with numerators $1, k^\lambda, k^\lambda k^\sigma, k^\lambda k^\sigma k^\tau$, and so on – in principle, one can have arbitrary powers of the integration momentum in the numerator

- In renormalizable theories (such as QED, QCD, the Standard Model), there can be at most $r = n$ factors of integration momentum in the numerator of an n -point function; so the number of needed tensor integrals is limited by n
- If needed, integrals with $r > n$ can be considered, too

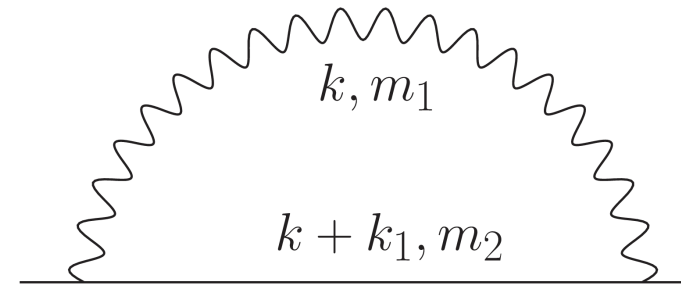
One, Two, Three, Four Points

- It also turns out that all one-loop integrals with more than four propagators (five, six, ... N-point functions) can be reduced to linear combinations of four-point integrals
- This means one has to consider one-, two-, three-, and four-point one-loop integrals in order to calculate a most general one-loop diagram
- Let us consider a two-point function and the integrals that can arise in it:

$$B_0 = \int \frac{d^4}{(2\pi)^4} \frac{1}{[k^2 - m_1^2][(k + k_1)^2 - m_2^2]} = B_0(k_1^2, m_1^2, m_2^2)$$

$$B^\mu = \int \frac{d^4}{(2\pi)^4} \frac{k^\mu}{[k^2 - m_1^2][(k + k_1)^2 - m_2^2]} = B_1(k_1^2, m_1^2, m_2^2) k_1^\mu$$

$$B^{\mu\nu} = \int \frac{d^4}{(2\pi)^4} \frac{k^\mu k^\nu}{[k^2 - m_1^2][(k + k_1)^2 - m_2^2]} = B_{00}(k_1^2, m_1^2, m_2^2) g^{\mu\nu} + B_{11}(k_1^2, m_1^2, m_2^2) k_1^\mu k_1^\nu$$



- This reduction can be done using the Feynman parameters (as you know), but it is in fact a very general thing that follows from Lorentz covariance

One, Two, Three, Four Points

- Analogously, one can write expressions for the reduction of three- and four-point one-loop tensor integrals to scalar functions (note the tensor integrals are symmetric in their indices by construction):

$$C_0 = \int \frac{d^4}{(2\pi)^4} \frac{1}{[k^2 - m_1^2][(k + k_1)^2 - m_2^2][(k + k_2)^2 - m_3^2]} = C_0(k_1^2, k_2^2, (k_1 + k_2)^2, m_1^2, m_2^2, m_3^2)$$

$$C^\mu = \sum_{i=1,2} C_i k_i^\mu, \quad C^{\mu\nu} = C_{00} g^{\mu\nu} + \sum_{i,j=1,2} C_{ij} k_i^\mu k_j^\nu,$$

$$C^{\mu\nu\rho} = \sum_{i=1,2} C_{00i} (g^{\mu\nu} k_i^\rho + g^{\mu\rho} k_i^\nu + g^{\nu\rho} k_i^\mu) + \sum_{i,j,k=1,2} C_{ijn} k_i^\mu k_j^\nu k_n^\rho,$$

$$D_0 = \int \frac{d^4}{(2\pi)^4} \frac{1}{[k^2 - m_1^2][(k + k_1)^2 - m_2^2][(k + k_2)^2 - m_3^2][(k + k_3)^2 - m_4^2]} \\ = D_0(k_1^2, k_2^2, k_3^2, (k_1 + k_2)^2, (k_1 + k_3)^2, (k_2 + k_3)^2, m_1^2, m_2^2, m_3^2, m_4^2)$$

$$D^\mu = \sum_{i=1,2,3} D_i k_i^\mu, \quad D^{\mu\nu} = D_{00} g^{\mu\nu} + \sum_{i,j=1,2,3} D_{ij} k_i^\mu k_j^\nu,$$

$$D^{\mu\nu\rho} = \sum_{i=1,2,3} D_{00i} (g^{\mu\nu} k_i^\rho + g^{\mu\rho} k_i^\nu + g^{\nu\rho} k_i^\mu) + \sum_{i,j,k=1,2,3} D_{ijn} k_i^\mu k_j^\nu k_n^\rho,$$

One, Two, Three, Four Points

and, finally,

$$D^{\mu\nu\rho\lambda} = \{g^{\mu\nu}g^{\rho\lambda}\}D_{0000} + \sum_{i,j=1,2,3} \{g^{\mu\nu}k_i^\rho k_j^\lambda\}D_{00ij} + \sum_{i,j,n,m=1,2,3} k_i^\mu k_j^\nu k_n^\lambda k_m^\rho D_{ijnm},$$

where the braces $\{\}$ mean that the object within has to be symmetrized with respect to all Lorentz indices: $\{g^{\mu\nu}g^{\rho\lambda}\} = g^{\mu\nu}g^{\rho\lambda} + g^{\rho\nu}g^{\mu\lambda} + g^{\lambda\nu}g^{\rho\mu}$, and so on.

- In this way, all tensor one-loop integrals can be reduced to scalar loop integrals (functions $B_0, B_1, B_{00}, B_{11}, C_0, C_i$, and so on)
- Good news: all these scalar integrals can be calculated analytically in terms of elementary functions (mostly radicals and logarithms) and of the dilogarithm function,

$$\text{Li}_2(z) = - \int_0^z \frac{\log(1-t)}{t} dt$$

- The expressions involved are, however, very cumbersome in the most general case; depending on the choice of the specific parametrization, they also may be subject to numerical instabilities (cancellations between large terms etc)



LoopTools

- LoopTools (developed by T. Hahn and collaborators, www.feynarts.de) is a numerical package that evaluates the scalar tensor coefficients that we considered above
- It works with analytic formulas for the tensor coefficients, so it is very fast (no numerical integration is involved)
- It has built-in methods to cross-check the results against large cancellations and other numerical errors (in particular, it implements two different methods to perform the reduction of the tensor integrals which can be compared)
- It is based on a previous package FF by G.J. van Oldenborgh and J. Vermaseren that was written in Fortran.
- LoopTools has Fortran, C/C++, and Mathematica interfaces for function calls

LoopTools: in Practice

- A numerical calculation of a one-loop diagram becomes easy* with the use of FORM and LoopTools:
 - First, one performs a tensor decomposition in FORM, expressing the whole Feynman diagram in terms of tensor coefficients B, C, D (rather than in terms of loop functions obtained via the Feynman parameters);
 - Second, one takes the obtained expression, substitutes the numerical values of kinematical parameters (masses, energies, etc), and calls the LoopTools routines that calculate the coefficients B, C, D
- The numerical results are ready!

* There (as always) can be issues; some of them will be discussed later

LoopTools: About Notation

- The notation that is used in LoopTools:
 - External loop momenta are defined to flow into the diagram

- Tensor coefficient functions are functions of masses and external momenta squared – here, $p_1^2, p_2^2, (p_1 + p_2)^2, m_1^2, m_2^2, m_3^2$

- These functions are, however, the coefficients of the momenta in the loop – here, k_1 and k_2 . For instance,

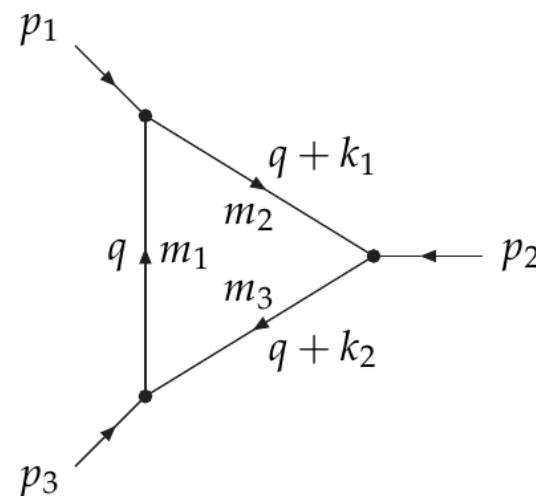
$$C_{12}(p_1^2, p_2^2, (p_1 + p_2)^2, m_1^2, m_2^2, m_3^2)$$

is the coefficient of $k_1^\mu k_2^\nu$ in the tensor reduction of $C^{\mu\nu}$, and so on

- Relations between p_i and k_i are

$$p_1 = k_1, \quad p_2 = k_2 - k_1, \quad \dots, \quad p_n = k_n - k_{n-1};$$

$$k_1 = p_1, \quad k_2 = p_1 + p_2, \quad \dots, \quad k_n = \sum_{i=1}^n p_i.$$



LoopTools: About Notation

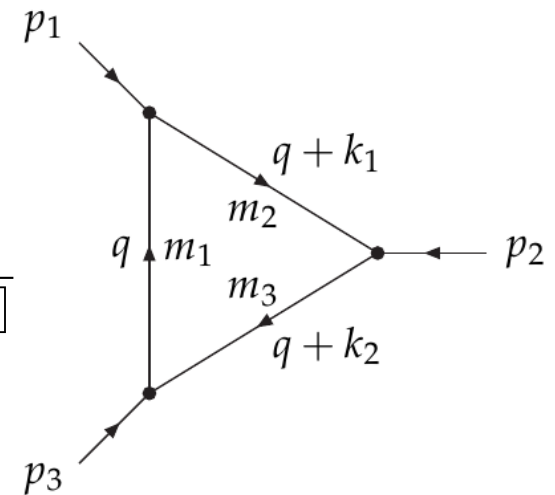
- The notation that is used in LoopTools:
 - Finally, the integral is considered in D dimensions, introducing the dimensional regularization and keeping the renormalization scale parameter, for example, the scalar three-point function is

$$C_0 = \frac{\mu^{4-D}}{r_\varepsilon} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{[k^2 - m_1^2][(k + k_1)^2 - m_2^2][(k + k_2)^2 - m_3^2]}$$

Here,

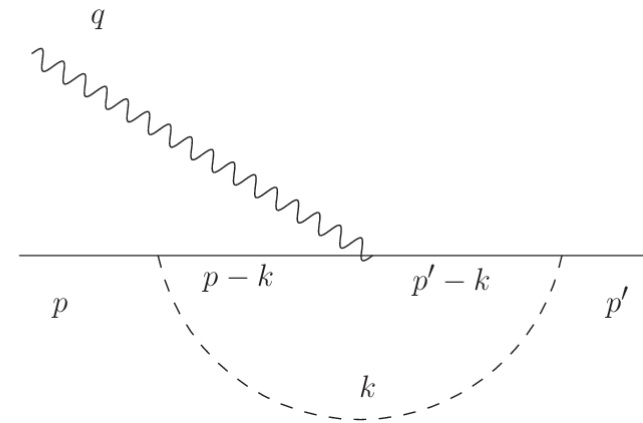
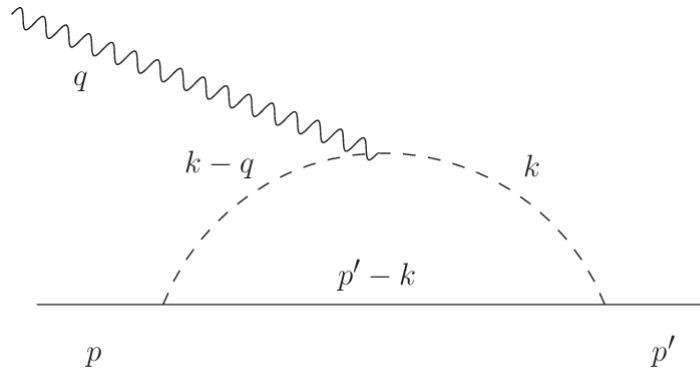
$$r_\varepsilon = \frac{\Gamma^2(1 - \varepsilon/2)\Gamma(1 + \varepsilon/2)}{\Gamma(1 - \varepsilon)}, \quad D = 4 - \varepsilon$$

- One has to keep in mind this normalization convention. Note that not all functions are divergent – those are only two-point functions, three-point tensor coefficients with two zeros, and four-point coefficient D_{0000}



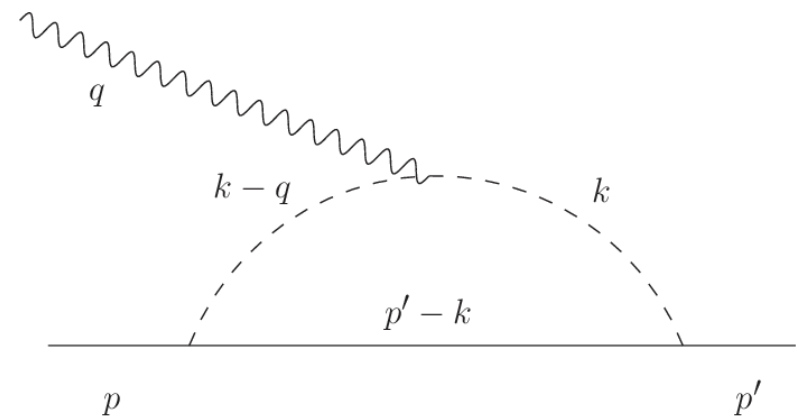
Example: Loop Graphs

- Let us consider the example that we had at the previous lecture:



$$i\Gamma^\nu(p, p') = \frac{em^2 g_A^2}{f_\pi} \left[2\tau^3 \int \frac{d^4 k}{(2\pi)^4} \frac{(2k - q)^\nu (\not{k} - \not{p}' + m)}{[k^2 - m_\pi^2][(k - q)^2 - m_\pi^2][(k - p')^2 - m^2]} \right. \\ \left. - \frac{1}{2}(3 - \tau^3) \int \frac{d^4 k}{(2\pi)^4} \frac{(\not{k} - \not{p}' + m)\gamma^\nu(\not{k} - \not{p} + m)}{[k^2 - m_\pi^2][(k - p)^2 - m^2][(k - p')^2 - m^2]} \right]$$

Example: Loop Graphs



- Start with the pion-coupling loop:

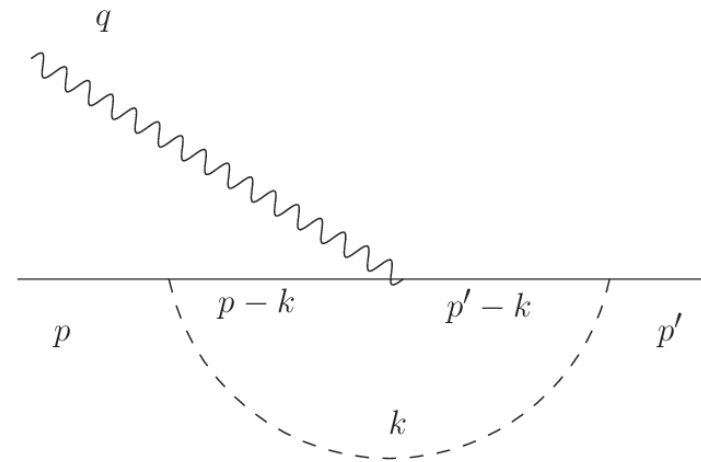
$$\int \frac{d^4 k}{(2\pi)^4} \frac{(2k - q)^\nu (\not{k} - \not{p} + m)}{[k^2 - m_\pi^2][(k - q)^2 - m_\pi^2][(k - p')^2 - m^2]}$$

- Here, we can choose $p_1 = q$, $p_2 = p$, $p_3 = -p'$, so
 $k_1 = q$, $k_2 = q + p = p'$, $m_1 = m_2 = m_\pi$, $m_3 = m$
- In order to match the LoopTools notation, however, we have to have $k + k_i$ in the denominators; this can be done by changing the sign of the loop momentum, giving

$$\int \frac{d^4 k}{(2\pi)^4} \frac{(2k + q)^\nu (\not{k} + \not{p} - m)}{[k^2 - m_\pi^2][(k + q)^2 - m_\pi^2][(k + p')^2 - m^2]}$$

- This form conforms the LoopTools notation and can be decomposed in the LoopTools functions; we will do it in a FORM script later

Example: Loop Graphs



- With the nucleon-coupling loop, we use

$$\int \frac{d^4 k}{(2\pi)^4} \frac{(\not{k} - \not{p}' + m)\gamma^\nu(\not{k} - \not{p} + m)}{[k^2 - m_\pi^2][(k - p)^2 - m^2][(k - p')^2 - m^2]}$$

and choose $p_1 = p$, $p_2 = q$, $p_3 = -p'$, getting

$$k_1 = p, \quad k_2 = q + p = p', \quad m_1 = m_\pi, \quad m_2 = m_3 = m$$

- After the sign change similarly to what we did before, we get an expression that conforms to the LoopTools notation

$$\int \frac{d^4 k}{(2\pi)^4} \frac{(\not{k} + \not{p}' - m)\gamma^\nu(\not{k} + \not{p} - m)}{[k^2 - m_\pi^2][(k + p)^2 - m^2][(k + p')^2 - m^2]}$$

- Note that we still have a different normalization of the integral – this has to be fixed at some stage

Exercise: the Schwinger Correction

- Using FORM, simplify the expression for the leading one-loop QED correction to the electron vertex function, and obtain the expression for the electron's anomalous magnetic moment in terms of LoopTools functions
- Calculate the value of the electron's a.m.m. numerically, calling the LoopTools routines, and check that you get the same answer as you previously got analytically using the Feynman parametrization
- Hint: choose some units for the mass parameters that enter the problem, e.g., measure everything in MeV
- Hint: evaluation might be unstable in the limit $q^2 \rightarrow 0$, and it may also be affected by the fact that one of the internal particles has zero mass, while the two others have equal masses. If you encounter numerical issues, try offsetting these troubles by slightly changing the parameters (e.g., take small nonzero q^2 , small nonzero value of the photon mass, the masses of the two electrons slightly different, etc.)

