Examples Sheet 7

Symmetries in Physics

Winter 2017/18

Lecturer: PD Dr. G. von Hippel

1. *A harmonic oscillator toy model for the BRST cohomology* (20 P.)

   1. Starting from the three-dimensional harmonic oscillator in the form

   $$\hat{H}_B = \sum_{i=1}^{3} \left(\frac{\hat{p}_i^2}{2m} + \frac{m\omega^2}{2} \hat{x}_i{}^2 \right),$$

   show how it can be written as

   $$\hat{H}_B = \hat{H}_1 + \hat{H}_2$$

   with

   $$\hat{H}_1 = \hat{p}^2 + \omega^2 \hat{x}^2 = \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$
   $$\hat{H}_2 = \hat{\bar{P}} \hat{P} + \omega^2 \hat{\bar{X}} \hat{X} = \omega \left(\hat{A}^\dagger \hat{A} + \hat{\bar{A}}^\dagger \hat{\bar{A}} + 1 \right)$$

   by appropriately identifying its component operators.

   2. Using the canonical commutation relations between the creation and annihilation operators, derive the spectrum and the Hilbert space of the three-dimensional harmonic oscillator.

   3. Consider the fermionic oscillator with Hamiltonian

   $$\hat{H}_F = \hat{\bar{\Pi}} \hat{\Pi} + \omega^2 \hat{\bar{Z}} \hat{Z} = \omega \left(\hat{\bar{b}}^\dagger \hat{\bar{b}} + \hat{b}^\dagger \hat{b} - 1 \right)$$

   where the operators obey canonical anticommutation relations, and derive its spectrum and Hilbert space.

   4. Write down the Hamiltonian $\hat{H} = \hat{H}_B + \hat{H}_F$ for a superoscillator.

   5. Show that the resulting theory is invariant under supersymmetry transformations generated by the BRST operator

   $$\hat{Q} = i \left(\hat{\bar{A}}^\dagger \hat{\bar{b}} - \hat{b}^\dagger \hat{A} \right)$$

   6. Show that the BRST operator is nilpotent, $\hat{Q}^2 = 0$.

   7. Write down the conjugate $\hat{Q}^\dagger$ of $\hat{Q}$, and show that it is also nilpotent and conserved.

8. Show that

$$\left\{\hat{Q},\hat{Q}^\dagger\right\} = \frac{\hat{H}'}{\omega}$$

with $\hat{H}' = \hat{H}_2 + \hat{H}_F$.

9. Conclude that $\operatorname{Ker}\hat{H}' = \operatorname{Ker}\hat{Q} \cap \operatorname{Ker}\hat{Q}^\dagger$ and $\operatorname{Ker}\hat{Q} = \operatorname{Ker}\hat{H}' \oplus \operatorname{Im}\hat{Q}$.
10. Defining the physical Hilbert space as $\mathcal{H}_{\text{phys}} = \operatorname{Ker}\hat{Q}/\operatorname{Im}\hat{Q}$, explicitly give the spectrum of physical states.

2. *Chiral symmetry of massless QCD* (20 P.)

1. Using the decomposition of the Dirac spinor into a pair of Weyl spinors, show that the Lagrangian

$$\mathcal{L} = \frac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) + \sum_{i=1}^{N_f} \overline{\psi}_i \gamma^\mu D_\mu \psi_i$$

for massless QCD with N_f quark flavours is invariant under separate $\mathrm{SU}(N_f)$ transformations of the left- and right-handed components $\psi_{L,R} = \frac{1}{2}(1 \mp \gamma_5)\psi$ of the Dirac spinor.

2. Write down the Noether currents for the two $\mathrm{SU}(N_f)$ symmetries.
3. By forming the sum and the difference of the Noether currents, show that the $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ symmetry can also be written as a $\mathrm{SU}(N_f)_V \times \mathrm{SU}(N_f)_A$, with the two factors corresponding to a vector and an axial vector Noether current, respectively.
4. Show that the vector Noether current remains conserved if a degenerate mass term $-m\sum\overline{\psi}_i\psi_i$ is added to the Lagrangian, but that the axial Noether current is no longer conserved.
5. The strong interactions lead to a quark condensate $\langle \operatorname{tr}(Q)\rangle \neq 0$ for the the trace of the matrix $Q_{ij}(x) = \overline{\psi}_i(x)\psi_j(x)$. How does Q transform under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ transformations? Show that the condensate leads to spontaneous breaking of the symmetry associated with the axial vector current, but leaves the vector symmetry intact.
6. Conclude that there must be $(N_f^2 - 1)$ massless Goldstone bosons associated with the spontaneous breaking of the axial symmetry of massless QCD.
7. Using your result for how the mass term breaks the axial symmetry, show that the pseudo-Goldstone bosons associated with the spontaneous breaking of the approximate axial symmetry of QCD with light quarks have masses of order $m_{PS}^2 \propto m$.
8. Which particles are the pseudo-Goldstone bosons for $N_f = 2, 3$? What is their parity and why?