Exercise sheet 11
Theoretical Physics 6a (QFT): WS 2017-2018
Lecturer : Prof. M. Vanderhaeghen

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Exercise 1. (50 points) : β-function in QED

In the lecture notes we have calculated the effective (running) coupling in QED at 1-loop level as:

$$e_R^2(Q^2) = e_R^2 \left\{ 1 + \frac{e_R^2}{12\pi^2} \ln \frac{Q^2}{m^2} + \mathcal{O}(e_R^4) \right\}, \tag{1}$$

where e_R^2 on the *rhs* can be interpreted as the coupling defined at $Q^2 = m^2$. The β-function in QED expresses in general how the coupling constant changes with mass scale Q as:

$$\beta(e_R) \equiv Q \frac{de_R}{dQ}. \tag{2}$$

(a)(10 points)
Use Eq. (1) to express the β-function at leading order in terms of the renormalized coupling $e_R(Q)$.
(b)(10 points)
Which power in e_R do the correction terms to the leading order result for the β-function have ?
(c)(10 points)
Discuss from the sign of the β function how it behaves at high energies (short distances), corresponding with $Q \to \infty$.
(d)(20 points)
By using the result obtained in (a) for the β-function, you obtain a differential equation for $e_R(Q^2)$. Solve this equation by expressing the running

coupling $e_R(Q^2)$ at scale Q in terms of the running coupling $e_R(\mu^2)$ at an arbitrary scale μ. Show that when taking $\mu = m$, you find back the result of Eq. (1) up to correction terms of $\mathcal{O}(e_R^6)$.

Exercise 2. (50 points) : Behavior of running coupling near a fixed point

Consider a theory with running coupling $g(Q^2)$, for which the β function is given by

$$\beta(g) \equiv Q\frac{dg}{dQ} = g(a^2 - g^2), \tag{3}$$

where a is a known constant.

(a)(10 points)

Plot β as function of g.

(b)(30 points)

The coupling at some reference scale $Q^2 = 1$ is given as:

$$g(Q^2 = 1) = g_0. \tag{4}$$

Solve the differential equation (3) to express $g(Q^2)$ at some arbitrary scale. Your result should be expressed in terms of Q^2 and depend on the parameters a and g_0. Check that for $Q^2 = 1$ your result gives $g(Q^2 = 1) = g_0$. Consider the cases $g_0 > 0$ and $g_0 < 0$ separately.

(c)(10 points)

For the case $g_0 > 0$, what will be the value of $g(Q^2)$ in the limit $Q^2 \to \infty$? In the opposite case where $g_0 < 0$, what will now be the value of $g(Q^2)$ in the limit $Q^2 \to \infty$?

The specific limiting couplings are called ultra-violet fixed points. Does their value depend on the chosen starting value (discuss this for $g_0 > 0$ and $g_0 < 0$ separately) ?