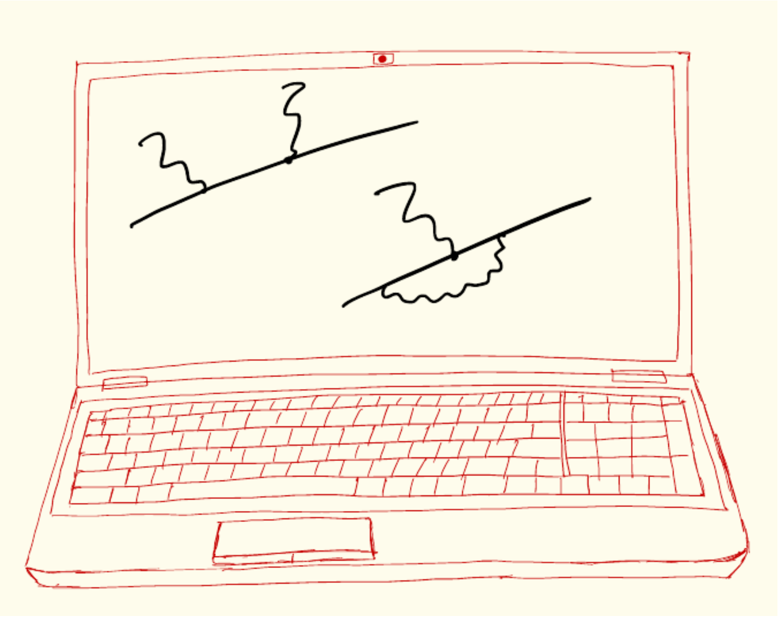
Computer Algebra for Feynman Graphs



9.

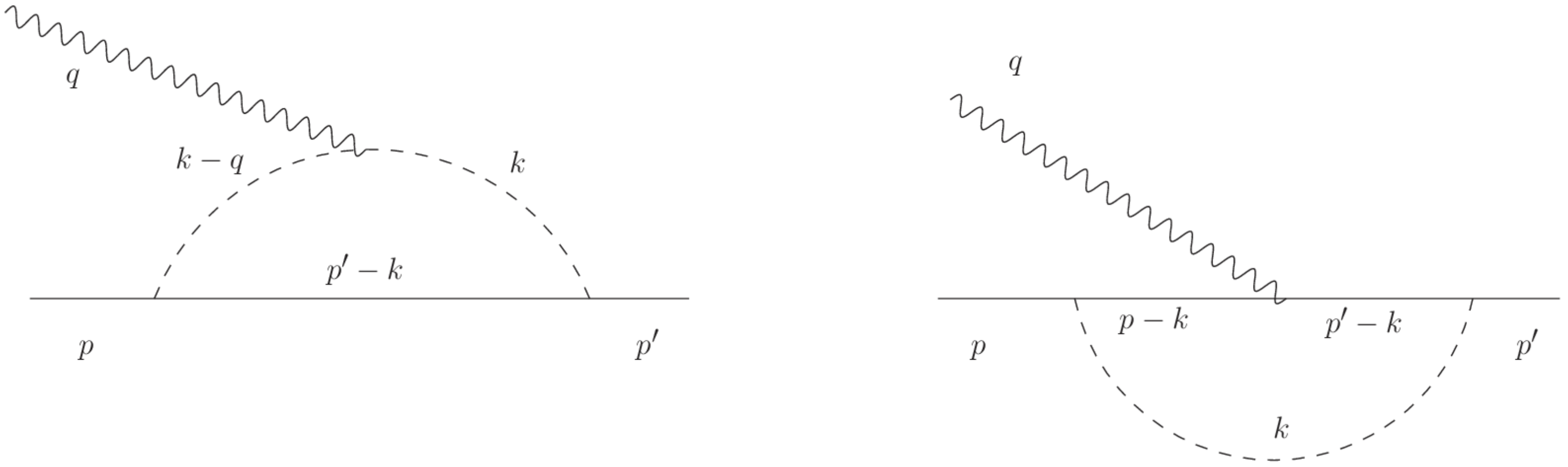
In this Lecture

- FORM examples
  - Feynman graphs: one-loop nucleon electromagnetic vertex with pseudoscalar pion-nucleon coupling

Example: Loop Graphs

- We want to compute these two one-loop graphs, which are the leading corrections to the nucleon e.m. vertex in theory with the pseudoscalar pion-nucleon coupling:

$$\mathcal{L}_{\mathrm{int}} = \frac{m g_A}{f_\pi} \bar{N} \gamma^5 \tau^a N \pi^a$$



$$i\Gamma^\nu(p,p') = \frac{e m^2 g_A^2}{f_\pi} \left[2\tau^3 \int \frac{d^4k}{(2\pi)^4} \frac{(2k-q)^\nu (\not{k} - \not{p}' + m)}{[k^2 - m_\pi^2][(k-q)^2 - m_\pi^2][(k-p')^2 - m^2]} \right.$$
$$\left. - \frac{1}{2}(3 - \tau^3) \int \frac{d^4k}{(2\pi)^4} \frac{(\not{k} - \not{p}' + m)\gamma^\nu(\not{k} - \not{p} + m)}{[k^2 - m_\pi^2][(k-p)^2 - m^2][(k-p')^2 - m^2]} \right]$$

Example: Loop Graphs

- If the initial and final nucleons are on-shell (free), we can use the Dirac equation:

$$i\Gamma^\nu(p,p') = \frac{em^2 g_A^2}{f_\pi}\left[2\tau^3 \int \frac{d^4k}{(2\pi)^4} \frac{(2k-q)^\nu(\not{k}-\not{p}'+m)}{[k^2-m_\pi^2][(k-q)^2-m_\pi^2][(k-p')^2-m^2]} \right.$$
$$\left. -\frac{1}{2}(3-\tau^3)\int \frac{d^4k}{(2\pi)^4} \frac{(\not{k}-\not{p}'+m)\gamma^\nu(\not{k}-\not{p}+m)}{[k^2-m_\pi^2][(k-p)^2-m^2][(k-p')^2-m^2]}\right]$$

$$i\Gamma^\nu(p,p') = \frac{em^2 g_A^2}{f_\pi}\left[2\tau^3 \int \frac{d^4k}{(2\pi)^4} \frac{(2k-q)^\nu \not{k}}{[k^2-m_\pi^2][(k-q)^2-m_\pi^2][(k-p')^2-m^2]} \right.$$
$$\left. -\frac{1}{2}(3-\tau^3)\int \frac{d^4k}{(2\pi)^4} \frac{\not{k}\gamma^\nu\not{k}}{[k^2-m_\pi^2][(k-p)^2-m^2][(k-p')^2-m^2]}\right]$$

- This might not be the most general consideration though, so let us keep in mind that we might want to retain the off-shell pieces at some stage

Example: Loop Graphs

- Now, we want to use the Feynman parameterization in order to combine the three denominators into one:

$$\frac{1}{A_1^{n_1}\dots A_N^{n_N}} = \frac{\Gamma(n_1+\dots+n_N)}{\Gamma(n_1)\dots\Gamma(n_N)} \prod_{i=1}^{N} \int_0^1 dx_i \; \frac{\delta(1-x_1-\dots-x_N)x_1^{n_1-1}\dots x_N^{n_N-1}}{(x_1A_1+\dots+x_NA_N)^{n_1+\dots+n_N}}$$

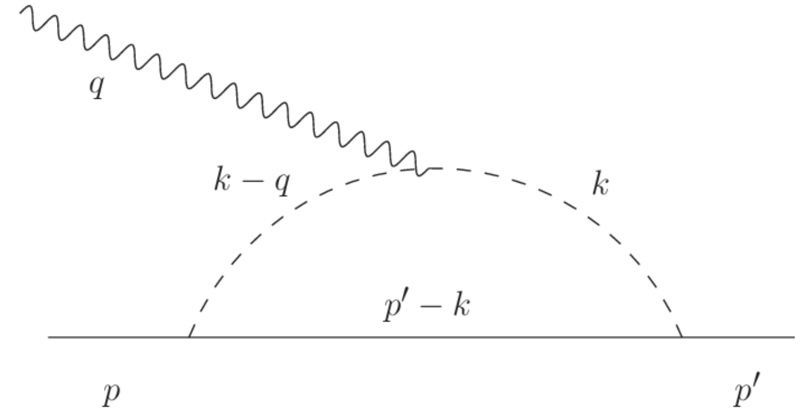
– this is the most general formula; we have three denominators, all of which are different, so in our case this becomes

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \left[Ax+By+C(1-x-y)\right]^{-3}$$

- Note that we can freely choose which of the denominators is A, B, or C; the choice can (and will) be different for the two different terms (= different loops), depending on what gives a more compact expression in the end

$$i\Gamma^\nu(p,p') = \frac{em^2g_A^2}{f_\pi}\left[2\tau^3\int \frac{d^4k}{(2\pi)^4}\frac{(2k-q)^\nu \not{k}}{[k^2-m_\pi^2][(k-q)^2-m_\pi^2][(k-p')^2-m^2]}\right.$$
$$\left. -\frac{1}{2}(3-\tau^3)\int\frac{d^4k}{(2\pi)^4}\frac{\not{k}\gamma^\nu\not{k}}{[k^2-m_\pi^2][(k-p)^2-m^2][(k-p')^2-m^2]}\right]$$

Example: Loop Graphs



- Start with the pion-coupling loop:

$$\int \frac{d^4k}{(2\pi)^4} \frac{(2k-q)^\nu \not{k}}{[k^2-m_\pi^2][(k-q)^2-m_\pi^2][(k-p')^2-m^2]}$$

$$\left([k^2-m_\pi^2][(k-q)^2-m_\pi^2][(k-p')^2-m^2]\right)^{-1} =$$

$$2\int_0^1 dx \int_0^{1-x} dy \left\{[k^2-m_\pi^2](1-x-y)+[(k-q)^2-m_\pi^2]y+[(k-p)^2-m^2]x\right\}^{-3}$$

- After cruncing the algebra, the combined denominator simplifies to $(k-k_1)^2-M_\pi^2$ with

$$k_1 = qy + p'x$$

$$M_\pi^2 = m^2x^2 + m_\pi^2(1-x) - q^2y(1-x-y) - (p'^2-m^2)x(1-x-y) - (p^2-m^2)xy$$

- Since the final and initial nucleons are on-shell, the two last terms vanish; a further simplification results if we redefine $y \to (1-x)y$

Example: Loop Graphs

- Now we can shift the momentum integration by k_1:

$$2\int_0^1 (1-x)dxdy \int \frac{d^4k}{(2\pi)^4} \frac{(2\tilde{k}-q)^\nu \tilde{\not{k}}}{(k^2-M_\pi^2)^3}$$

- Here, $\tilde{k} = k + k_1$ with $k_1 = q(1-x)y + p'x$, and

$$M_\pi^2 = m^2x^2 + m_\pi^2(1-x) - q^2y(1-y)(1-x)^2$$

- The momentum integration can now be performed. We will have divergences, so we need to go to D dimensions in order to apply dimensional regularization
- We will also need the loop functions which converge for n>2:

$$J_n(M) = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-M^2)^n} = \frac{i(-1)^n}{(4\pi)^{D/2}} \frac{\Gamma(n-D/2)}{\Gamma(n)} M^{D-2n}$$

- For small n the loop functions diverge, for instance, we will have n=2 and we will have to expand in powers of [4-D]; the exact expressions for J_n, however, will not be needed – we will just write down the vertices in terms of those

Example: Loop Graphs

- We need one more formula that relates tensor integrals with loop functions:

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 - M^2)^n} = \frac{g^{\mu\nu}}{2(n-1)} J_{n-1}(M)$$

- We also have to remember that all such tensor integrals with odd powers of integration momenta vanish:

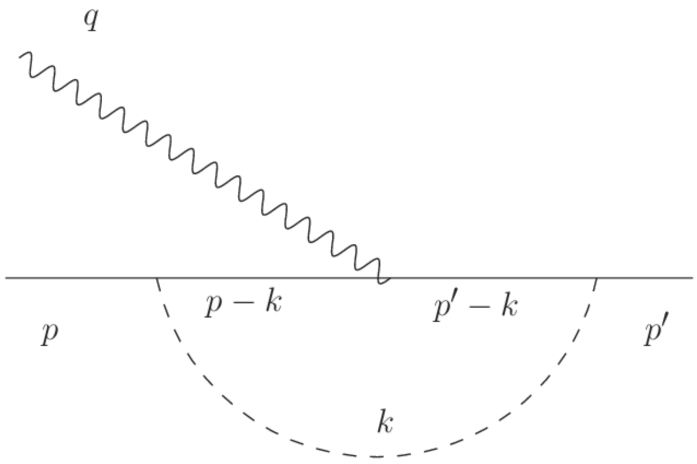
$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 - M^2)^n} = \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\lambda}{(k^2 - M^2)^n} = \cdots = 0$$

- We also need to remember that $\gamma^\mu \gamma_\mu = D, \qquad g^\mu_\mu = D$, and similarly for other identities with gamma matrices
- It is also useful to remember the off-shell Gordon identity that is just the consequence of the Dirac matrix algebra (and four-momenta conservation):

$$\gamma^\mu = \frac{1}{2m} \left\{ (p + p')^\mu + \frac{1}{2} \left[\not{q}, \gamma^\mu \right] - (\not{p}' - m)\gamma^\mu - \gamma^\mu (\not{p} - m) \right\}$$

- The two last terms vanish for on-shell nucleons

Example: Loop Graphs



- With the nucleon-coupling loop, we use

$$\int \frac{d^4k}{(2\pi)^4} \frac{\not{k}\gamma^\nu \not{k}}{[k^2 - m_\pi^2][(k-p)^2 - m^2][(k-p')^2 - m^2]}$$

$$\left([k^2 - m_\pi^2][(k-p)^2 - m^2][(k-p')^2 - m^2]\right)^{-1} =$$

$$2\int_0^1 dx \int_0^{1-x} dy \left\{[k^2 - m_\pi^2]x + [(k-p)^2 - m^2]y + [(k-p')^2 - m^2](1-x-y)\right\}^{-3}$$

- Making the same simplifications as for the first loop, we get the final result for this loop integral:

$$2\int_0^1 (1-x)dxdy \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{\not{k}}\gamma^\nu \tilde{\not{k}}}{(k^2 - M_N^2)^3}$$

where $\tilde{k} = k + k_2$ with $k_2 = p(1-x)y + p'(1-x)(1-y)$, and

$$M_N^2 = m^2(1-x)^2 + m_\pi^2 x - q^2 y(1-y)(1-x)^2 = M_\pi^2(m \leftrightarrow m_\pi)$$

Note that these are also simplified assuming on-shell nucleons!

Example: Loop Graphs

- The expression that we have to insert in the code is, up to a constant factor,

$$i\Gamma^{\nu}(p,p') = \left[4\tau^3 \int_0^1 (1-x)dxdy \int \frac{d^Dk}{(2\pi)^D} \frac{(2\tilde{k}-q)^{\nu}\tilde{\not k}}{(k^2-M_{\pi}^2)^3} - (3-\tau^3)\int_0^1 (1-x)dxdy \int \frac{d^Dk}{(2\pi)^D} \frac{\tilde{\not k}\gamma^{\nu}\tilde{\not k}}{(k^2-M_N^2)^3}\right]$$

- Remember that the shifted momenta are different in the two integrals!
- Now, we will program in FORM.
  - We will treat the two loops separately (which is not necessary though)
  - We will forget about the integration and about denominators in each integral and work with the numerators
  - In the latter, we will disentangle the structures that correspond to scalar and tensor integrals and replace them with loop functions
  - This will be our final result (which still has to be integrated over the Feynman parameters)

Example: Loop Graphs

- A few thoughts about what we expect to get, based on the Lorentz covariance arguments: the photon-nucleon vertex (on-shell) has to have the form

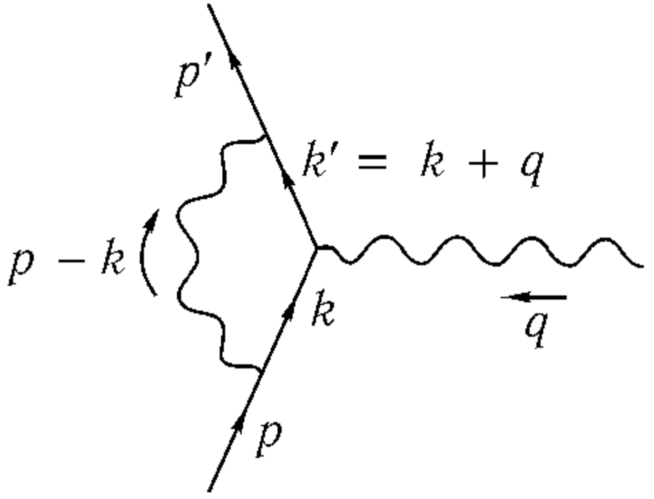
$$i\Gamma^{\nu}(p,p') = F_1(q^2)(p+p')^{\nu} + F_2(q^2)[\gamma^{\nu},\gamma^{\mu}]q_{\mu} + F_3(q^2)q^{\nu}$$

- For on-shell nucleons, the third term has to vanish – a consequence of gauge invariance, follows from the requirement $\Gamma(p,p')^{\nu}q_{\nu} = 0$; we can check that!
- Besides that, there can be some off-shell structures (important if the loop vertices are parts of larger Feynman graphs); they are proportional to $[\not{p} - m]$ on the right or $[\not{p}' - m]$ on the left (or both) and therefore vanish for on-shell nucleons (recall the off-shell Gordon identity). Note that we threw away some of these terms already when we used the Dirac equation in the initial integrals; if we want to trace the complete off-shell vertices we have to restore them (as well as pieces proportional to $p^2 - m^2$ and $p'^2 - m^2$ in M_{π}^2 and M_N^2)

[Example file: vertices.frm]

Exercise: the Schwinger Correction

- Using FORM, simplify the expression for the leading one-loop QED correction to the electron vertex function, and obtain the expression for the electron's anomalous magnetic moment
- Hint: the a.m.m. is the value of $F_2(q^2)$ (see previous slide) at $q^2 = 0$, up to a constant normalisation factor (which is for you to figure out)
- Calculate the QED loop diagram, using the appropriate Feynman parameterisation; assume the initial and final electrons on-shell

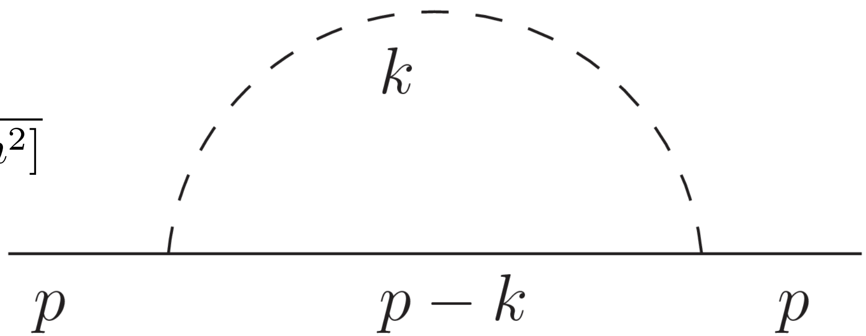


- Check that $F_3(q^2)$ vanishes; disregard $F_1(q^2)$, and find the value of the electron a.m.m. at leading order in the expansion in powers of the fine structure constant – the Schwinger correction

Exercise: Ward-Takahashi Identity

- The requirement that $\Gamma(p,p')^\nu q_\nu = 0$ is in fact the consequence of a more general statement called the Ward-Takahashi identity: if one considers the self-energy loop due to the interaction with the pion and the self-energy correction due to that loop,

$$i\Sigma(p) = -3\frac{m^2 g_A^2}{f_\pi^2}\int \frac{d^4k}{(2\pi)^4}\frac{\not{k}-\not{p}+m}{[k^2-m_\pi^2][(k-p)^2-m^2]}$$



these have to fulfill:

$$\Gamma^\nu(p,p')q_\nu = e[\Sigma(p')-\Sigma(p)]$$

- Exercise (very advanced!!!): check the Ward-Takahashi identity
- Hint: one has to calculate the self-energy correction without assuming the nucleons on-shell (and go back and do the same for the vertex loops)
- It may be easier to check this first when one of the two nucleons (either the final or the initial) is on-shell

Exercise: Ward-Takahashi Identity

- The Ward-Takahashi identity

$$\Gamma^{\nu}(p,p')q_{\nu} = e[\Sigma(p') - \Sigma(p)]$$

can be represented in a (simplified) form which is valid at $q = 0$ and can be obtained by taking the derivative with respect to q_{ν}:

$$\Gamma^{\nu}(p,p) = e\frac{\partial}{\partial p'_{\nu}}\Sigma(p')\bigg|_{p'=p}$$

- A simplified variation of the previous exercise would be to check the derivative form of the Ward-Takahashi identity; note that this is a matrix identity, one has to remember that when taking the derivatives!