Examples Sheet 5

Symmetries in Physics

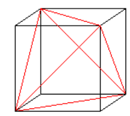
Winter 2017/18

Lecturer: PD Dr. G. von Hippel

Vacation sheet – Return 11.01.2018 – with up to 20 bonus points.

1. *Raising of degeneracies in crystals* (15 P.)

   Let $\mathrm{T} \subset \mathrm{SO}(3)$ be the tetrahedral group, i.e. the group of rotational symmetries of a regular tetrahedron. A regular tetrahedron can be inscribed into a cube as shown below.

   1. Show that the tetrahedral group can be presented as $\mathrm{T} = \langle b, c | b^2, c^3, (bc)^3 \rangle$, where b is a rotation around an axis through two opposing faces of the cube, and c is a rotation around a spatial diagonal of the cube. [*Hint:* It may be useful to consider the rotations as permutations of the vertices.]
   2. Find the conjugacy classes of T.
   3. Construct the character table of T. [*Hint:* It may be useful to consider how the subgroups relate to the conjugacy classes.]
   4. Let $\hat{H}_0$ be the rotationally invariant Hamiltonian of an electron in an atom (such the outermost shell of an alkali atom). What is the degeneracy pattern of $\hat{H}_0$, assuming there are no additional symmetries?
   5. Now assume that the atom of the preceding question is put into the center of a cube, at four vertices of which identical ions are located, such that they form a regular tetrahedron. Let V be the electrostatic potential generated by those ions. What is the symmetry group under which $\hat{H} = \hat{H}_0 + V$ is invariant?
   6. Show that for ρ a representation of SO(3), the matrices $\rho(g)$, $g \in \mathrm{T}$, form a representation of T. [This is known as a *subduced representation*.]
   7. Using characters determine the branching rules for the three lowest-dimensional irreps of SO(3) when the symmetry is broken to T.
   8. Use the result of the preceding question to predict what happens to the degenerate levels of an P- and a D-orbital ($l = 1, 2$), respectively.

2. *Isospin in $NN \to d\pi$ processes* (5 P.)

   Show using the Wigner-Eckart theorem that isospin symmetry predicts the ratio of

   $$\frac{\sigma(p + n \to d + \pi^0)}{\sigma(p + p \to d + \pi^+)} = \frac{1}{2}$$

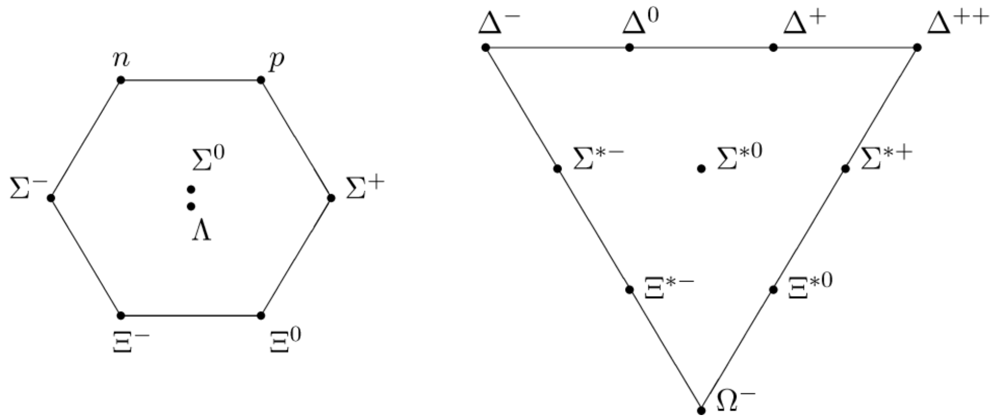
   for the cross-sections for the fusion via the strong interactions into deuterium of a proton and a neutron and two protons, respectively.

3. *The Gell-Mann–Okubo mass formula* (10 P.)

   1. Define the hypercharge Y of a an isospin multiplet to be equal to twice the average charge of the members of the multiplet:

      $$Y = 2\overline{Q}.$$

      Derive the hypercharges of the members of the baryon octet and decuplet.



   2. Assuming that isospin is an exact symmetry, but SU(3) flavour symmetry is not exact, explain why the Hamiltonian of the SU(3) quark model must be of the form

      $$\hat{H} = \hat{H}_0 + \hat{H}_8,$$

      where $\hat{H}_0$ is an SU(3) singlet, and $\hat{H}_8$ is a function of $\hat{I}^2$ and $\hat{Y}$ only.

   3. Show that a Hamiltonian of the form

      $$\hat{H}_8 = \alpha\hat{Y} + \beta\hat{I}^2$$

      cannot simultaneously explain the observed equal spacings of the masses in the decuplet and the mass difference of the σ^0 and Λ^0 baryons.

   4. Derive the value of γ for which

      $$\hat{H}_8 = \alpha\hat{Y} + \beta\left(\hat{I}^2 + \gamma\hat{Y}^2\right)$$

      is compatible with the equal spacings of the decuplet masses.

   5. Conclude the Gell-Mann–Okubo mass formula

      $$2(m_N + m_\Xi) = 3m_\Lambda + m_\Sigma$$

      and verify it against the experimental values.

4. *Algebraic solution of the hydrogen atom* (10+20 P.)

   In the lectures, we discussed the dynamical SO(4) symmetry of the Kepler-Coulomb potential, leaving most explicit calculations in the quantum mechanical case as exercises. In this question, you are asked to fill in the missing pieces. *By solving the "starred" parts of this question, you can earn up to 20 bonus points.*

   1. Show that the components of the Runge-Lentz-Pauli operator satisfy

   $$[\hat{H}, \hat{M}_i] = 0$$

   2. * Show that the components of the Runge-Lentz-Pauli operator satisfy

   $$[\hat{M}_i, \hat{M}_j] = i\epsilon_{ijk}\left(-\frac{2\hat{H}}{m}\right)\hat{L}_k$$

   3. * Show that the Runge-Lentz-Pauli operator and orbital angular momentum operator satisfy

   $$\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{L}} = \hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{M}} = 0$$

   4. * Show that the Runge-Lentz-Pauli operator squares to

   $$\hat{\boldsymbol{M}}^2 = \frac{2\hat{H}}{m}\left(\hat{\boldsymbol{L}}^2 + \hbar^2\hat{\mathbb{1}}\right) + e^4$$

   where the prefactor $\hbar^2 = 1$ can be restored on dimensional grounds.

Merry Christmas and a Happy New Year!