10.5 Zusammenstellung von Rechenregeln der Vektoranalysis

Es seien f und g hinreichend oft (i. Allg. stetig oder zweimal stetig) differenzierbare skalare Felder und $\vec{F}$ und $\vec{G}$ entsprechende Vektorfelder.

1. Summenregeln

$$\begin{aligned}
\vec{\nabla}(f+g) &= \vec{\nabla}f + \vec{\nabla}g, \\
\vec{\nabla}\cdot(\vec{F}+\vec{G}) &= \vec{\nabla}\cdot\vec{F} + \vec{\nabla}\cdot\vec{G}, \\
\vec{\nabla}\times(\vec{F}+\vec{G}) &= \vec{\nabla}\times\vec{F} + \vec{\nabla}\times\vec{G}.
\end{aligned}$$

2. Produktregeln

$$\begin{aligned}
\vec{\nabla}(fg) &= (\vec{\nabla}f)g + f\vec{\nabla}g, \\
\vec{\nabla}\cdot(f\vec{G}) &= (\vec{\nabla}f)\cdot\vec{G} + f\vec{\nabla}\cdot\vec{G}, \\
\vec{\nabla}\times(f\vec{G}) &= (\vec{\nabla}f)\times\vec{G} + f\vec{\nabla}\times\vec{G}, \\
\vec{\nabla}(\vec{F}\cdot\vec{G}) &= (\vec{F}\cdot\vec{\nabla})\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F} + \vec{F}\times(\vec{\nabla}\times\vec{G}) + \vec{G}\times(\vec{\nabla}\times\vec{F}), \\
\vec{\nabla}\cdot(\vec{F}\times\vec{G}) &= \vec{G}\cdot(\vec{\nabla}\times\vec{F}) - \vec{F}\cdot(\vec{\nabla}\times\vec{G}), \\
\vec{\nabla}\times(\vec{F}\times\vec{G}) &= (\vec{G}\cdot\vec{\nabla})\vec{F} - (\vec{F}\cdot\vec{\nabla})\vec{G} + \vec{F}(\vec{\nabla}\cdot\vec{G}) - \vec{G}(\vec{\nabla}\cdot\vec{F}).
\end{aligned}$$

3. Zweite Ableitungen

$$\begin{aligned}
\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) &= 0, \\
\vec{\nabla}\times(\vec{\nabla}f) &= \vec{0}, \\
\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \Delta\vec{F}.
\end{aligned}$$

10.6 Ableitungen in sphärischen und Zylinderkoordinaten

$$
\begin{aligned}
f(x,y,z) = g(\rho,\varphi,z) = h(r,\theta,\varphi),&\\
\vec{V} &= V_x(x,y,z)\hat{e}_x + V_y(x,y,z)\hat{e}_y + V_z(x,y,z)\hat{e}_z\\
&= G_\rho(\rho,\varphi,z)\hat{e}_\rho + G_\varphi(\rho,\varphi,z)\hat{e}_\varphi + G_z(\rho,\varphi,z)\hat{e}_z\\
&= H_r(r,\theta,\varphi)\hat{e}_r + H_\theta(r,\theta,\varphi)\hat{e}_\theta + H_\varphi(r,\theta,\varphi)\hat{e}_\varphi.
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla} f &= \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z\\
&= \frac{\partial g}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial g}{\partial \varphi}\hat{e}_\varphi + \frac{\partial g}{\partial z}\hat{e}_z\\
&= \frac{\partial h}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial h}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin(\theta)}\frac{\partial h}{\partial \varphi}\hat{e}_\varphi,\\
\vec{\nabla}\cdot\vec{V} &= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\\
&= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho G_\rho) + \frac{1}{\rho}\frac{\partial G_\varphi}{\partial \varphi} + \frac{\partial G_z}{\partial z}\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 H_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}(\sin(\theta)H_\theta) + \frac{1}{r\sin(\theta)}\frac{\partial H_\varphi}{\partial \varphi},\\
\vec{\nabla}\times\vec{V} &= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{e}_x + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{e}_z\\
&= \left(\frac{1}{\rho}\frac{\partial G_z}{\partial \varphi} - \frac{\partial G_\varphi}{\partial z}\right)\hat{e}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial G_z}{\partial \rho}\right)\hat{e}_\varphi + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho G_\varphi) - \frac{\partial G_\rho}{\partial \varphi}\right)\hat{e}_z\\
&= \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial \theta}(\sin(\theta)H_\varphi) - \frac{\partial H_\theta}{\partial \varphi}\right]\hat{e}_r + \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial H_r}{\partial \varphi} - \frac{\partial}{\partial r}(rH_\varphi)\right]\hat{e}_\theta\\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial \theta}\right]\hat{e}_\varphi,\\
\Delta f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\\
&= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial g}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 g}{\partial \varphi^2} + \frac{\partial^2 g}{\partial z^2}\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial h}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 h}{\partial \varphi^2}.
\end{aligned}
$$