Exercise sheet 3
Theoretical Physics 6a (QFT): WS 2017-2018
Lecturer : Prof. M. Vanderhaeghen

13.11.2017

Exercise 1 (40 points) : Dirac Field

The Free Dirac Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

where the normal mode expansion for the fields are

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \sum_s \left\{ a(\vec{k},s)u(\vec{k},s)e^{-ikx} + b^\dagger(\vec{k},s)v(\vec{k},s)e^{ikx} \right\}$$
$$\bar{\psi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \sum_s \left\{ b(\vec{k},s)\bar{v}(\vec{k},s)e^{-ikx} + a^\dagger(\vec{k},s)\bar{u}(\vec{k},s)e^{ikx} \right\}$$

(a)(5 points) Show that the sum of the spinors can be written as

$$\sum_{s=\pm 1/2} u(\vec{k},s)\bar{u}(\vec{k},s) = p\!\!\!/ + m$$
$$\sum_{s=\pm 1/2} v(\vec{k},s)\bar{v}(\vec{k},s) = p\!\!\!/ - m$$

(b)(5 points) Considering

$$[a(\vec{k},s), a^\dagger(\vec{k}',s')]_+ = [b(\vec{k},s), b^\dagger(\vec{k}',s')]_+ = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')\delta_{s\,s'}$$

and the other combinations zero, show that

$$[\psi_a(x), \psi_b^\dagger(x')]_+ = \delta^{(3)}(x-x')\delta_{a\,b} \qquad [\psi_a(x), \psi_b(x')]_+ = [\psi_a^\dagger(x), \psi_b^\dagger(x')]_+ = 0$$

(c)(10 points) Show that the momentum operator is

$$P = \int \frac{d^3k}{(2\pi)^3} \sum_s \vec{k} \left\{ a^\dagger(\vec{k},s)a(\vec{k},s) + b^\dagger(\vec{k},s)b(\vec{k},s) \right\}$$

(d)(10 points) Show that the conserved charge is

$$Q = \int \frac{d^3k}{(2\pi)^3} \sum_s \left\{ a^\dagger(\vec{k},s)a(\vec{k},s) - b^\dagger(\vec{k},s)b(\vec{k},s) \right\}$$

and explain the meaning of the relative minus sign in the expression above.

(e)(10 points) Show that if the Dirac field were quantized according to the Bose-Einstein statistics, that is, through commutators as for the Klein-Gordon field, one would get the following Hamiltonian:

$$H = \int \frac{d^3k}{(2\pi)^3} \sum_s E_{\vec{k}} \left\{ a^\dagger(\vec{k},s)a(\vec{k},s) - b^\dagger(\vec{k},s)b(\vec{k},s) \right\}$$

and explain why this would lead to unphysical results for the energy spectrum.

Exercise 2. (30 points) : Weyl representation

In the standard Dirac representation, Dirac matrices have the form

$$\gamma_s^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\gamma}_s = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix},$$

while in the so-called *Weyl representation*, they have the form

$$\gamma_W^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

with $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$.

(a)(10 points) Write down a unitary matrix S connecting both representations $\gamma_s^\mu = S\gamma_W^\mu S^{-1}$.

(b)(10 points) Write the γ_5 matrix in both representations.

(c)(10 points) In the Weyl representation with $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, show that the bispinors ψ_L and ψ_R are independent for massless particles, and write down the eigenstates of the chirality operator γ_5 with their corresponding eigenvalues.

Exercise 3. (30 points) : Axial current

For a Dirac field, the transformations

$$\psi(x) \to \psi'(x) = e^{i\alpha\gamma_5}\psi(x), \qquad \psi^\dagger(x) \to \psi^{\dagger'}(x) = \psi^\dagger(x)e^{-i\alpha\gamma_5},$$

where α is here an arbitrary real parameter, are called chiral phase transformations.

(a)(15 points) Show that the Dirac Lagrangian density $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under chiral phase transformations in the zero-mass limit $m = 0$ only, and that the corresponding conserved current in this limit is the axial vector current $J_A^\mu \equiv \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$.

(b)(15 points) Deduce the equations of motion for the fields

$$\psi_L(x) \equiv \frac{1}{2}(\mathbb{1} - \gamma_5)\psi(x), \qquad \psi_R(x) \equiv \frac{1}{2}(\mathbb{1} + \gamma_5)\psi(x),$$

for non-vanishing mass, and show that they decouple in the limit $m = 0$.

Hence, the Lagrangian density $\mathcal{L} = i\bar{\psi}_L\partial\!\!\!/\psi_L$ describes massless fermions with negative helicity and massless anti-fermions with positive helicity only. This field is called the Weyl field and can be used to describe the neutrinos as far as the latter can be considered as massless.