Examples Sheet 3

Symmetries in Physics

Winter 2017/18

Lecturer: PD Dr. G. von Hippel

1. *The Weyl group* (15 P.)

   Let
   $$s_{\boldsymbol{\alpha}_i}(x) : \mathbb{R}^r \to \mathbb{R}^r, \boldsymbol{x} \mapsto \boldsymbol{x} - 2\frac{\boldsymbol{\alpha}_i \cdot \boldsymbol{x}}{\boldsymbol{\alpha}_i^2}\boldsymbol{\alpha}_i$$
   be the Weyl reflection corresponding to the root $\boldsymbol{\alpha}_i$. The Weyl group is the subgroup of $\mathrm{O}(r)$ generated by the Weyl reflections.

   1. Show that $s_{\boldsymbol{\alpha}_i}$ permutes the positive roots other than $\boldsymbol{\alpha}_i$ if $\boldsymbol{\alpha}_i$ is simple.
   2. Show that $\{s_{\boldsymbol{\alpha}_i} | \boldsymbol{\alpha}_i \text{ simple}\}$ generates the Weyl group.
   3. Conclude that the Weyl group can be presented as $\langle g_i | (g_i g_j)^{r_{ij}} \rangle$, where $g_i = s_{\boldsymbol{\alpha}_i}$ for $\boldsymbol{\alpha}_i$ simple, $r_{ii} = 1$, and $r_{ij} = 2, 3, 4, 6$ for $n_{ij} = 0, 1, 2, 3$. [*Hint:* What is the product of two reflections?]

2. *The Chevalley basis* (10 P.)

   The *Chevalley basis* of a simple Lie algebra is defined by
   $$h_i = H_{\boldsymbol{\alpha}_i}, \qquad e_i = E_{\boldsymbol{\alpha}_i}, \qquad f_i = E_{-\boldsymbol{\alpha}_i}$$
   with $\boldsymbol{\alpha}_i$ the simple roots.

   1. Show the commutation relations
   $$[h_i, h_i] = 0, \quad [h_i, e_j] = K_{ji} e_j, \quad [h_i, f_j] = -K_{ji} f_i, \quad [e_j, f_j] = \delta_{ij} h_j,$$
   where K is the Cartan matrix.
   2. Show that the remaining root vectors are given by commutators of the e_i, f_j subject to the *Serre relations*
   $$\underbrace{[e_i, [\ldots [e_i,}_{1-K_{ji}} e_j] \ldots] = 0, \qquad \underbrace{[f_i, [\ldots [f_i,}_{1-K_{ji}} f_j] \ldots] = 0.$$
   [*Hint:* Consider the lemmas on root strings.]

3. *Working backwards from the Dynkin diagram* (15 P.)

   1. Starting from the Dynkin diagram for A_2, reconstruct the Cartan matrix.
   2. Starting from the Cartan matrix for A_2, reconstruct a pair of simple roots. [*Hint:* You may pick $\boldsymbol{\alpha}_1 = (1, 0)^t$.]
   3. Starting from the simple roots of A_2, reconstruct the full root system.
   4. Repeat the above steps for G_2.