Exercise sheet 4
Theoretical Physics 6a (QFT): WS 2017-2018
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20.11.2017

Exercise 1. (50 points) : Dirac bilinears

Since a spinor turns into minus itself after a rotation over 2π, physical quantities must be bilinears in ψ, so that physical quantities turn into themselves after a rotation over 2π. These bilinears have the general form $\bar{\psi}\Gamma\psi$. There are 16 independent covariant ones related to 16 complex 4×4 matrices:

- $\Gamma_S = \mathbb{1}$ (scalar);
- $\Gamma_P = \gamma_5$ (pseudoscalar);
- $\Gamma_V^\mu = \gamma^\mu$ (vector);
- $\Gamma_A^\mu = \gamma^\mu \gamma_5$ (axial vector);
- $\Gamma_T^{\mu\nu} = \sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ (tensor).

Without referring to any explicit representation for the Γ matrices,

(a)(10 points) show that $\Gamma^2 = \pm\mathbb{1}$.

(b)(10 points) show that for any Γ except Γ_S, we have $\mathrm{Tr}[\Gamma] = 0$. *Hint*: show first that for any Γ except Γ_S, there always exists a Γ' such that $\{\Gamma, \Gamma'\} = 0$.

(c)(10 points) check that the product of 2 different Γ's is proportional to some Γ different from Γ_S;

(d)(10 points) show that the Γ's are linearly independent, *i.e.* $\sum_i a_i \Gamma_i = 0 \Leftrightarrow a_i = 0$.

(e)(10 points) and using the Lorentz transformation of the Dirac spinor $\psi'(x') = S(a)\psi(x)$ with $x'^\mu = a^\mu{}_\nu x^\nu$, check that the bilinears transform according to their name, *i.e.* $\bar{\psi}'\psi' = \bar{\psi}\psi$, $\bar{\psi}'\gamma_5\psi' = \det(a)\bar{\psi}\gamma_5\psi$, $\bar{\psi}'\gamma^\mu\psi' = a^\mu{}_\nu\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}'\gamma^\mu\gamma_5\psi' = \det(a)a^\mu{}_\nu\bar{\psi}\gamma^\mu\gamma_5\psi$ and $\bar{\psi}'\sigma^{\mu\nu}\psi' = a^\mu{}_\rho a^\nu{}_\sigma\bar{\psi}\sigma^{\rho\sigma}\psi$.

Exercise 2. (50 points) : Fierz Transformation

(a)(10 points) Normalize the 16 matrices Γ^A to the convention

$$tr[\Gamma^A, \Gamma^B] = 4\delta^{AB}$$

This gives $\Gamma^A = \{1, \gamma^0, i\gamma^j, \dots\}$; write all 16 elements of this set.

(b)(20 points) Write the general Fierz identity as an equation

$$(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) = \sum_{C,D} C^{AB}_{CD}(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2) \tag{1}$$

with unknown coefficients C^{AB}_{CD}. Using the completeness of the 16 Γ^A matrices, show that

$$C^{AB}_{CD} = \frac{1}{16}tr[\Gamma^C\Gamma^A\Gamma^D\Gamma^B] \tag{2}$$

(c)(20 points) Work out explicitly the Fierz transformation laws for the products $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$ and $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$.