Practice Exam
Theoretical Physics 3 : QM SS2017
Lecturer : Prof. M. Vanderhaeghen

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Exercise 1 (20 points). Scattering off a cliff.

A particle of mass m and kinetic energy $E_{kin} > 0$ approaches an abrupt potential drop V_0 (Figure 1).

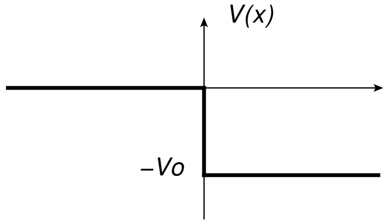

Figure 1: **Scattering off a cliff.**

a) **(14 points)** What is the probability that it will reflect back, if $E_{kin} = V_0/3$?

b) **(6 points)** Imagine now that this particle is a free neutron entering a nucleus. The neutron experiences a sudden drop in potential energy, from $V = 0$ outside to around -12 MeV inside. Suppose that this neutron coming from a fission event strikes such a nucleus with kinetic energy 4 MeV. What is the probability that it will be absorbed, thereby initiating another fission?

Exercise 2 (25 points). The generalized uncertainty principle.

The generalized uncertainty principle states that

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4}\langle C \rangle^2$$

where $\hat{C} \equiv -i[\hat{A}, \hat{B}]$.

a) **(6 points)** Check the principle for $\hat{A} = L_z$ and $\hat{B} = \vec{r}$, with $\vec{r} = (x, y, z)$, and evaluate the commutator for each component. It might be easiest to compute for $\hat{B} = x$, then generalize to $\vec{r}$.

b) **(7 points)** Check the principle now for $\hat{A} = L^2$ and $\hat{B} = L_z$. What is the meaning of the result?

c) **(12 point)** Finally, using the previous results, for $\hat{A} = L^2$ and $\hat{B} = [L^2, \vec{r}]$. It might be easiest to compute first for $\hat{B} = [L^2, z]$ and then generalize the result from z to $\vec{r}$ using the fact that $\vec{r} \cdot \vec{L} = \vec{r} \cdot (\vec{r} \times \vec{p}) = 0$.

Exercise 3 (25 points). A particle with spin 3/2.

Consider a particle with spin 3/2.

a) **(3 points)** Find the basis $|s, s_z\rangle$ of eigenstates of S_z for such a particle.

b) **(8 points)** Using the previous basis, find the matrices representing S_x and S_y.
Hint: Use $S_\pm = S_x \pm iS_y$ and
$S_\pm|s, s_z\rangle = \hbar\sqrt{s(s+1) - s_z(s_z \pm 1)}|s, s_z \pm 1\rangle$.

c) **(8 points)** Solve the characteristic equation to determine the eigenvalues of S_x.

d) **(6 points)** Write down, at least, one eigenvector of S_x.

Exercise 4 (30 points). Electron in static electric or magnetic field.

This problem considers the modification (using first order perturbation theory) of the energy spectrum of the hydrogen atom placed in a static electric field or in a static magnetic field.

a) **(10 points) Electron in a static magnetic field (Zeeman effect)**

Consider an electron moving in the $n = 2$ state of hydrogen. We neglect in the following all spin effects (i.e. we consider only orbital angular momentum effects).
The electron magnetic moment due to its orbital motion $\vec{\mu_l}$ interacts with an external magnetic field $\vec{\mathbf{B}}$ as:

$$\hat{H}'_B = -\vec{\mu_l} \cdot \vec{\mathbf{B}}$$

The magnetic moment vector $\vec{\mu_l}$ is expressed in terms of the electron orbital angular momentum vector $\vec{\mathbf{L}}$ as $\vec{\mu_l} = -\frac{e}{2m}\vec{\mathbf{L}}$, with $e > 0$, and m the electron mass.
Consider the case of a constant magnetic field along the z-axis:

$$\vec{\mathbf{B}} = B_0\,\hat{\mathbf{e}}_{\mathbf{z}},$$

with $\hat{\mathbf{e}}_{\mathbf{z}}$ the unit vector along the z-axis, and with B_0 a constant.
Use as unperturbed eigenstates $|n\,l\,m_l\rangle$ the notation:

$$\begin{aligned} |1\rangle &\equiv |2\,0\,0\rangle, \\ |2\rangle &\equiv |2\,1\,0\rangle, \\ |3\rangle &\equiv |2\,1\,+1\rangle, \\ |4\rangle &\equiv |2\,1\,-1\rangle. \end{aligned} \tag{1}$$

1) (4 points) What is the 4×4 matrix form of $\hat{H}'_B$ in the above basis?
Use the shorthand notation $\omega_0 \equiv eB_0/(2m)$ (*Larmor frequency*) in your result.

2) (4 points) Calculate the first order corrections to each of the four $n = 2$ levels due to $\hat{H}'_B$.

3) (2 points) Make a qualitative sketch of the total energy of the $n = 2$ levels as function of the externally applied magnetic field B_0. Comment on their degeneracies.

b) **(20 points) Electron in a static electric field (Stark effect)**

Consider next the $n = 2$ electron moving in a static electric field. The electric dipole moment $\vec{\mathbf{d}} = -e\vec{\mathbf{r}}$ of the electron interacts with an external electric field $\vec{\mathbf{E}}$ as:

$$\hat{H}'_E = -\vec{\mathbf{d}} \cdot \vec{\mathbf{E}}$$

Consider the case of a constant electric field along the x-axis :

$$\vec{\mathbf{E}} = E_0\, \hat{\mathbf{e}}_{\mathbf{x}},$$

with $\hat{\mathbf{e}}_{\mathbf{x}}$ the unit vector along the x-axis, and E_0 a constant.

1) (8 points) What is the 4×4 matrix form of $\hat{H}'_E$ in the unperturbed basis of Eq. (1)?
We recall that the hydrogen atom wave functions are given by:

$$\psi_{nlm_l}(r, \theta, \phi) = R_{n,l}(r)\; Y_{l,m_l}(\theta, \phi).$$

You are given the spherical harmonics:

$$\begin{aligned} Y_{0,0}(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}, \\ Y_{1,0}(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta, \\ Y_{1,\pm1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}, \end{aligned}$$

and you are given the radial integral:

$$\int_0^\infty dr\, r^3 R_{2,0}(r) R_{2,1}(r) = 3\sqrt{3}\, a,$$

with a the Bohr radius.
Hint : There are a lot of integrals in this problem, but several of them are zero. Evaluate first the different angular integrals and use symmetry relations to argue that several of them are zero. If the ϕ integral vanishes, there's no need to perform the r and θ integrals.

2) (3 points) Express next your result for the 4×4 matrix form of $\hat{H}'_E$ in terms of

$$\Omega_e \equiv eE_0 \frac{a}{\hbar}.$$

What is the dimension of Ω_e? (explain the result)

3) (6 points) Diagonalize the above matrix and calculate the first order corrections to the $n = 2$ levels due to $\hat{H}'_E$ (you only need to calculate the eigenvalues, not the eigenstates).

4) (3 points) Make a qualitative sketch of the total energy of the $n = 2$ levels as function of the externally applied electric field E_0. Comment on their degeneracies.