

# CHAPTER 3 : FORMALISM

## ⇒ 3.1 HILBERT SPACE

- QUANTUM THEORY  $\nearrow$  WAVE FUNCTIONS  $\Rightarrow$  DESCRIBE STATE OF SYSTEM  
 $\searrow$  OPERATORS  $\Rightarrow$  DESCRIBE OBSERVABLES  
e.g. ENERGY, MOMENTUM, ...
- $\hookrightarrow$  STATE VECTOR  $|\alpha\rangle$  LIVES IN N-DIM SPACE

$$|\alpha\rangle \Leftrightarrow a = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$$

INNER PRODUCT  $\langle \beta | \alpha \rangle = b^\dagger a = \sum_{i=1}^N b_i^* a_i$

$\hookrightarrow$  OPERATORS  $\Rightarrow$  REPRESENTED BY MATRICES IN N-DIM SPACE

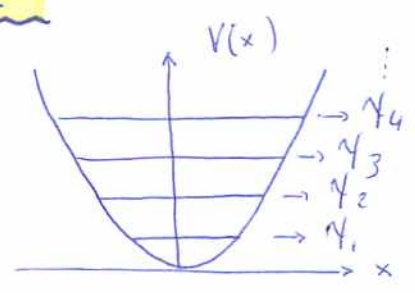
$$|\beta\rangle = \hat{T} |\alpha\rangle$$

$$\Downarrow$$
$$b = T a$$

$$T = \begin{pmatrix} T_{11} & \dots & T_{1N} \\ \vdots & & \vdots \\ T_{N1} & \dots & T_{NN} \end{pmatrix}$$

HILBERT SPACE

↳ e.g. H.O.



VECTORS IN Q.M. ARE SQUARE INTEGRABLE FUNCTIONS  $f(x)$

PHYSICAL STATE OF SYSTEM :  $\int_{-\infty}^{+\infty} dx |\Psi(x,t)|^2 = 1$

↳  $f(x)$  IS SQUARE INTEGRABLE ON INTERVAL  $[a, b]$

↑

$$\int_a^b dx |f(x)|^2 < \infty$$

↳ SET OF ALL SQUARE INTEGRABLE FUNCTIONS ON A SPECIFIED INTERVAL IS CALLED A HILBERT SPACE ( $\infty$  DIMENSIONAL !)

INNER PRODUCT OF 2 FUNCTIONS

$$\langle f | g \rangle \equiv \int_a^b dx f^*(x) g(x)$$

IF BOTH  $f$  &  $g$  ARE SQUARE-INTEGRABLE

↳ INNER PRODUCT EXISTS

PROOF: SCHWARZ INEQUALITY

$$|\langle f | g \rangle|^2 \leq \langle f | f \rangle \langle g | g \rangle$$

$$\hookrightarrow \langle g | f \rangle = \langle f | g \rangle^* \quad \checkmark \text{ VERIFIED FROM DEFINITION}$$

$$\hookrightarrow \langle f | f \rangle \geq 0 \text{ \& REAL}$$

$$\langle f | f \rangle = 0 \iff f(x) = 0$$

∴  $\langle f | g \rangle$  SATISFIES ALL PROPERTIES OF AN INNER PRODUCT

### • ORTHOGONALITY, NORMALIZATION

SET OF FUNCTIONS  $\{f_n\}$  IS ORTHONORMAL,

$$\begin{array}{c} \updownarrow \\ \langle f_m | f_n \rangle = \delta_{mn}. \end{array}$$

e.g. STATIONARY STATES  $\{\psi_n\}$  OF H.O.

### • COMPLETENESS

↳ SET OF FUNCTIONS IS COMPLETE

$\updownarrow$   
ANY OTHER FUNCTION IN HILBERT SPACE CAN BE EXPRESSED AS A LINEAR COMBINATION OF THEM

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

e.g. STATIONARY STATES  $= \{\psi_n\}$  OF H.O.

↳ IF  $\{f_n\}$  ARE ORTHONORMAL

$$c_n = \langle f_n | f \rangle$$

# ⇒ 3.2 OBSERVABLES

## • HERMITIAN OPERATORS

↳ EXPECTATION VALUE OF OBSERVABLE  $Q(x, p)$

e.g.  $H(x, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

$$\langle Q \rangle = \int dx \Psi^* \hat{Q} \Psi$$
$$= \langle \Psi | \hat{Q} \Psi \rangle$$

↳ MEASUREMENT ⇒ REAL OUTCOME

$\langle Q \rangle = \langle Q \rangle^*$  REAL EXPECTATION VALUES.

↳  $\langle \Psi | \hat{Q} \Psi \rangle^* = \langle \hat{Q} \Psi | \Psi \rangle$

$$= \langle \Psi | \hat{Q} \Psi \rangle$$

OPERATORS  $\hat{Q}$  SATISFYING

$\langle \hat{Q} \Psi | \Psi \rangle = \langle \Psi | \hat{Q} \Psi \rangle$   $\forall \Psi(x)$

ARE HERMITIAN OPERATORS

∴ OBSERVABLES ARE REPRESENTED BY HERMITIAN OPERATORS

↳  $\langle \hat{Q} f | g \rangle = \langle f | \hat{Q} g \rangle$   $\forall f(x), g(x)$

SHOW THAT THIS IS EQUIVALENT TO ABOVE

↳ EXAMPLE : MOMENTUM OPERATOR  $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

$$\langle f | \hat{p} g \rangle = \int_{-\infty}^{+\infty} dx f^*(x) (-i\hbar) \frac{d}{dx} g(x)$$

↓ INTEGRATION BY PARTS

$$= -i\hbar \cancel{f^* g} \Big|_{-\infty}^{+\infty} + i\hbar \int_{-\infty}^{+\infty} dx \left( \frac{d}{dx} f^*(x) \right) g(x)$$

*f & g ARE SQUARE-INTEGRABLE*

$$= \int_{-\infty}^{+\infty} dx \left( -i\hbar \frac{d}{dx} f(x) \right)^* g(x)$$

$$= \langle \hat{p} f | g \rangle$$

■

$\left( -i\hbar \frac{d}{dx} \right)$  IS HERMITIAN

$\left( \frac{d}{dx} \right)$  IS NOT!

• DETERMINATE STATES

↳ IN DETERMINATE STATE: EVERY MEASUREMENT OF OBSERVABLE Q RETURNS THE SAME VALUE

e.g. STATIONARY STATES ARE DETERMINATE STATES OF HAMILTONIAN

$$\langle \psi_m | \hat{H} \psi_m \rangle = E_m$$

↑  
ENERGY VALUE

↳ STANDARD DEVIATION  $\sigma$  OF Q IS ZERO IN A DETERMINATE STATE  $|\hat{Q}\psi\rangle = q|\psi\rangle$

$$\langle \hat{Q} \rangle = \langle \psi | \hat{Q} \psi \rangle = q$$

$$\begin{aligned} \sigma^2 &= \langle (\hat{Q} - \langle \hat{Q} \rangle)^2 \rangle \\ &= \langle \psi | (\hat{Q} - q)^2 \psi \rangle \end{aligned}$$

$$= \langle (\hat{Q} - q) \psi | (\hat{Q} - q) \psi \rangle$$

$$\supseteq |\hat{Q}\psi\rangle = q|\psi\rangle$$

$$= 0$$

|| DETERMINATE STATE IS EIGENFUNCTION OF  $\hat{Q}$

$$\hat{Q}|\psi\rangle = q|\psi\rangle$$

↳ EIGENVALUES OF HERMITIAN OPERATOR ARE REAL

COLLECTION OF ALL EIGENVALUES OF OPERATOR  $\Rightarrow$  SPECTRUM

IF 2 OR MORE LIN. INDEP EIGENFUNCTIONS SHARE SAME EIGENVALUE  $\Rightarrow$  SPECTRUM IS DEGENERATE

↳ e.g. ENERGY SPECTRUM

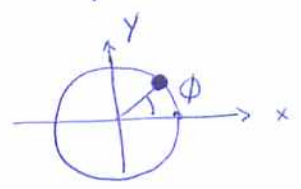
EIGENVALUES OF  $\hat{H}$

$$\hat{H} |\psi_m\rangle = E_m |\psi_m\rangle$$

(TIME INDEP. SCHRÖDINGER EQ.)

↳ EXAMPLE

$$\hat{Q} = i \frac{\partial}{\partial \phi}$$



$$\phi \in [0, 2\pi] \Rightarrow f(\phi) = f(\phi + 2\pi) \quad (*)$$

$\Rightarrow$  HERMITIAN

$$\begin{aligned} \langle f | \hat{Q} g \rangle &= \int_0^{2\pi} d\phi f^*(\phi) i \frac{\partial}{\partial \phi} g(\phi) \\ &= \cancel{i \int_0^{2\pi} d\phi f^* g} - i \int_0^{2\pi} d\phi \left( \frac{\partial f}{\partial \phi} \right)^* g(\phi) \quad (*) \\ &= \int_0^{2\pi} d\phi \left( i \frac{\partial}{\partial \phi} f \right)^* g \\ &= \langle \hat{Q} f | g \rangle \quad \forall f, g \end{aligned}$$

⇒ EIGENVALUES

$$i \frac{d}{d\phi} f(\phi) = q f(\phi)$$

$$f(\phi) = A e^{-iq\phi}$$

$$f(\phi) = f(\phi + 2\pi) \Rightarrow e^{-iq2\pi} = 1$$

⇓

$$q = 0, \pm 1, \pm 2, \dots$$

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SPECTRUM OF  $\hat{Q}$



# ⇒ 3.3 EIGENFUNCTIONS OF HERMITIAN OPERATOR

SPECTRUM

- ↗ DISCRETE e.g. H.O., INFINITE SQUARE WELL  
↳ EIGENSTATES NORMALIZABLE
- ↘ CONTINUOUS e.g.  $V(x) = 0$   
↳ EIGENSTATES NOT NORMALIZABLE

## • DISCRETE SPECTRUM OF HERMITIAN OPERATOR

↳ EIGENVALUES ARE REAL

$$\hat{Q} |f\rangle = q |f\rangle$$

$\hat{Q}$  IS HERMITIAN  $\langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle$

$$q \langle f | f \rangle = q^* \langle f | f \rangle$$

$$q = q^*$$

↳ EIGENFUNCTIONS BELONGING TO DISTINCT EIGENVALUES ARE ORTHOGONAL ( $\hat{Q} = \hat{Q}^+$ )

$$\hat{Q} |f\rangle = q |f\rangle$$

$$\hat{Q} |g\rangle = q' |g\rangle$$

$$\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle$$

$$q' \langle f | g \rangle = q^* \langle f | g \rangle = q \langle f | g \rangle$$

IF NOT USE GRAM SCHMIDT ORTHOGONALIZATION PROCEDURE TO CONSTRUCT ORTHOG. EIGENF.

IF  $q \neq q'$   
↓  
 $\langle f | g \rangle = 0$  □

↳ IN FINITE DIM. VECTOR SPACE

EIGENVECTORS OF HERMITIAN OPERATOR ARE COMPLETE

(i.e. SPAN THE WHOLE SPACE)

↳ FOR HILBERT SPACE ( $\infty$  DIM)

AXIOM : EIGENFUNCTIONS OF AN OBSERVABLE OPERATOR ARE COMPLETE :

ANY FUNCTION IN HILBERT SPACE CAN BE WRITTEN AS A LIN. COMBINATION OF THEM

• CONTINUOUS SPECTRUM

↳ EIGENFUNCTIONS NOT NORMALIZABLE

↳ HOW TO UNDERSTAND PROPERTIES SUCH AS REALITY (OF EIGENVALUES), ORTHOGONALITY, COMPLETENESS

↳ EXAMPLE : EIGENVALUES & EIGENFUNCTIONS OF MOMENTUM OPERATOR

$$\underbrace{-i\hbar \frac{d}{dx}}_{\hat{p}} \psi_p(x) = p \psi_p(x)$$

↳ EIGENFUNCTION

$$\psi_p(x) = A e^{\frac{i}{\hbar} p x} \quad \left( \Rightarrow \lambda = \frac{2\pi\hbar}{p} \right)$$

(NOT SQUARE INTEGRABLE)

$\Rightarrow \psi_p$  IS NOT A FUNCTION IN HILBERT SPACE

⇒ BUT, WE CAN UNDERSTAND ORTHONORMALITY IN FOLLOWING SETS:

$$\langle f_{p'} | f_p \rangle = \int_{-\infty}^{+\infty} dx f_{p'}^*(x) f_p(x) = |A|^2 \int_{-\infty}^{+\infty} dx e^{-\frac{i}{\hbar} (p' - p) x}$$

$$= |A|^2 2\pi \hbar \delta(p - p')$$

CHOOSE  $A = \frac{1}{\sqrt{2\pi\hbar}}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p x}$$

$$\langle f_{p'} | f_p \rangle = \delta(p - p')$$

REPLACES KRONECKER DELTA IN CONTINUOUS CASE



CALLED: DIRAC ORTHONORMALITY

⇒ EIGENFUNCTIONS ARE COMPLETE  $\sum \rightarrow \int$

$$f(x) = \int_{-\infty}^{+\infty} dp c(p) f_p(x)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp c(p) e^{\frac{i}{\hbar} p x} \quad (\text{FOURIER TF.})$$

$$\langle f_{p'} | f \rangle = \int_{-\infty}^{+\infty} dp c(p) \underbrace{\langle f_{p'} | f_p \rangle}_{\delta(p - p')} = \underline{\underline{c(p')}}$$

↳ EXAMPLE : EIGENVALUES & EIGENFUNCTIONS OF  
POSITION OPERATOR.

$$\hat{x} g_y(x) = y g_y(x)$$

$$\underline{\underline{g_y(x) = A \delta(x-y)}}$$

$$\begin{aligned} \langle g_{y'} | g_y \rangle &= \int dx g_{y'}^*(x) g_y(x) \\ &= |A|^2 \int dx \delta(x-y') \delta(x-y) \\ &= |A|^2 \delta(y-y') \end{aligned}$$

CHOOSE  $A = 1$

$$\| g_y(x) = \delta(x-y)$$

$$\| \langle g_{y'} | g_y \rangle = \delta(y-y')$$

⇒ 'COMPLETENESS'

$$f(x) = \int_{-\infty}^{+\infty} dy c(y) g_y(x)$$

$$= \int_{-\infty}^{+\infty} dy c(y) \delta(x-y) = c(x)$$

$$\| c(y) = f(y)$$

## ⇒ 3.4 STATISTICAL INTERPRETATION

- OBSERVABLE  $Q(x, p)$  HERMITIAN  $Q^\dagger = Q$

↓  
MEASUREMENT GIVES ONE OF THE REAL EIGENVALUES OF  $Q$

- ↳ DISCRETE SPECTRUM  $q_n$   
PROBABILITY TO FIND EIGENVALUE  $q_n$

$$\hookrightarrow |c_n|^2 \Rightarrow c_n = \langle f_n | \Psi \rangle$$

$$\Psi(x, t) = \sum_n c_n f_n(x)$$

- ↳ CONTINUOUS SPECTRUM  $q(z)$

PROBABILITY TO FIND EIGENVALUE IN RANGE  $dz$

$$\hookrightarrow |c(z)|^2 dz \Rightarrow c(z) = \langle f_z | \Psi \rangle$$

∴ BY DOING THE MEASUREMENT  $\leadsto$  MEASURING A PARTICULAR EIGENVALUE

⇓  
THE WAVE FUNCTION COLLAPSES  
TO THE CORRESPONDING  
EIGENSTATE

• TOTAL PROBABILITY / EXPECTATION VALUES

$$\hookrightarrow \sum_n |c_n|^2 = 1 \quad (\text{TOTAL PROBABILITY})$$

$$\hookrightarrow \Psi = \sum_n c_n f_n$$

$$\langle \Psi | \Psi \rangle = \sum_n \sum_{n'} c_n^* c_{n'} \underbrace{\langle f_n | f_{n'} \rangle}_{\delta_{nn'}}$$

$$= \sum_n |c_n|^2$$

$$= 1 \quad (\text{NORMALIZATION OF W.F.})$$

$$\hookrightarrow \langle Q \rangle = \langle \Psi | \hat{Q} \Psi \rangle$$

$$= \sum_n \sum_{n'} c_n^* c_{n'} \underbrace{\langle f_n | \hat{Q} f_{n'} \rangle}_{q_{n'} | f_{n'} \rangle}$$

$$= \sum_n \sum_{n'} c_n^* c_{n'} q_{n'} \delta_{nn'}$$

$$\langle Q \rangle = \sum_n q_n |c_n|^2$$

## MOMENTUM SPACE WAVE FUNCTION

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p x} \quad \text{EIGENFUNCTIONS OF } \hat{p}$$

$$\hat{p} f_p(x) = p f_p(x)$$

$$C(p) = \langle f_p | \Psi \rangle \quad \text{PROB. AMPL. TO FIND } \Psi \text{ IS MOMENTUM EIGENSTATE}$$

$$C(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx e^{-\frac{i}{\hbar} p x} \Psi(x, t)$$

$$C(p) \equiv \bar{\Phi}(p, t) \quad \text{IS CALLED } \underline{\text{MOMENTUM SPACE WAVE FUNCTION.}}$$

(FOURIER TF OF POSITION SPACE WF  $\Psi(x, t)$ )

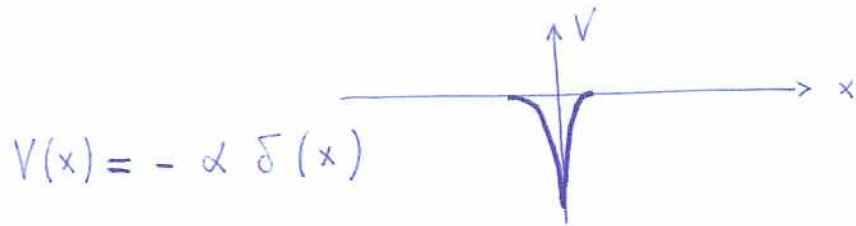
$$\bar{\Phi}(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx e^{-\frac{i}{\hbar} p x} \Psi(x, t)$$

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp e^{\frac{i}{\hbar} p x} \bar{\Phi}(p, t)$$

∞∞ PROBABILITY THAT A MEASUREMENT OF MOMENTUM YIELDS A VALUE BETWEEN  $p$  AND  $p+dp$

$$|\bar{\Phi}(p, t)|^2 dp$$

- EXAMPLE : MOM. SPACE W.F. OF PARTICLE OF MASS  $m$  IN  $\delta$ -FUNCTION POT.



$$V(x) = -\alpha \delta(x)$$

↳ BEFORE  $E = -\frac{\hbar^2 k^2}{2m}$   $k = \frac{m\alpha}{\hbar^2}$

↳ BOUND STATE ENERGY

$$\underline{\Psi}(x,t) = \sqrt{k} e^{-k|x|} e^{-\frac{i}{\hbar} Et}$$

↳  $\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx e^{-\frac{i}{\hbar} px} \underline{\Psi}(x,t)$

$$= \frac{\sqrt{k}}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} Et} \int_{-\infty}^{+\infty} dx e^{-\frac{i}{\hbar} px} e^{-k|x|}$$

$$= \frac{\sqrt{k}}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} Et} \left\{ \int_{-\infty}^0 dx e^{(k - \frac{i}{\hbar} p)x} + \int_0^{+\infty} dx e^{(-k - \frac{i}{\hbar} p)x} \right\}$$

$$\frac{1}{k - \frac{i}{\hbar} p} e^{(k - \frac{i}{\hbar} p)x} \Big|_{-\infty}^0$$

$$+ \frac{1}{-k - \frac{i}{\hbar} p} e^{-(k + \frac{i}{\hbar} p)x} \Big|_0^{+\infty}$$

$$= \frac{\sqrt{k}}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} Et} \left\{ \frac{1}{k - \frac{i}{\hbar} p} + \frac{1}{k + \frac{i}{\hbar} p} \right\}$$



$$\Phi(p, t) = \frac{\sqrt{k}}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} Et} \frac{2k}{k^2 + \frac{p^2}{\hbar^2}}$$

$$\Phi(p, t) = \sqrt{\frac{2}{\pi}} \frac{(\hbar k)^{3/2}}{p^2 + (\hbar k)^2} e^{-\frac{i}{\hbar} Et}$$

↳ PROBABILITY TO FIND PARTICLE WITH MOMENTUM  $> \hbar k$

$$P(p > \hbar k) = \int_{\hbar k}^{+\infty} dp |\Phi(p, t)|^2$$

$$= \frac{2}{\pi} (\hbar k)^3 \int_{\hbar k}^{+\infty} dp \frac{1}{(p^2 + (\hbar k)^2)^2}$$

↓ HELP:  $2p_0^3 \int dp \frac{1}{(p^2 + p_0^2)^2} = \frac{pp_0}{p^2 + p_0^2} + \arctan\left(\frac{p}{p_0}\right)$

$$P(p > \hbar k) = \frac{1}{\pi} \left[ \frac{p \hbar k}{p^2 + (\hbar k)^2} + \arctan\left(\frac{p}{\hbar k}\right) \right]_{\hbar k}^{\infty}$$

$$= \frac{1}{\pi} \left[ \frac{\pi}{2} - \frac{1}{2} - \frac{\pi}{4} \right]$$

$$= \frac{1}{4} - \frac{1}{2\pi} \approx 0.09$$

# ⇒ 3.5 THE UNCERTAINTY PRINCIPLE

## • PROOF OF UNCERTAINTY PRINCIPLE

↳ OBSERVABLE A ( $\hat{A}$  IS HERMITIAN)

$$\langle \hat{A} \rangle$$

$$\sigma_A^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle$$

$$= \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)^2 \Psi \rangle$$

}  $\hat{A}$  HERMITIAN

$$= \langle (\hat{A} - \langle \hat{A} \rangle) \Psi | (\hat{A} - \langle \hat{A} \rangle) \Psi \rangle$$

$$\equiv \langle f | f \rangle \quad \text{WITH } |f\rangle = |(\hat{A} - \langle \hat{A} \rangle) \Psi \rangle$$

↳ OBSERVABLE B ( $\hat{B}$  IS HERMITIAN)

$$\langle \hat{B} \rangle$$

$$\sigma_B^2 \equiv \langle g | g \rangle \quad \text{WITH } |g\rangle = |(\hat{B} - \langle \hat{B} \rangle) \Psi \rangle$$

↳ SCHWARTZ INEQUALITY

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$$

↳  $\forall z$  : COMPLEX NUMBER

$$|z|^2 = (\text{Re } z)^2 + (\text{Im } z)^2 \geq (\text{Im } z)^2 = \left[ \frac{1}{2i} (z - z^*) \right]^2$$

LET  $z = \langle f | g \rangle$

$$\frac{1}{2i} (z - z^*) = \frac{1}{2i} (\langle f | g \rangle - \langle f | g \rangle^*)$$

$$= \frac{1}{2i} (\langle f | g \rangle - \langle g | f \rangle)$$

$$\hookrightarrow \langle f | g \rangle = \langle (\hat{A} - \langle \hat{A} \rangle) \Psi | (\hat{B} - \langle \hat{B} \rangle) \Psi \rangle$$

$$= \langle \Psi | (\hat{A} - \langle \hat{A} \rangle) (\hat{B} - \langle \hat{B} \rangle) \Psi \rangle$$

$$= \langle \Psi | (\hat{A} \hat{B} - \langle \hat{A} \rangle \hat{B} - \hat{A} \langle \hat{B} \rangle + \langle \hat{A} \rangle \langle \hat{B} \rangle) \Psi \rangle$$

$$= \langle \Psi | \hat{A} \hat{B} \Psi \rangle - \langle \hat{A} \rangle \langle \Psi | \hat{B} \Psi \rangle$$

$$- \langle \hat{B} \rangle \langle \Psi | \hat{A} \Psi \rangle + \langle \hat{A} \rangle \langle \hat{B} \rangle \underbrace{\langle \Psi | \Psi \rangle}_1$$

$$= \langle \Psi | \hat{A} \hat{B} \Psi \rangle - 2 \langle \hat{A} \rangle \langle \hat{B} \rangle + \langle \hat{A} \rangle \langle \hat{B} \rangle$$

$$= \langle \Psi | \hat{A} \hat{B} \Psi \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$$

$$\langle g | f \rangle = \langle \Psi | \hat{B} \hat{A} \Psi \rangle - \langle \hat{B} \rangle \langle \hat{A} \rangle$$

$$\hookrightarrow \langle f | g \rangle - \langle g | f \rangle = \langle \Psi | (\hat{A} \hat{B} - \hat{B} \hat{A}) \Psi \rangle$$

$$= \langle \Psi | [\hat{A}, \hat{B}] \Psi \rangle$$

$$\underline{\underline{[\hat{A}, \hat{B}] = \hat{A} \hat{B} - \hat{B} \hat{A} \quad \text{COMMUTATOR}}}$$

$$\hookrightarrow \sigma_A^2 \sigma_B^2 \geq \left[ \frac{1}{2i} (\langle f|g \rangle - \langle g|f \rangle) \right]^2$$

$$\sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

UNCERTAINTY PRINCIPLE

$\hookrightarrow$  NOTE COMMUTATOR OF 2 HERMITIAN OPERATORS IS ANTI-HERMITIAN!

$$\begin{aligned} [\hat{A}, \hat{B}]^\dagger &= (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\ &= (\hat{B}^\dagger \hat{A}^\dagger - \hat{A}^\dagger \hat{B}^\dagger) \quad \left. \begin{array}{l} \hat{A}^\dagger = \hat{A} \\ \hat{B}^\dagger = \hat{B} \end{array} \right\} \\ &= \hat{B}\hat{A} - \hat{A}\hat{B} \\ &= -[\hat{A}, \hat{B}] \end{aligned}$$

$\rightsquigarrow$  EIGENVALUES OF ANTI-HERMITIAN OPERATOR ARE IMAGINARY

$\rightsquigarrow$   $\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle$  IS REAL NUMBER!

# POSITION-MOMENTUM UNCERTAINTY : $\sigma_x \sigma_p$

$$\hat{A} = \hat{x} \quad \text{POSITION OPERATOR}$$

$$\hat{B} = \hat{p} \quad \text{MOMENTUM OPERATOR}$$

$$= -i\hbar \frac{d}{dx}$$

$$[\hat{A}, \hat{B}] = [\hat{x}, \hat{p}] = i\hbar$$

$$\sigma_x^2 \sigma_p^2 \geq \left( \frac{1}{2i} i\hbar \right)^2 = \left( \frac{\hbar}{2} \right)^2$$

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

EQUIVALENT NOTATION

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

ORIGINAL UNCERTAINTY PRINCIPLE OF HEISENBERG

↳ PAIR OF OBSERVABLES  $A, B$  FOR WHICH  
 $[\hat{A}, \hat{B}] \neq 0 \Rightarrow A, B$  ARE IN COMPATIBLE OBSERVABLES.  
↓  
THEY DO NOT HAVE  
A COMPLETE SET OF SHARED EIGENFUNCTIONS.

↳ COMPATIBLE OBSERVABLES  
 $[\hat{A}, \hat{B}] = 0 \hookrightarrow$  DO HAVE A COMPLETE SET  
OF SHARED EIGENFUNCTIONS

↳ (SEE PROBLEM 3.15!)

• MINIMUM UNCERTAINTY WAVE PACKET

↳ GROUND STATE OF H.O.

$$\sigma_x \sigma_p = \frac{\hbar}{2} \rightarrow \text{HITS THE UNCERTAINTY LIMIT}$$

↳ EQUALITY MEANS :

1) SCHWARTZ INEQUALITY BECOMES EQUALITY

$$\langle f | f \rangle \langle g | g \rangle = |\langle f | g \rangle|^2$$

$$\Downarrow$$
$$g(x) = c f(x)$$

$$2) |z|^2 = |\langle f | g \rangle|^2 = (\text{Im} \langle f | g \rangle)^2$$
$$\Downarrow$$

$\langle f | g \rangle$  IS PURELY IMAGINARY  $\Rightarrow c = ia$

$\therefore g(x) = ia f(x)$        $a$  IS REAL

↳  $f(x) \Rightarrow |(\hat{x} - \langle \hat{x} \rangle) \Psi\rangle$

$g(x) \Rightarrow |(\hat{p} - \langle \hat{p} \rangle) \Psi\rangle$

$$g(x) = ia f(x)$$

$$\left(-i\hbar \frac{d}{dx} - \langle p \rangle\right) \Psi = ia (x - \langle x \rangle) \Psi$$

SOLUTION

TRY:  $\underline{\Psi}(x) = A e^{\frac{i}{\hbar} \langle p \rangle x} \phi(x)$

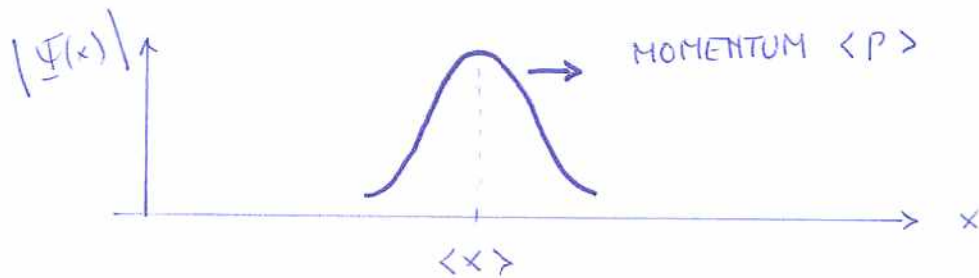
$$A e^{\frac{i}{\hbar} \langle p \rangle x} \left( -i\hbar \frac{d}{dx} \right) \phi = i a A e^{\frac{i}{\hbar} \langle p \rangle x} (x - \langle x \rangle) \phi$$

↓

$$\frac{d}{dx} \phi(x) = -\frac{a}{\hbar} (x - \langle x \rangle) \phi(x)$$

SOLUTION  $\phi(x) = e^{-\frac{a}{2\hbar} (x - \langle x \rangle)^2}$

∴  $\underline{\Psi}(x) = A e^{\frac{i}{\hbar} \langle p \rangle x} \cdot e^{-\frac{a}{2\hbar} (x - \langle x \rangle)^2}$



↳ CALLED : **MINIMUM UNCERTAINTY GAUSSIAN WAVEPACKET**

## ENERGY - TIME UNCERTAINTY PRINCIPLE

$$\hookrightarrow \frac{d}{dt} \langle \hat{Q} \rangle$$

$$= \frac{d}{dt} \langle \Psi | \hat{Q} | \Psi \rangle$$

$$= \left\langle \frac{\partial \Psi}{\partial t} \middle| \hat{Q} | \Psi \right\rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} | \Psi \rangle + \langle \Psi | \hat{Q} \frac{\partial \Psi}{\partial t} \rangle$$

$$\downarrow \quad i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$$

$$= \frac{i}{\hbar} \langle \hat{H} \Psi | \hat{Q} | \Psi \rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} | \Psi \rangle - \frac{i}{\hbar} \langle \Psi | \hat{Q} \hat{H} | \Psi \rangle$$

$$= \frac{i}{\hbar} \langle \Psi | \hat{H} \hat{Q} | \Psi \rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} | \Psi \rangle - \frac{i}{\hbar} \langle \Psi | \hat{Q} \hat{H} | \Psi \rangle$$

$$= \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle$$

$$\frac{d}{dt} \langle \hat{Q} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle$$



↳ GENERALIZED UNCERTAINTY PRINCIPLE

$$\hat{A} = \hat{H}$$

$$\hat{B} = \hat{Q}$$

WHICH DOES NOT DEPEND ON TIME  $\frac{\partial \hat{Q}}{\partial t} = 0$

$$\sigma_H^2 \sigma_Q^2 \geq \left( \frac{1}{2i} \langle [\hat{H}, \hat{Q}] \rangle \right)^2$$

$$= \left( \frac{\hbar}{2} \right)^2 \left( \frac{d\langle Q \rangle}{dt} \right)^2$$

$$\sigma_H \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|$$

$$\sigma_H \equiv \Delta E$$

AND

$$\Delta t \equiv \frac{\sigma_Q}{\left| \frac{d\langle Q \rangle}{dt} \right|}$$

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

ENERGY - TIME  
UNCERTAINTY PRINCIPLE

MEANING:

$$\sigma_Q = \left| \frac{d\langle Q \rangle}{dt} \right| \Delta t$$

↑  
RATE OF CHANGE OF  $\langle Q \rangle$   
PER UNIT OF TIME

Δt : TIME IT TAKES FOR EXPECTATION VALUE  $\langle Q \rangle$   
TO CHANGE BY ONE STANDARD DEVIATION

↳ EXAMPLE :

- FOR STATIONARY STATE :

ENERGY IS UNIQUELY DETERMINED  $\Rightarrow \Delta E = 0$   
 $\Downarrow$   
 $\Delta t = \infty$

(ALL OBSERVABLES ARE CONSTANT IN TIME  $\frac{d\langle Q \rangle}{dt} = 0$ )

- LINEAR COMBINATION OF 2 STATIONARY STATES.

$$\underline{\Psi}(x, t) = a \psi_1(x) e^{-\frac{i}{\hbar} E_1 t} + b \psi_2(x) e^{-\frac{i}{\hbar} E_2 t}$$

$a, b, \psi_1, \psi_2$  REAL.

$$|\underline{\Psi}(x, t)|^2 = a^2 \psi_1^2 + b^2 \psi_2^2 + 2ab \psi_1 \psi_2 \cos\left(\frac{E_2 - E_1}{\hbar} t\right)$$

PERIOD OF OSCILLATION  $\tau = \frac{2\pi \hbar}{E_2 - E_1}$

$\Delta E = E_2 - E_1$  (ROUGHLY)

$\Delta t = \tau$  (ROUGHLY)

∴  $\Delta E \Delta t \sim 2\pi \hbar = (4\pi) \frac{\hbar}{2} \gg \frac{\hbar}{2}$

(see PROBLEM 3.18!)

## ⇒ 3.6 DIRAC NOTATION

- STATE  $|\Psi\rangle$

$|x\rangle$  POSITION EIGENSTATE

$$\Psi(x, t) = \langle x | \Psi \rangle$$

↳ W.F. IN COORDINATE SPACE

- $|p\rangle$  MOMENTUM EIGENSTATE

$$\hat{p} |p\rangle = p |p\rangle$$

$$\Phi(p, t) = \langle p | \Psi \rangle$$

↳ W.F. IN MOMENTUM SPACE

- $|m\rangle$  STATIONARY STATE

$$\hat{H} |m\rangle = E_m |m\rangle$$

$$c_m(t) = \langle m | \Psi \rangle$$

- OPERATORS : TRANSFORM ONE VECTOR INTO ANOTHER

$$|\beta\rangle = \hat{Q} |\alpha\rangle$$

BASIS  $|e_n\rangle$  :

$$|\alpha\rangle = \sum_n a_n |e_n\rangle$$

$$|\beta\rangle = \sum_n b_n |e_n\rangle$$

$$Q_{nm} \equiv \langle e_n | \hat{Q} |e_m\rangle$$

# DIRAC'S BRA & KET NOTATION

↳

$$|\alpha\rangle \Leftrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

↓

'KET'

$$\langle\alpha| \Leftrightarrow (a_1^* \dots a_m^*)$$

↓

↳ 'BRA'

$$|\alpha\rangle = \sum_m a_m |e_m\rangle$$

$$\langle\alpha| = \sum_m a_m^* \langle e_m|$$

↳ PROJECTION OPERATOR  $\hat{P} \equiv \underline{\underline{|\alpha\rangle\langle\alpha|}}$

$$\hat{P}|\beta\rangle = \langle\alpha|\beta\rangle |\alpha\rangle$$

↳ IF  $\{|e_m\rangle\}$  IS ORTHONORMAL BASIS (DISCRETE)

$$\langle e_m | e_m \rangle = \delta_{mm}$$

$$\sum_m |e_m\rangle \langle e_m| = 1$$

⇒ COMPLETENESS

$$\begin{aligned}
 & \sum_m |e_m\rangle \underbrace{\langle e_m | \alpha \rangle}_{a_m} \\
 &= \sum_m a_m |e_m\rangle \\
 &= |\alpha\rangle
 \end{aligned}$$

$$\circ \sum_m |e_m\rangle \langle e_m| = 1 \quad \square$$

$\hookrightarrow$  if  $\{|e_z\rangle\}$  is DIRAC ORTHONORMALIZED BASIS

$$\begin{aligned}
 \langle e_z | e_{z'} \rangle &= \delta(z - z') \\
 \int dz |e_z\rangle \langle e_z| &= 1
 \end{aligned}$$

$$|\Phi\rangle = \int dz |e_z\rangle \underbrace{\langle e_z | \Phi \rangle}_{\Phi(z)}$$

$$\underbrace{\langle e_{z'} | \Phi \rangle}_{\Phi(z')} = \int dz \underbrace{\langle e_{z'} | e_z \rangle}_{\delta(z - z')} \Phi(z) \stackrel{!}{=} \Phi(z') \quad \square$$