

2.1

CHAPTER 2 :

TIME - INDEPENDENT SCHRÖDINGER EQUATION

⇒ 2.1 STATIONARY STATES

- SOLVE SCHRÖDINGER EQ.

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

FOR TIME-INDEPENDENT POTENTIAL $V(x,t) = V(x)$

- METHOD OF SEPARATION OF VARIABLES. TO SOLVE PARTIAL DIFFERENTIAL EQ.
TRY SOLUTION OF FORM

$$\underline{\Psi}(x,t) = \psi(x) \cdot \varphi(t)$$

$$\hookrightarrow \frac{\partial^2 \Psi}{\partial x^2} = \frac{d^2 \psi}{dx^2} \cdot \varphi$$

$$\hookrightarrow \frac{\partial \Psi}{\partial t} = \psi \cdot \frac{d\varphi}{dt}$$

↳ SCHRÖDINGER EQ.

$$i\hbar \psi \frac{d\psi}{dt} = -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} \psi + V(x) \psi \psi$$

↓ DIVIDE BY $\psi \cdot \psi$

$$i\hbar \frac{1}{\psi} \frac{d\psi}{dt} = -\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{d^2\psi}{dx^2} + V(x)$$

ONLY
FUNCTION OF t

ONLY
FUNCTION OF x

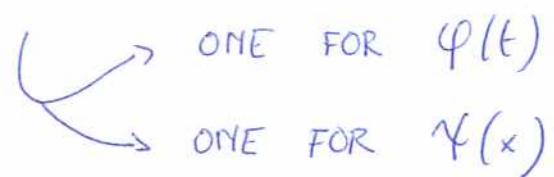
ONLY POSSIBLE IF BOTH ARE CONSTANT

DENOTING CONSTANT BY E

$$\left\{ \begin{array}{l} i\hbar \frac{1}{\psi} \frac{d\psi}{dt} = E \\ -\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{d^2\psi}{dx^2} + V(x) = E \end{array} \right.$$

o o USING METHOD OF SEPARATION OF VARIABLES.

TO TURN PARTIAL DIFFERENTIAL EQUATION INTO 2 ORDINARY DIFFERENTIAL EQUATIONS



↳

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = E \Psi}$$

GENERAL SOLUTION

$$\Psi(t) = C \cdot e^{-\frac{i}{\hbar} E t}$$

↑
CONSTANT

BECAUSE WE WILL NORMALIZE $\Psi = \Psi(x) \cdot \Psi(t)$,
 WE CAN SET (WITHOUT LOSS OF GENERALITY) $C = 1$

$$\Psi(t) = e^{-\frac{i}{\hbar} E t}$$

↳

$\Psi(x)$ SATISFIES

$$\boxed{-\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + V \Psi = E \Psi}$$

TIME INDEPENDENT SCHRÖDINGER EQUATION

↳

GENERAL SOLUTION OF FORM

$$\boxed{\Psi(x, t) = \Psi(x) e^{-\frac{i}{\hbar} E t}}$$

2.4
• SOLUTION $\Psi(x, t) = \psi(x) e^{-\frac{i}{\hbar} Et}$

IS A STATIONARY STATE

↳ PROBABILITY DENSITY IS TIME INDEPENDENT

$$|\Psi(x, t)|^2 = |\psi(x)|^2$$

ALSO EXPECTATION VALUES ARE TIME INDEPENDENT

$$\begin{aligned} \text{e.g. } \langle x \rangle &= \int dx \, x |\Psi(x, t)|^2 \\ &= \int dx \, x |\psi(x)|^2 \end{aligned}$$

$$m \frac{d}{dt} \langle x \rangle = \langle p \rangle = 0$$

↳ AVERAGE MOMENTUM IN STATIONARY STATE = 0.

↳ STATIONARY STATE IS STATE OF DEFINITE TOTAL ENERGY E

HAMILTONIAN: CLASSICAL MECH

$$H(x, p) = \frac{p^2}{2m} + V(x)$$

↓
QUANTUM MECH

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

$$p \rightarrow -i\hbar \frac{\partial}{\partial x}$$

TIME INDEPENDENT SCHRÖDINGER EQ.

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x) \Psi(x) = E \Psi(x)$$

$$\hat{H} \Psi(x) = E \Psi(x)$$

EXPECTATION VALUE OF TOTAL ENERGY

$$\langle H \rangle = \int dx \Psi^*(x) \hat{H} \Psi(x)$$

$$= E \int dx \Psi^* \Psi$$

$$= E \underbrace{\int dx |\Psi|^2}_1$$

$$\underline{\underline{\langle H \rangle = E}}$$

↳ GENERAL SOLUTION : LINEAR COMBINATION OF SEPARABLE SOLUTIONS

$$\Psi_1(x, t) = \psi_1(x) e^{-\frac{i}{\hbar} E_1 t}$$

$$\Psi_2(x, t) = \psi_2(x) e^{-\frac{i}{\hbar} E_2 t}$$

$$\Psi(x, t) = \sum_{n=1}^{\infty} c_n \underbrace{\psi_n(x) e^{-\frac{i}{\hbar} E_n t}}_{\Psi_n(x, t)}$$

↑
EACH SEPARABLE SOLUTION IS A STATIONARY STATE

↳ EXAMPLE : SUM OF 2 STATIONARY STATES

INITIAL STATE ($t=0$) $\Psi(x, 0) = c_1 \psi_1(x) + c_2 \psi_2(x)$

GENERAL STATE $\Psi(x, t) = c_1 \psi_1(x) e^{-\frac{i}{\hbar} E_1 t} + c_2 \psi_2(x) e^{-\frac{i}{\hbar} E_2 t}$

PROBABILITY DENSITY $|\Psi(x, t)|^2 = c_1^2 \psi_1^2 + c_2^2 \psi_2^2 + 2c_1 c_2 \psi_1 \psi_2 \cos \left[(E_1 - E_2) \frac{t}{\hbar} \right]$

↑
OSCILLATION IN TIME OF $|\Psi|^2$ WITH FREQUENCY $(E_1 - E_2)/\hbar$

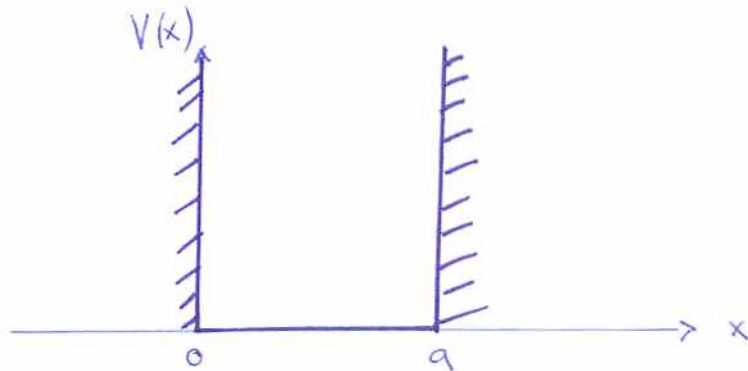
VISUALIZATION:

↳ SEE WEBPHYSICS.DAVIDSON.EDU/MJB/NCS_AAPT_QM_2002/WELCOME.HTML

MAX $|\Psi|_{\text{MAX}}^2 = (c_1 \psi_1 + c_2 \psi_2)^2$
MIN $|\Psi|_{\text{MIN}}^2 = (c_1 \psi_1 - c_2 \psi_2)^2$

⇒ 2.2 INFINITE SQUARE WELL

- $V(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \infty & \text{OTHERWISE} \end{cases}$



- FOR $x < 0$ OR $x > a$: $\psi(x) = 0$

FOR $0 \leq x \leq a$

↳ $\psi(x)$ IS SOLUTION OF $-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi$

↓ $k = \frac{\sqrt{2mE}}{\hbar}$

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

GENERAL SOLUTION :

$$\psi(x) = A \sin kx + B \cos kx.$$

A, B: REAL CONSTANTS.

• A & B DETERMINED BY BOUNDARY CONDITIONS

$$\Psi(0) = \Psi(a) = 0$$

Ψ MUST BE A CONTINUOUS FUNCTION (IS ZERO OUTSIDE THE WELL)

$$\Psi(0) = 0 \Rightarrow B = 0$$

$$\Psi(a) = 0 \Rightarrow A \sin ka = 0$$

⇓

$$ka = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$$

↳ $A = 0$ SOLUTION IS NOT ACCEPTABLE BECAUSE IT IS NOT NORMALIZABLE

↳ SAME APPLIES TO $k = 0$

↳ SOLUTIONS FOR $ka = -m\pi$

ARE EQUIVALENT TO THOSE FOR $ka = +m\pi$

BECAUSE $\sin(-x) = -\sin x$

& MINUS SIGN CAN BE ABSORBED IN A

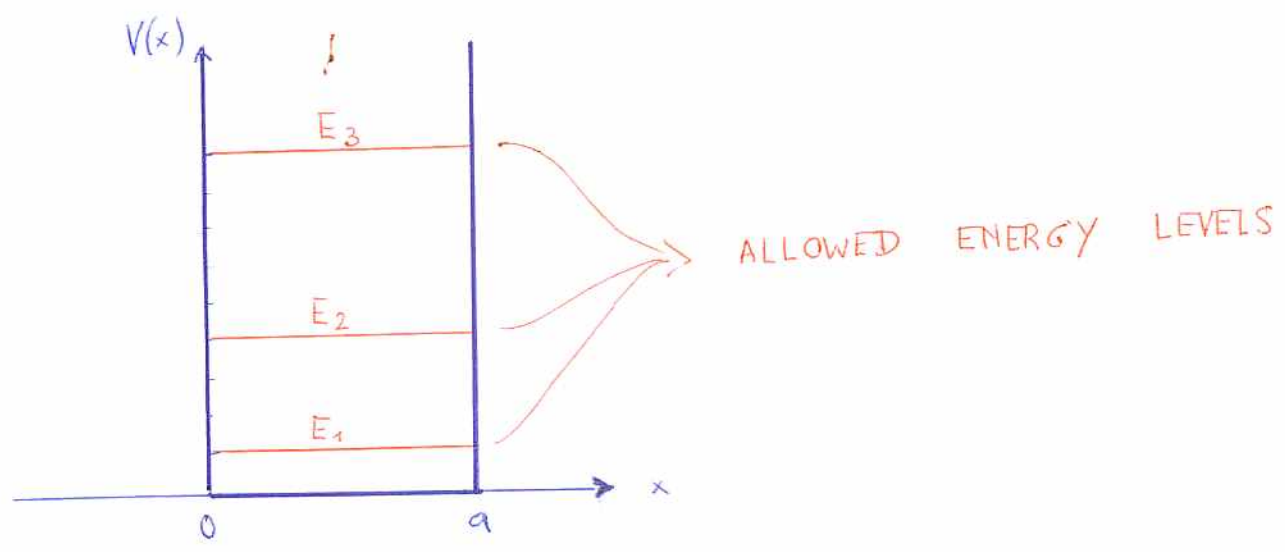
↳ DISTINCT SOLUTIONS:

$$\boxed{k_m = m \frac{\pi}{a}}, \quad \underline{\underline{m = 1, 2, 3, \dots}}$$

↳ BOUNDARY CONDITIONS DETERMINE
ALLOWED VALUES FOR k
" " " E

$$E_m = \frac{\hbar^2 k_m^2}{2m} = m^2 \frac{\hbar^2 \pi^2}{2ma^2} = m^2 E_1$$

ENERGY IS QUANTIZED



• NORMALIZATION

$$\int_{-\infty}^{+\infty} dx |\psi(x)|^2 = 1$$

⇓

$$|A|^2 \int_0^a dx \sin^2 \left(m \frac{\pi}{a} x \right) = 1$$

$$|A|^2 \cdot \frac{a}{2} = 1$$

$$A = \sqrt{\frac{2}{a}}$$

ABSOLUTE
(PHASE HAS NO PHYSICAL SIGNIFICANCE)

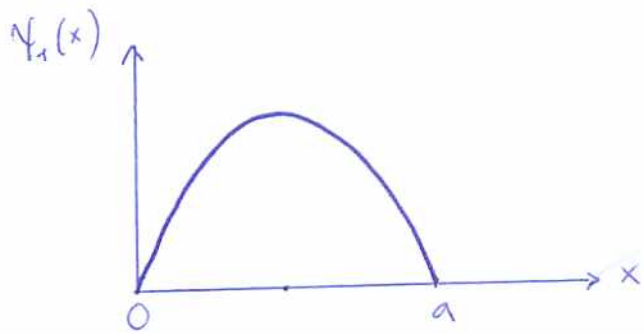
MATH HELP :

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

PROOF : SECOND TERM
GIVES 0 UPON
INTEGRATION

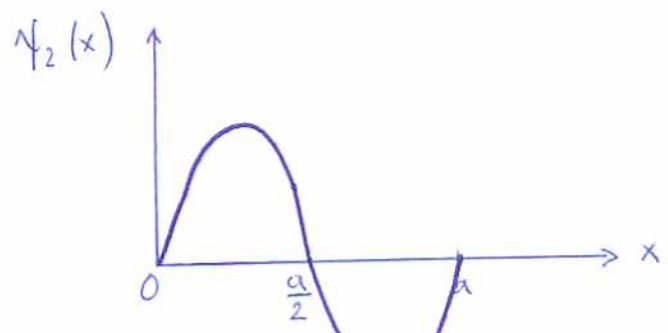
SOLUTIONS

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$



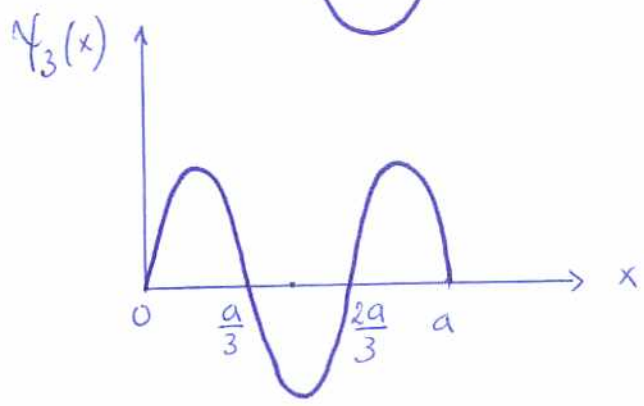
$$\Psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right)$$

GROUND STATE
(STATE OF LOWEST ENERGY)



$$\Psi_2(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi}{a}x\right)$$

1 NODE (ZERO) AT $x = \frac{a}{2}$



$$\Psi_3(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi}{a}x\right)$$

2 NODES AT $x = \frac{a}{3}, \frac{2a}{3}$

Ψ_2, Ψ_3, \dots ARE CALLED EXCITED STATES

SOLUTIONS ARE STANDING WAVES

• PROPERTIES OF SOLUTIONS

↳ EVEN OR ODD w.r.t CENTER OF WELL

$$\begin{aligned}
 m = 1, 3, 5, \dots &\Rightarrow \text{EVEN} \\
 m = 2, 4, 6, \dots &\Rightarrow \text{ODD}
 \end{aligned}
 \quad \left(\begin{array}{l} \text{SYMMETRY} \\ \text{PROPERTY} \end{array} \right)$$

↳ EXCITED STATES HAVE NODES (ZEROS)

$$\Psi_m \text{ HAS } m - 1 \text{ NODES.}$$

THE HIGHER THE ENERGY, THE MORE NODES

↳ SOLUTIONS ARE ORTHOGONAL TO EACH OTHER

$\int dx \Psi_m^*(x) \cdot \Psi_m(x) = 0$	$n \neq m$
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PROOF: $\int dx \Psi_m^*(x) \Psi_m(x)$

$$= \frac{2}{a} \int_0^a dx \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}x\right)$$

MATH HELP
 $\sin x \cdot \sin y$

$$= -\frac{1}{2} [\cos(x+y) - \cos(x-y)]$$

$$= -\frac{1}{a} \int_0^a dx \left[\cos\left(\frac{(m+n)\pi}{a}x\right) - \cos\left(\frac{(m-n)\pi}{a}x\right) \right]$$

↳ SOLUTIONS ARE COMPLETE

i.e. ANY OTHER FUNCTION CAN BE WRITTEN AS A LINEAR COMBINATION OF THEM

$$f(x) = \sum_{n=1}^{\infty} c_n \Psi_n(x)$$

$$f(x) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a} x\right)$$

(FOURIER SERIES)

COEFFICIENTS c_n DETERMINED FROM ORTHONORMALITY:

$$\int dx \Psi_m^*(x) f(x) = \sum_{n=1}^{\infty} c_n \underbrace{\int dx \Psi_m^*(x) \Psi_n(x)}_{\delta_{nm}} = c_m$$

$$c_n = \int dx \Psi_n^*(x) f(x)$$

$$= -\frac{1}{a} \left[\frac{a}{\pi(m+n)} \sin\left(\frac{(m+n)\pi}{a} x\right) - \frac{a}{\pi(m-n)} \sin\left(\frac{(m-n)\pi}{a} x\right) \right]_0^a$$

↳ ONLY VALID FOR $m \neq n$!

$$= -\frac{1}{\pi} \left\{ \sin((m+n)\pi) - \sin((m-n)\pi) \right\}$$

$$\stackrel{!}{=} 0$$

↳ FOR $m = n$: NORMALIZATION CONDITION

◦◦◦ COMBINE ORTHOGONALITY & NORMALIZATION

INTO :

$$\int dx \psi_m^*(x) \psi_n(x) = \delta_{nm}$$

$$\delta_{nm} = \begin{cases} 1 & , n = m \\ 0 & , n \neq m \end{cases}$$

ψ_m 's ARE CALLED ORTHONORMAL

↳ STATIONARY STATE

$$\Psi_m(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar} E_m t}$$

WITH $E_m = \frac{\hbar^2}{2m} \left(\frac{n\pi}{a}\right)^2$

↳ MOST GENERAL SOLUTION OF TIME DEPENDENT SCHRÖDINGER EQ. IN INFINITE SQUARE WELL POTENTIAL:

$$\Psi(x,t) = \sum_{n=1}^{\infty} C_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar} E_n t}$$

IF WE KNOW THE INITIAL $\Psi(x, t=0)$

WE CAN DETERMINE C_n



WE ALSO KNOW SOLUTION AT ALL TIMES t

$$C_n = \int dx \psi_n^*(x) \Psi(x, 0)$$

$$= \sqrt{\frac{2}{a}} \int_0^a dx \sin\left(\frac{n\pi}{a}x\right) \Psi(x, 0)$$

◦ KNOWING $\Psi(x, t) \Rightarrow$ CALCULATE ALL OBSERVABLES

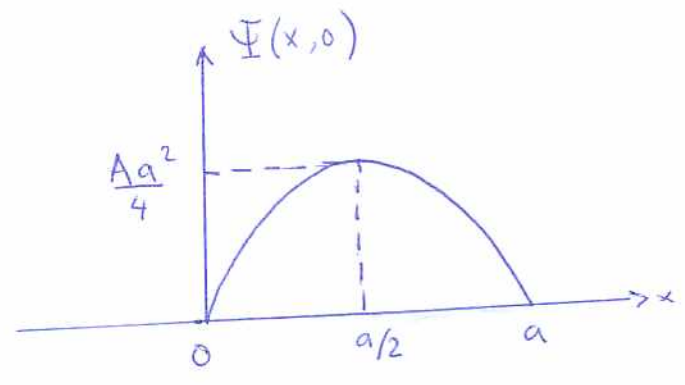
$\langle x \rangle, \langle p \rangle, \dots$

$\sigma_x, \sigma_y, \dots$

● EXAMPLE

PARTICLE IN INFINITE SQUARE WELL

INITIAL W.F. $\Psi(x, 0) = A x (a - x)$, $0 \leq x \leq a$
= 0 , ELSEWHERE



DETERMINE W.F. AT ANY TIME

↳ NORMALIZATION

$$\begin{aligned}
 1 &= \int dx |\Psi(x, 0)|^2 \\
 &= A^2 \int_0^a dx x^2 (a-x)^2 = A^2 \int_0^a dx x^2 (a^2 - 2ax + x^2) \\
 &= A^2 \left(\frac{a^5}{3} - \frac{2a^5}{4} + \frac{a^5}{5} \right) \\
 &= A^2 \frac{a^5}{30}
 \end{aligned}$$

$$A = \sqrt{\frac{30}{a^5}}$$

↳

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar} E_n t}$$

$$c_n = \sqrt{\frac{2}{a}} \int_0^a dx \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)$$

$$c_n = \sqrt{\frac{2}{a}} A \int_0^a dx \sin\left(\frac{n\pi}{a}x\right) x(a-x)$$

\uparrow
 $\sqrt{\frac{30}{a^5}}$

$$= \frac{2\sqrt{15}}{a^3} \left\{ a \int_0^a dx x \sin\left(\frac{n\pi}{a}x\right) - \int_0^a dx x^2 \sin\left(\frac{n\pi}{a}x\right) \right\}$$

$$= \frac{2\sqrt{15}}{a^3} \left\{ -\frac{a^2}{n\pi} x \cos\left(\frac{n\pi}{a}x\right) \Big|_0^a + \frac{a^2}{n\pi} \int_0^a dx \cos\left(\frac{n\pi}{a}x\right) \right\}$$

$$+ \frac{a}{n\pi} x^2 \cos\left(\frac{n\pi}{a}x\right) \Big|_0^a - \frac{2a}{n\pi} \int_0^a dx x \cos\left(\frac{n\pi}{a}x\right) \Big\}$$

$$= \frac{2\sqrt{15}}{a^3} \left\{ -\frac{a^3}{n\pi} \cos(n\pi) + \frac{a^3}{n\pi} \cos(n\pi) \right\}$$

$$- \frac{2a^2}{(n\pi)^2} x \sin\frac{n\pi}{a}x \Big|_0^a + \frac{2a^2}{(n\pi)^2} \int_0^a dx \sin\left(\frac{n\pi}{a}x\right)$$

$$c_n = \frac{2\sqrt{15}}{a^3} \cdot (-1)^n \frac{2a^2}{(n\pi)^2} \cdot \frac{a}{n\pi} \cos\left(\frac{n\pi}{a}x\right) \Big|_0^a \quad \leftarrow 2.17$$

$$c_n = \frac{4\sqrt{15}}{(n\pi)^3} [1 - \cos(n\pi)]$$

$$c_n = \begin{cases} 0 & , \quad n \text{ EVEN} \\ \frac{8\sqrt{15}}{(n\pi)^3} & , \quad n \text{ ODD} \end{cases}$$

$$\Psi(x,t) = \sqrt{\frac{30}{a}} \left(\frac{2}{\pi}\right)^3 \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar} E_n t}$$

$$E_n = \frac{\hbar^2}{2m} \left(\frac{n\pi}{a}\right)^2$$

c_n : OVERLAP OF WAVEFUNCTION WITH n -th STATIONARY STATE

LARGEST OVERLAP WITH c_1 $|c_1|^2 = \left(\frac{8\sqrt{15}}{\pi^3}\right)^2$

$$c_n \sim \frac{1}{n^3}$$

$$= 0.99855\dots$$

$$(99.9\%)$$

↳ $|c_m|^2$ PROBABILITY TO FIND Ψ IN STATE m

$$\sum_{n=1}^{\infty} |c_n|^2 = 1$$

PROOF: $1 = \int dx |\Psi(x, 0)|^2$

$$= \int dx \left(\sum_{m=1}^{\infty} c_m \psi_m(x) \right)^* \left(\sum_{n=1}^{\infty} c_n \psi_n(x) \right)$$

$$= \sum_n \sum_m c_m c_m^* \int dx \underbrace{\psi_m^*(x) \psi_n(x)}_{\delta_{nm} \text{ ORTHONORMAL}}$$

$$= \sum_{n=1}^{\infty} |c_n|^2$$

■ QED

↳ EXPECTATION VALUE OF ENERGY

$$\langle H \rangle = \int dx \Psi \hat{H} \Psi \quad \hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

$$= \sum_n \sum_m c_m^* c_n \int dx \psi_m^*(x) \hat{H} \psi_n(x)$$

$$= \sum_n \sum_m c_m^* c_n E_n \int dx \psi_m^*(x) \psi_n(x) \quad E_n \psi_n(x)$$

$$\langle H \rangle = \sum_{n=1}^{\infty} E_n |c_n|^2$$

• CHECK OF UNCERTAINTY PRINCIPLE FOR INFINITE SQUARE WELL 2.17

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2$$

FOR STATIONARY STATE $\Psi_m(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$.

$$\hookrightarrow \langle x \rangle = \int_0^a dx \Psi_m^*(x) x \Psi_m(x)$$

$$= \frac{2}{a} \int_0^a dx x \sin^2\left(\frac{n\pi}{a}x\right)$$

$$\frac{1}{2} \left(1 - \cos\left(\frac{2n\pi}{a}x\right)\right)$$

$$= \frac{2}{a} \int_0^a dx \frac{x}{2}$$

$$\underline{\underline{\langle x \rangle = \frac{1}{2} a}} \quad \text{INDEPENDENT OF } n$$

$$\hookrightarrow \langle p \rangle = m \frac{d}{dt} \langle x \rangle = 0$$

$$\text{CHECK } = m \int_0^a dx \Psi_m^*(x) \left(-i\hbar \frac{d}{dx}\right) \Psi_m(x)$$

$$= -i\hbar m \frac{2}{a} \cdot \frac{2n\pi}{a} \int_0^a dx \sin\left(\frac{n\pi}{a}x\right) \cdot \cos\left(\frac{n\pi}{a}x\right)$$

$$\underline{\underline{\langle p \rangle = 0}}$$

$$\begin{aligned}
 \hookrightarrow \langle x^2 \rangle &= \frac{2}{a} \int_0^a dx \, x^2 \underbrace{\sin^2 \left(\frac{n\pi}{a} x \right)}_{\frac{1}{2} \left(1 - \cos \left(\frac{2n\pi}{a} x \right) \right)} \\
 &= \frac{1}{a} \left\{ \frac{a^3}{3} - \frac{a}{2n\pi} x^2 \sin \left(\frac{2n\pi}{a} x \right) \Big|_0^a \right. \\
 &\quad \left. + \frac{a}{2n\pi} \cdot 2 \int_0^a dx \, x \sin \left(\frac{2n\pi}{a} x \right) \right\} \\
 &= \frac{1}{a} \left\{ \frac{a^3}{3} - \frac{a}{n\pi} \cdot \frac{a}{2n\pi} x \cos \left(\frac{2n\pi}{a} x \right) \Big|_0^a \right. \\
 &\quad \left. + \frac{a}{n\pi} \cdot \frac{a}{2n\pi} \int_0^a dx \, \cos \left(\frac{2n\pi}{a} x \right) \right\} \\
 &= \frac{1}{a} \left\{ \frac{a^3}{3} - \frac{a^3}{2(n\pi)^2} \cdot \underbrace{\cos(2n\pi)}_1 \right\}
 \end{aligned}$$

$$\langle x^2 \rangle = a^2 \left[\frac{1}{3} - \frac{1}{2(n\pi)^2} \right]$$

$$\hookrightarrow \langle p^2 \rangle = \int dx \psi_m^*(x) \left(-\hbar^2 \frac{d^2}{dx^2} \right) \psi_m(x)$$

$$= \int_0^a dx \psi_m^*(x) (2m E_n) \psi_m(x)$$

$$= 2m E_n$$

$$\langle p^2 \rangle = \hbar^2 \left(\frac{n\pi}{a} \right)^2$$

$$\hookrightarrow \sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \frac{a^2}{4} \left[\frac{1}{3} - \frac{2}{(n\pi)^2} \right]$$

$$\sigma_x = \frac{a}{2} \sqrt{\frac{1}{3} - \frac{2}{(n\pi)^2}}$$

$$\hookrightarrow \sigma_y^2 = \langle p^2 \rangle - \langle p \rangle^2$$

$$\sigma_y = \hbar \frac{n\pi}{a}$$

$$\hookrightarrow \sigma_x \cdot \sigma_p = \frac{\hbar}{2} \cdot \sqrt{\frac{(n\pi)^2}{3} - 2} \geq \frac{\hbar}{2}$$

SMALLEST FOR $n=1$: $\sqrt{\frac{\pi^2}{3} - 2} = 1.136$

⇒ 2.3 HARMONIC OSCILLATOR

• CLASSICAL H.O.



$$m \frac{d^2 x}{dt^2} = -kx$$

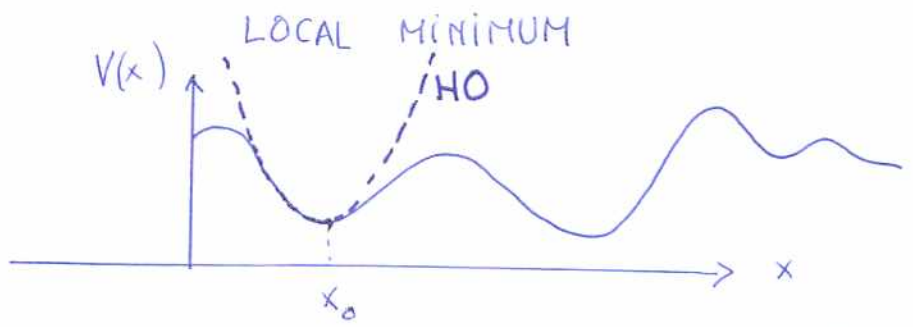
↳ SOLUTION: $x(t) = A \sin(\omega t) + B \cos(\omega t)$

$\omega = \sqrt{\frac{k}{m}}$ ANGULAR FREQUENCY

↳ POTENTIAL ENERGY $F = -\frac{dV}{dx}$

$V(x) = \frac{1}{2} k x^2$ PERFECT H.O.

↳ IN REALITY: H.O. IS GOOD APPROX. AROUND



FOR x AROUND x_0 : $V(x) = V(x_0) + \cancel{V'(x_0)}(x-x_0) + V''(x_0) \frac{1}{2}(x-x_0)^2 + \dots$

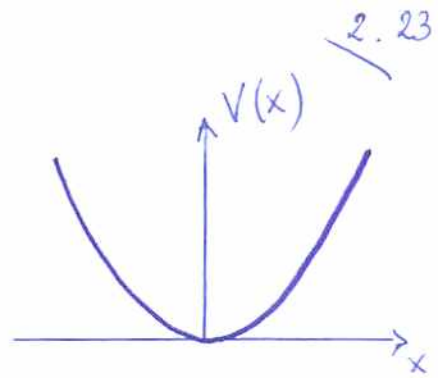
TAYLOR SERIES

LOCAL MIN.

H.O. $V(x) - V(x_0) \approx \frac{1}{2} V''(x_0) (x-x_0)^2 \rightarrow k \equiv V''(x_0)$

QUANTUM H.O.

$$V(x) = \frac{1}{2} m \omega^2 x^2$$



TIME INDEPENDENT SCHRÖDINGER EQ.

$$-\frac{\hbar^2}{2m} \frac{d^2 \Psi(x)}{dx^2} + \frac{1}{2} m \omega^2 x^2 \Psi(x) = E \Psi(x)$$

2 METHODS TO SOLVE \rightarrow ALGEBRAIC METHOD
 \rightarrow POWER SERIES METHOD

① ALGEBRAIC METHOD TO QUANTUM H.O.

$$\hookrightarrow \langle P \rangle = \int dx \Psi^*(x,t) \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi(x,t)$$

$$= \int dx \Psi^*(x) \left(-i\hbar \frac{d}{dx} \right) \Psi(x)$$

\hat{P} : MOMENTUM OPERATOR

SCHRÖDINGER EQ.

$$\frac{\hat{P}^2}{2m} \Psi + \frac{1}{2} m \omega^2 x^2 \Psi = E \Psi$$

$$\hat{H} \Psi = E \Psi$$

\hookrightarrow HAMILTONIAN

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

↳ IDEA TRY TO FACTOR \hat{H}

CONSIDER $(i\hat{p} + m\omega x) \cdot (-i\hat{p} + m\omega x)$

$$= \hat{p}^2 + im\omega (\hat{p}x - x\hat{p}) + m^2\omega^2 x^2$$

$$= 2m\hat{H} + \text{EXTRA TERM } im\omega (\hat{p}x - x\hat{p})$$

ATTENTION \hat{p} AND x DO NOT COMMUTE

COMMUTATOR $[x, \hat{p}] \equiv x\hat{p} - \hat{p}x$

$$[x, \hat{p}]f(x) = \left(-i\hbar x \frac{d}{dx} + i\hbar \frac{d}{dx} x \right) f(x)$$

$$= -i\hbar x \frac{df}{dx} + i\hbar f + i\hbar x \frac{df}{dx}$$

$$= i\hbar f(x)$$

$$[x, \hat{p}] = i\hbar$$

CANONICAL COMMUTATION RELATION

↳ DEFINE OPERATORS WHICH CORRESPOND WITH THE ABOVE FACTORS

c.e.

$$a_+ \equiv \frac{1}{\sqrt{\hbar\omega 2m}} (-i\hat{p} + m\omega x)$$

$$a_- \equiv \frac{1}{\sqrt{\hbar\omega 2m}} (+i\hat{p} + m\omega x)$$

↑
NORMALIZATION FACTOR
TO GIVE SIMPLE PHYSICAL INTERPRETATION
TO a_+ & a_- (SEE FURTHER ON)

↳ $a_- a_+ = \frac{1}{\hbar\omega 2m} \{ 2m \hat{H} + im\omega (-i\hbar) \}$

⇓

$$\hat{H} = \hbar\omega \left(a_- a_+ - \frac{1}{2} \right)$$

↳ ANALOGOUSLY WE CAN CALCULATE $a_+ a_-$

$$a_+ a_- = \frac{1}{\hbar\omega 2m} (-i\hat{p} + m\omega x)(+i\hat{p} + m\omega x)$$

$$= \frac{1}{\hbar\omega 2m} \left(2m \hat{H} + im\omega \underbrace{[x, \hat{p}]}_{i\hbar} \right)$$

$$\hat{H} = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right)$$

$$a_- a_+ - \frac{1}{2} = a_+ a_- + \frac{1}{2}$$

↓

$$\boxed{[a_-, a_+] = 1.}$$

↳ SCHRÖDINGER EQ.

$$\hat{H}\psi = E\psi$$

$$\boxed{\begin{aligned} \hat{H} &= \hbar\omega (a_+ a_- + \frac{1}{2}) \\ &= \hbar\omega (a_- a_+ - \frac{1}{2}) \end{aligned}}$$

$$\hbar\omega (a_+ a_- + \frac{1}{2}) \psi = E\psi$$

OR EQUIVALENTLY

$$\hbar\omega (a_- a_+ - \frac{1}{2}) \psi = E\psi$$

CRUCIAL STEP

IF ψ IS SOLUTION OF SCHRÖDINGER EQ. WITH ENERGY E

↓

$a_+ \psi$ IS SOLUTION WITH ENERGY $E + \hbar\omega$

$a_- \psi$ IS SOLUTION WITH ENERGY $E - \hbar\omega$

PROOF $\hat{H}\psi = E\psi$

$$\textcircled{1} \quad \hat{H}(a_+ \psi) = \hbar\omega (a_+ a_- + \frac{1}{2}) (a_+ \psi)$$

$$= \hbar\omega (a_+ a_- a_+ + \frac{1}{2} a_+) \psi$$

~~~~~

$$\downarrow a_- a_+ = 1 + a_+ a_-$$

$$\begin{aligned}
 \hat{H} (a_+ \psi) &= \hbar\omega (a_+ + a_+ a_+ a_- + \frac{1}{2} a_+) \psi \quad \text{2.27} \\
 &= a_+ \hbar\omega (a_+ a_- + \frac{1}{2} + 1) \psi \\
 &\quad \downarrow \quad \hat{H} = \hbar\omega (a_+ a_- + \frac{1}{2}) \\
 &= a_+ (\hat{H} + \hbar\omega) \psi \\
 &= a_+ (E + \hbar\omega) \psi \\
 &= (E + \hbar\omega) (a_+ \psi).
 \end{aligned}$$

∴  $(a_+ \psi)$  is SOLUTION WITH ENERGY  $E + \hbar\omega$

$$\begin{aligned}
 \textcircled{2} \quad \hat{H} (a_- \psi) &= \hbar\omega (a_- a_+ - \frac{1}{2}) (a_- \psi) \\
 &= \hbar\omega (a_- a_+ a_- - \frac{1}{2} a_-) \psi \\
 &= a_- \underbrace{\hbar\omega (a_+ a_- - \frac{1}{2})}_{\hat{H} - \hbar\omega} \psi \\
 &= a_- (\hat{H} - \hbar\omega) \psi \\
 &= a_- (E - \hbar\omega) \psi \\
 &= (E - \hbar\omega) (a_- \psi)
 \end{aligned}$$

∴  $(a_- \psi)$  is SOLUTION WITH ENERGY  $E - \hbar\omega$

↳ BY APPLYING  $a_{\pm}$  OPERATOR ON

A GIVEN SOLUTION WITH ENERGY  $E$

ONE GENERATES A NEW SOLUTION WITH ENERGY  $E \pm \hbar\omega$

$$a_+ \leftrightarrow E + \hbar\omega$$

$$a_- \leftrightarrow E - \hbar\omega$$

$a_+$  CALLED 'RAISING' OPERATOR

↳ RAISES ENERGY BY 'QUANTUM'  $\hbar\omega$

$a_-$  CALLED 'LOWERING' OPERATOR

↳ LOWERS ENERGY BY 'QUANTUM'  $\hbar\omega$

- ONE CAN APPLY  $a_+$  SUCCESSIVELY TO GET STATES OF HIGHER ENERGY

e.g.  $\hat{H} (a_+^2 \psi) = (E + 2\hbar\omega) (a_+^2 \psi)$

⋮

$$\hat{H} (a_+^n \psi) = (E + n\hbar\omega) (a_+^n \psi)$$

- ONE CAN APPLY  $a_-$  SUCCESSIVELY TO GET STATES OF LOWER ENERGY

e.g.  $\hat{H} (a_-^2 \psi) = (E - 2\hbar\omega) (a_-^2 \psi)$

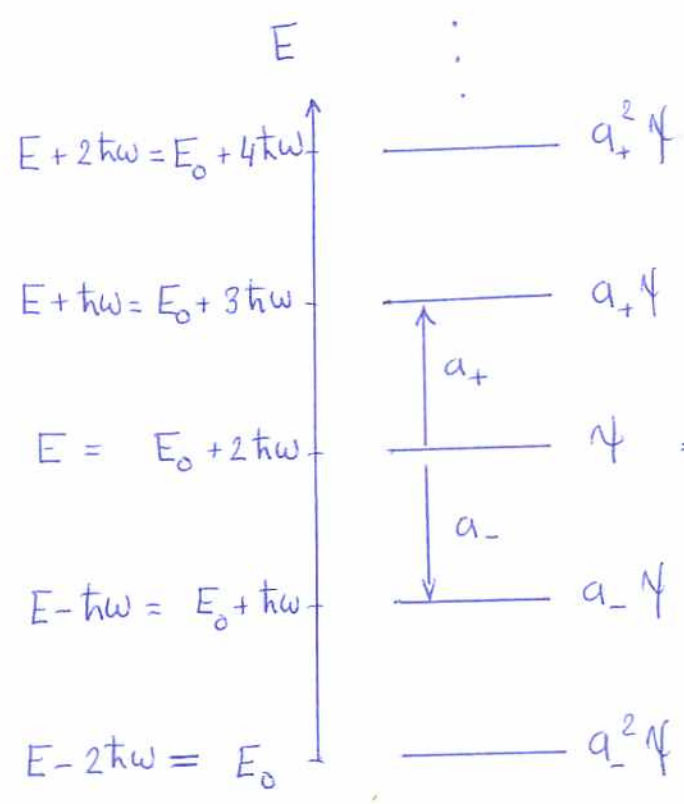
⋮

BUT PAY ATTENTION !

BY REPEATED APPLICATION OF  $a_-$  ONE WOULD EVENTUALLY REACH A STATE WITH NEGATIVE ENERGY

↓  
UNPHYSICAL

∴ THERE MUST BE A STATE OF LOWEST ENERGY  $E_0$   
↳ GROUND STATE



⇒ LET'S SAY WE KNOW THIS SOLUTION

$\underbrace{a_-^2 \psi}_{\psi_0}$   
GROUND STATE  
= STATE WITH LOWEST ENERGY

FOR GROUND STATE

$a_- \psi_0 = 0$

i.e. ONE CANNOT GET A STATE OF LOWER ENERGY THAN  $\psi_0$

$\hat{H} (a_- \psi_0) = 0$

NOT NORMALIZABLE SOLUTION OF SCHRÖDINGER EQ.

↳ USE CONDITION  $a_- \Psi_0 = 0$  TO DETERMINE  
GROUND STATE  $\Psi_0$

$$a_- \Psi_0 = 0$$

↑

$$\frac{1}{\sqrt{\hbar m \omega}} (i \hat{p} + m \omega x) \Psi_0 = 0$$

$$\left( \hbar \frac{d}{dx} + m \omega x \right) \Psi_0 = 0$$

$$\frac{d\Psi_0}{dx} = - \frac{m\omega}{\hbar} x \Psi_0$$

SOLUTION:  $\Psi_0(x) = A e^{-\frac{m\omega}{2\hbar} x^2}$

USE NORMALIZATION TO DETERMINE A :

$$1 = \int_{-\infty}^{+\infty} dx |\Psi_0(x)|^2 = |A|^2 \int_{-\infty}^{+\infty} dx e^{-\frac{m\omega}{\hbar} x^2}$$

GAUSSIAN (MATH HELP)  
INTEGRAL:  
 $\int_{-\infty}^{+\infty} dx e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$

$$= |A|^2 \cdot \sqrt{\frac{\pi \hbar}{m\omega}}$$

$$A = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4}$$

GROUND STATE OF QUANTUM H.O.

$$\Psi_0(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

↳ GROUND STATE ENERGY  $E_0$

$$E_0 = \langle \hat{H} \rangle = \int_{-\infty}^{+\infty} dx \Psi_0^*(x) \hat{H} \Psi_0(x)$$

$$\downarrow$$

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2$$

EVALUATE ....

BUT THERE IS A SHORTER WAY TO DETERMINE  $E_0$

$$\hat{H} = \hbar\omega \left( a_+ a_- + \frac{1}{2} \right)$$

$$\hat{H} \Psi_0 = E_0 \Psi_0 = \hbar\omega \left( a_+ a_- + \frac{1}{2} \right) \Psi_0$$

$$\downarrow \quad \underline{a_- \Psi_0 = 0}$$

THIS DEFINES THE GROUND STATE  $\Psi_0$ .

$$= \frac{1}{2} \hbar\omega \Psi_0$$

$$E_0 = \frac{1}{2} \hbar\omega$$

GROUND STATE ENERGY OF QUANTUM H.O.

IS NOT ZERO ! (IN CLASSICAL LIMIT  $\hbar \rightarrow 0$   
 $E_0 \rightarrow 0$ )

↳ ANY OTHER SOLUTION

BY REPEATED APPLICATION OF  $a_+$  ON  $\psi_0$ .

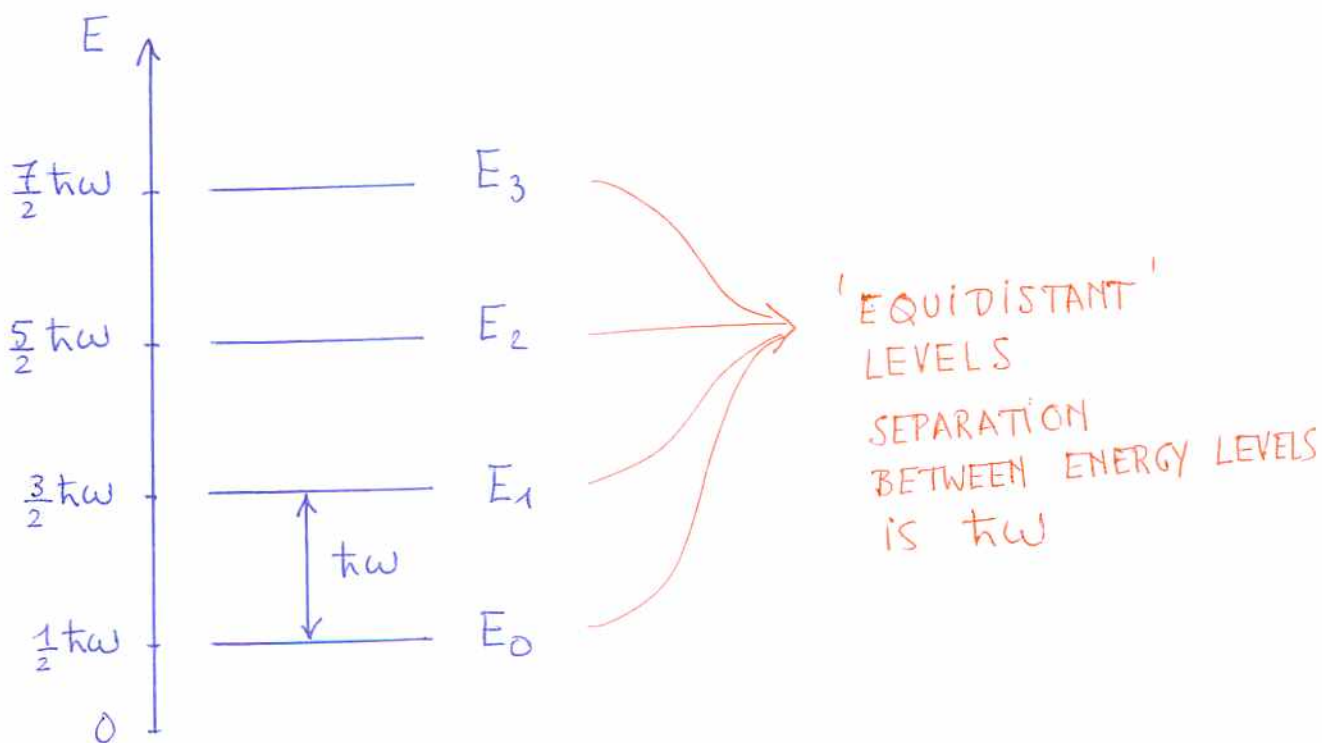
$$\psi_m(x) = A_m a_+^m \psi_0$$

↑  
NORMALIZATION  
CONSTANT

$\psi_m$  HAS ENERGY  $E_m = E_0 + m \hbar \omega$

$$E_m = \hbar \omega \left( \frac{1}{2} + m \right)$$

SPECTRUM OF QUANTUM H.O.





e.g. FIRST EXCITED STATE

$$\Psi_1(x) = A_1 a_+ \Psi_0(x)$$

$$= A_1 \frac{1}{\sqrt{\hbar \omega 2m}} \left( -i \hat{p} + m \omega x \right) \underbrace{\Psi_0(x)}^{\left( \frac{m \omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m \omega}{2 \hbar} x^2}}$$

$$= A_1 \frac{1}{\sqrt{\hbar \omega 2m}} \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \left( -\hbar \frac{d}{dx} + m \omega x \right) e^{-\frac{m \omega}{2 \hbar} x^2}$$

$$= A_1 \frac{1}{\sqrt{\hbar \omega 2m}} \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} 2 m \omega x e^{-\frac{m \omega}{2 \hbar} x^2}$$

$$\boxed{\Psi_1(x) = A_1 \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \sqrt{\frac{2 m \omega}{\hbar}} x e^{-\frac{m \omega}{2 \hbar} x^2}}$$

NORMALIZATION  $1 = \int_{-\infty}^{+\infty} dx |\Psi_1(x)|^2$

$$= |A_1|^2 \cdot \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \cdot \frac{2 m \omega}{\hbar} \int_{-\infty}^{+\infty} dx x^2 e^{-\frac{m \omega}{\hbar} x^2}$$

↓

MATH HELP:  $\int_{-\infty}^{+\infty} dx x^2 e^{-\lambda x^2} = \frac{1}{2} \sqrt{\frac{\pi}{\lambda}}$

$$= |A_1|^2 \cdot \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \cdot \frac{2 m \omega}{\hbar} \cdot \frac{1}{2} \left( \frac{\pi \hbar}{m \omega} \right)^{1/2} \frac{\hbar}{m \omega}$$

$$= |A_1|^2$$

$$A_1 = 1$$



2.50

↓ INTEGRATION BY PARTS.

$$\begin{aligned} |c_n|^2 &= \int_{-\infty}^{+\infty} dx \psi_n^* \frac{1}{\sqrt{\hbar\omega 2m}} \left( +\frac{\hbar}{i} \frac{d}{dx} + m\omega x \right) a_+ \psi_n \\ &= \int_{-\infty}^{+\infty} dx \psi_n^* a_- a_+ \psi_n \\ &= (n+1) \end{aligned}$$

∴  $a_+ \psi_n = \sqrt{n+1} \psi_{n+1}$

ANALOGOUSLY  $|d_n|^2 = n$

$a_- \psi_n = \sqrt{n} \psi_{n-1}$

⇒  $\psi_1 = a_+ \psi_0$

$$\psi_2 = \frac{1}{\sqrt{2}} a_+ \psi_1 = \frac{1}{\sqrt{2}} a_+^2 \psi_0$$

$$\psi_3 = \frac{1}{\sqrt{3}} a_+ \psi_2 = \frac{1}{\sqrt{3 \cdot 2}} a_+^3 \psi_0 = \frac{1}{\sqrt{3!}} a_+^3 \psi_0$$

⋮

$$\psi_n = \frac{1}{\sqrt{n}} a_+ \psi_{n-1} = \frac{1}{\sqrt{n(n-1)}} a_+^2 \psi_{n-2} = \frac{1}{\sqrt{n!}} a_+^n \psi_0$$

↑

$$A_n = \frac{1}{\sqrt{n!}} \quad \square \quad \text{QED}$$

↳ SOLUTIONS OF H.O. ARE ORTHONORMAL

$$\int_{-\infty}^{+\infty} dx \Psi_m^*(x) \Psi_n(x) = \delta_{mn}$$

PROOF

$$\int_{-\infty}^{+\infty} dx \Psi_m^*(x) a_+ a_- \Psi_m(x)$$

BY  
PARTIAL  
INTEGRATION  
TWICE

$$= m \int_{-\infty}^{+\infty} dx \Psi_m^*(x) \Psi_m(x)$$

$$= \int_{-\infty}^{+\infty} dx \underbrace{(a_+ a_- \Psi_m)^*}_{m \Psi_m^*} \Psi_m$$

$$= m \int_{-\infty}^{+\infty} dx \Psi_m^* \Psi_m$$

FOR  $n \neq m \Rightarrow \int_{-\infty}^{+\infty} dx \Psi_m^* \Psi_n = 0.$

$m = m \Rightarrow \int_{-\infty}^{+\infty} dx |\Psi_m|^2 = 1$  NORMALIZATION

□ QED

↳

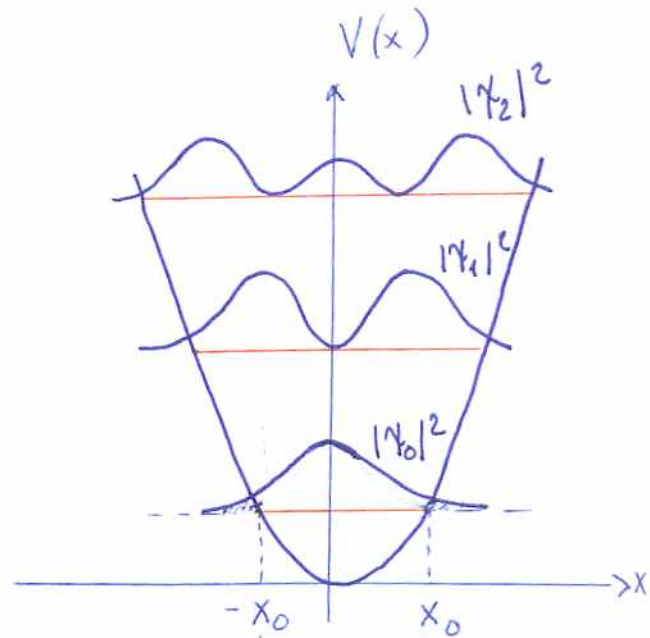
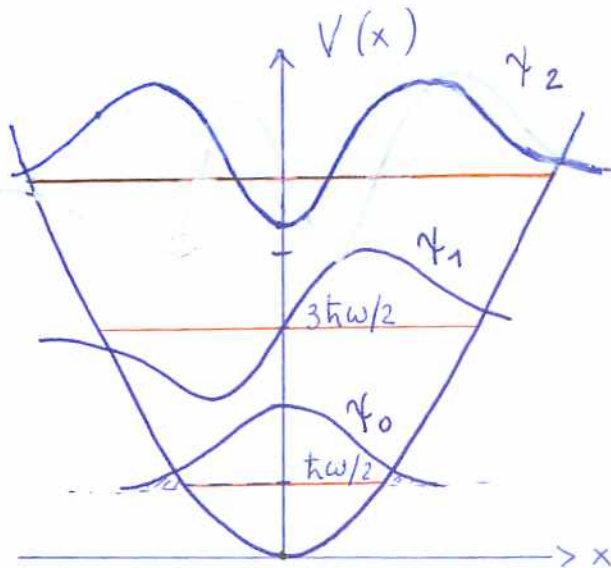
PLOT OF LOWEST FEW WAVE FUNCTIONS.

$$\psi_0(x) \sim e^{-\frac{m\omega}{2\hbar} x^2} \quad \text{EVEN IN } x$$

$$|\psi_0|^2 \sim e^{-\frac{m\omega}{\hbar} x^2}$$

$$\psi_1 \sim x e^{-\frac{m\omega}{2\hbar} x^2} \quad \text{ODD IN } x$$

$$|\psi_1|^2 \sim x^2 e^{-\frac{m\omega}{\hbar} x^2}$$



CLASSICAL  
TURNING POINTS

$$\begin{aligned} V(x_0) &= \frac{1}{2} m \omega^2 x_0^2 \\ &= E_0 = \frac{1}{2} \hbar \omega \end{aligned}$$

$$\Downarrow$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

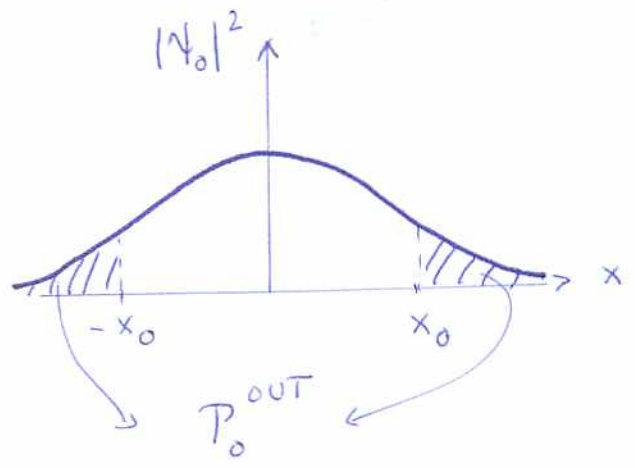
- FOR ENERGY LEVEL  $E_m$   
THE CLASSICAL TURNING POINTS  $x_m$  ARE

$$V(x_m) = \frac{1}{2} m \omega^2 x_m^2 = E_m$$

$$x_m = \sqrt{\frac{2 E_m}{m \omega^2}}$$

- IN Q.M. THERE IS A NON-ZERO PROBABILITY TO FIND PARTICLE OUTSIDE THE CLASSICALLY ALLOWED REGION

FOR  $\psi_0$  :



$$P_0^{OUT} = 2 \int_{x_0}^{\infty} dx |\psi_0(x)|^2$$

$$= 2 \cdot \left(\frac{m \omega}{\pi \hbar}\right)^{1/2} \int_{x_0}^{\infty} dx e^{-\frac{m \omega}{\hbar} x^2}$$

↓ DIMENSIONLESS VAR  $\xi = x \left(\frac{\hbar}{m \omega}\right)^{1/2}$

$$= \frac{2}{\sqrt{\pi}} \int_1^{\infty} d\xi e^{-\xi^2} = 0.1573 \quad \text{15.7\%} \quad \nabla$$

↳ CLASSICAL ↔ QUANTUM PROBABILITY DENSITIES  
FOR H.O.


• CLASSICAL



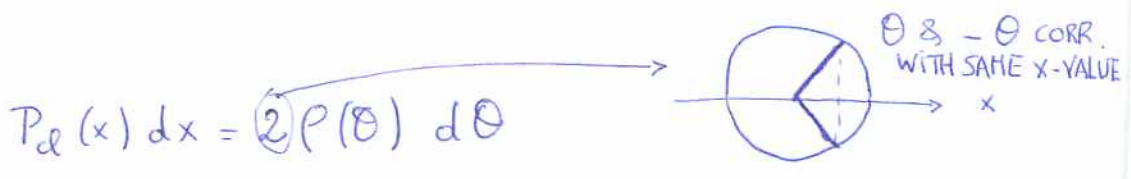
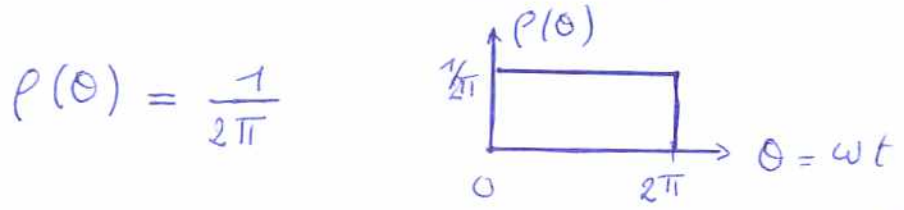
$$x(t) = A \cos(\omega t) \qquad \omega = \sqrt{\frac{k}{m}}$$

↓  
PERIOD  $T = \frac{2\pi}{\omega}$

FOR  $t \gg T$  : H.O. GOES BACK & FORTH  
MANY, MANY TIMES

$P_{cl}(x) dx$  : PROBABILITY TO FIND PARTICLE (  IN FIGURE )  
BETWEEN  $x$  &  $x + dx$

↳ FOR  $t \gg T$  : EACH VALUE OF  $\theta = \omega t$   
WILL HAVE SAME PROBABILITY



$$\left| \frac{dx}{d\theta} \right| = A \sin(\omega t) = A \sqrt{1 - \frac{x^2}{A^2}}$$

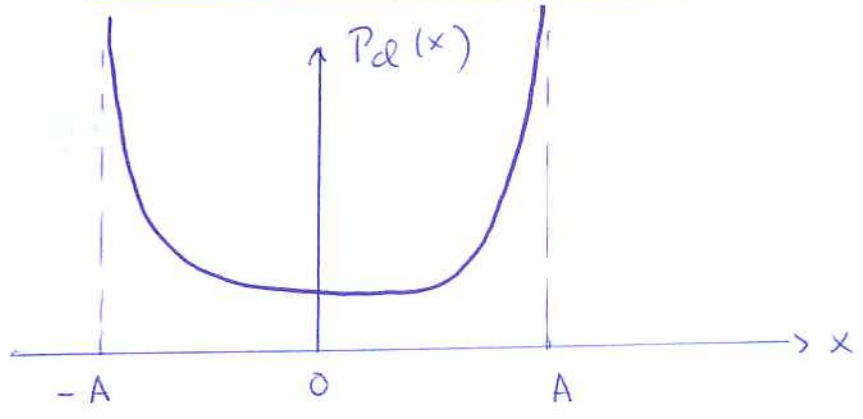
$$P_{cl}(x) = \frac{2}{2\pi} \left| \frac{d\theta}{dx} \right|$$

$$= \frac{2W}{2\pi} \cdot \left| \frac{dt}{dx} \right|$$

$$= \frac{2}{T} \cdot \frac{1}{|v|}$$

↳ VELOCITY

$$P_{cl}(x) = \frac{1}{\pi} \cdot \frac{1}{A} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{A^2}}}$$





• QUANTUM

HOW DOES THIS PROBABILITY DENSITY COMPARE WITH QUANTUM?

$$P_{QM}(x) = |\Psi(x, t)|^2$$

IF H.O. IS IN STATIONARY STATE  $\Psi_m$

WITH ENERGY  $E_m = \hbar\omega (m + \frac{1}{2})$

$$P_{QM} = |\Psi_m(x)|^2$$

FOR ENERGY  $E_m$ : CLASSICAL TURNING POINTS A:

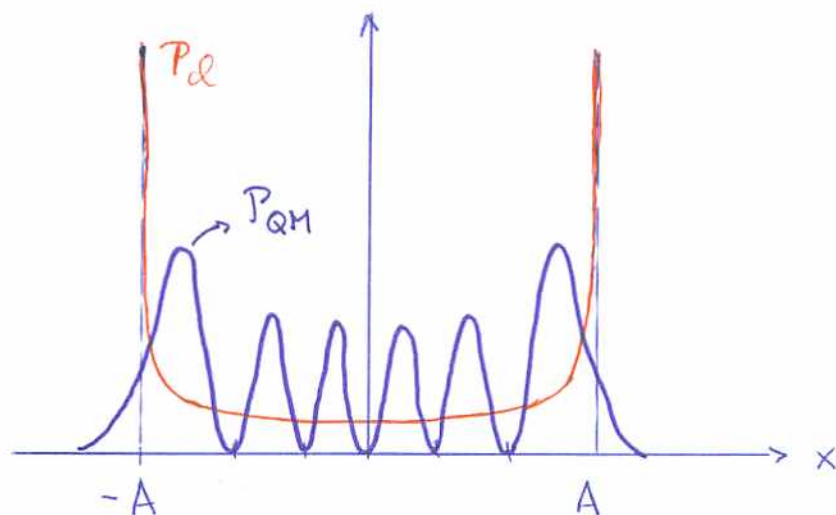
$$\frac{1}{2} m \omega^2 A^2 = \frac{\hbar\omega}{2} (2m + 1)$$

$$A = \sqrt{\frac{\hbar}{m\omega}} \sqrt{2m + 1}$$

$$P_{CL}(x) = \frac{1}{2\pi} \frac{1}{\sqrt{\hbar/m\omega}} \frac{1}{\sqrt{2m+1}} \frac{1}{\sqrt{1 - \frac{x^2}{(\hbar/m\omega)(2m+1)}}}$$

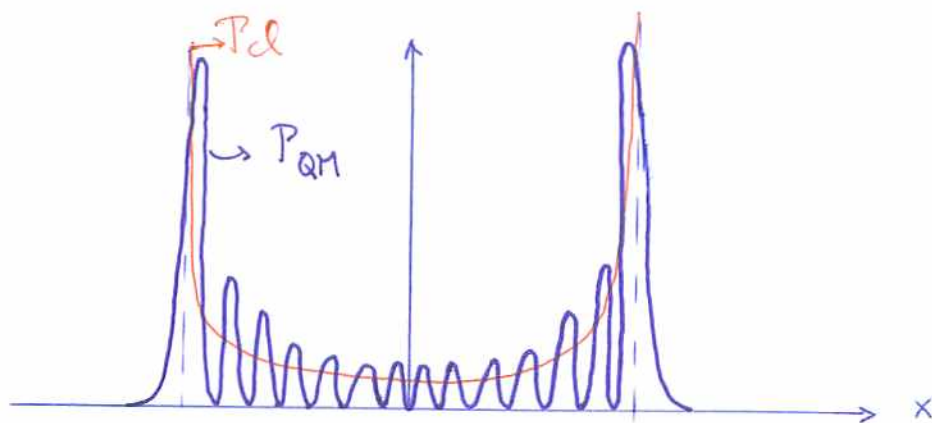
GRAPHICAL e.g.  $n=5$  (5 NODES)

2.42



$$n=5: \quad A = \sqrt{\frac{\hbar}{m\omega}} \underbrace{\sqrt{11}}_{3.32}$$

FOR  $n \gg 1$



QUANTUM PROBABILITY OSCILLATES AROUND CLASSICAL PROBABILITY

↳ CORRESPONDENCE PRINCIPLE

• VISUALIZATION

SEE [WEBPHYSICS.DAVIDSON.EDU/MJB/NCS\\_AAPT\\_QM\\_2002/WELCOME.HTML](http://WEBPHYSICS.DAVIDSON.EDU/MJB/NCS_AAPT_QM_2002/WELCOME.HTML)

WELCOME.HTML

# ⇒ 2.4 THE FREE PARTICLE

$V(x) = 0$  EVERYWHERE

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi$$

⇓

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

$$E \equiv \frac{\hbar^2 k^2}{2m}$$

k: WAVEVECTOR

↳ GENERAL SOLUTION  $\psi(x) = A e^{ikx} + B e^{-ikx}$

SIMILAR AS INSIDE REGION OF INFINITE SQUARE WELL

DIFFERENCE: NO BOUNDARY CONDITIONS TO CONSTRAIN k

↓

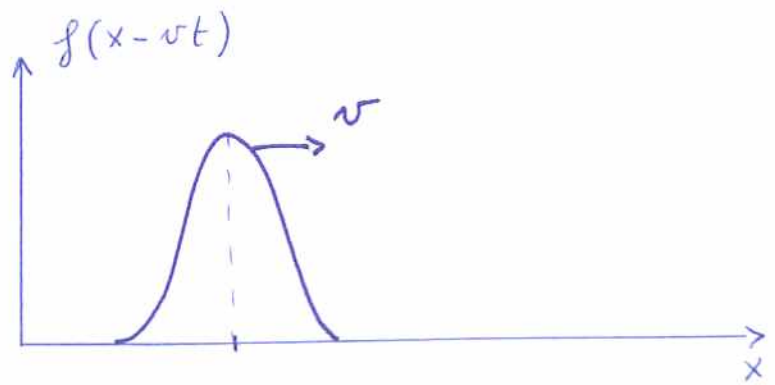
FREE PARTICLE CAN MOVE WITH ANY VALUE OF E

↳ TIME DEPENDENCE

$$\begin{aligned} \bar{\Psi}_k(x, t) &= A e^{-\frac{i}{\hbar}Et} e^{ikx} + B e^{-\frac{i}{\hbar}Et} e^{-ikx} \\ &= A e^{+ik(x - \frac{\hbar k}{2m}t)} + B e^{-ik(x + \frac{\hbar k}{2m}t)} \end{aligned}$$

↳ SOLUTION DEPENDS ON CONTINUOUS VARIABLE k  
STATIONARY STATE

$f(x \mp vt)$  CORRESPONDS WITH WAVE TRAVELLING WITH VELOCITY  $v$  IN  $\pm$  X-DIRECTION



$$k = \pm \frac{\sqrt{2mE}}{\hbar}$$

$k > 0$  : WAVE TRAVELING TO RIGHT  
 $k < 0$  : WAVE TRAVELING TO LEFT

↳ MOMENTUM CARRIED BY WAVE

$$p = \hbar k$$

$$k = \frac{2\pi}{\lambda} \Rightarrow p = \frac{h}{\lambda}$$

DE BROGLIE FORMULA

↳ SPEED OF WAVE

QUANTUM :

$$v_{QM} = \frac{\hbar k}{2m} = \sqrt{\frac{E}{2m}}$$

CLASSICAL PARTICLE WITH KIN. ENERGY  $E$

$$E = \frac{1}{2} m v^2 \Rightarrow v_{cl} = \sqrt{\frac{2E}{m}} = 2 v_{QM}$$

▽  
 0 QUANTUM WAVE TRAVELS AT HALF THE SPEED OF CLASSICAL PARTICLE ? SEE LATER

↳ NORMALIZATION

$$\Psi_k(x, t) = A e^{i(kx - \frac{\hbar k^2}{2m} t)}$$

$$\int_{-\infty}^{+\infty} dx |\Psi(x, t)|^2 = |A|^2 \underbrace{\int_{-\infty}^{+\infty} dx}_{\infty} = 1$$

NOT NORMALIZABLE !



UNPHYSICAL ⇒ A FREE PARTICLE CANNOT EXIST IN A STATIONARY STATE

BUT : WE CAN CONSTRUCT NORMALIZABLE SOLUTIONS AS LINEAR COMBINATIONS OF STATIONARY STATES

BECAUSE STATIONARY STATE DEPENDS ON CONTINUOUS VARIABLE  $k$



GENERAL SOLUTION IS INTEGRAL OVER  $k$ . (INSTEAD OF A SUM OVER A DISCRETE INDEX  $n$ )

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)}$$



CONVENIENT NORMALIZATION FACTOR

⇒ WAVE PACKET

ROLE OF  $C_n$  BEFORE ↔  $\frac{1}{\sqrt{2\pi}} \phi(k)$

↳ HOW TO DETERMINE  $\Phi(k)$

USUALLY ONE IS GIVEN AN INITIAL WAVEFUNCTION

$\Psi(x, t=0) \Rightarrow$  FROM WHICH ONE DETERMINES  $\Phi(k)$

$$\Psi(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \Phi(k) e^{ikx}$$

NORMALIZED



$\Phi(k)$  DETERMINED BY FOURIER TRANSFORM

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{-ikx} \Psi(x, 0)$$



GENERAL SOLUTION OBTAINED BY PLUGGING  $\Phi(k)$  INTO  $\Psi(x, t)$

# FOURIER TRANSFORM

2.47

PROOF OF :

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk e^{ikx} F(k)$$
$$\Updownarrow$$
$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{-ikx} f(x)$$

↳ CONSIDER FIRST FINITE INTERVAL  $x \in [-a, a]$   
AND LET  $a \rightarrow \infty$  AT END

GENERAL SOLUTION ( $t=0$ )

$$f(x) = \sum_{n=0}^{\infty} \left\{ a_n \sin\left(\frac{n\pi}{a}x\right) + b_n \cos\left(\frac{n\pi}{a}x\right) \right\}$$

PERIODIC BOUNDARY CONDITION  $f(a) = f(-a)$ 

$$\cos(kx) = \cos(k(x+2a))$$

$$\sin(kx) = \sin(k(x+2a))$$

$$2ka = n \cdot 2\pi$$

$$\underline{\underline{k = \frac{n\pi}{a}}}$$

2.48

USING  $e^{\pm ikx} = \cos kx \pm i \sin kx$

$$\cos\left(\frac{n\pi}{a}x\right) = \frac{1}{2} \left( e^{i\frac{n\pi}{a}x} + e^{-i\frac{n\pi}{a}x} \right)$$

$$\sin\left(\frac{n\pi}{a}x\right) = \frac{1}{2i} \left( e^{i\frac{n\pi}{a}x} - e^{-i\frac{n\pi}{a}x} \right)$$

$$f(x) = \sum_{n=0}^{\infty} \left\{ \frac{1}{2} (-ia_n + b_n) e^{i\frac{n\pi}{a}x} + \frac{1}{2} (+ia_n + b_n) e^{-i\frac{n\pi}{a}x} \right\}$$

DEFINE  $c_n \equiv \begin{cases} b_0, & n=0 \\ \frac{1}{2} (-ia_n + b_n), & n=1, 2, 3, \dots \\ \frac{1}{2} (ia_{-n} + b_{-n}), & n=-1, -2, -3, \dots \end{cases}$

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i\frac{n\pi}{a}x}$$



↳

$$C_n = \frac{1}{2a} \int_{-a}^a dx e^{-i \frac{n\pi}{a} x} f(x)$$

PROOF:

$$\begin{aligned} & \int_{-a}^a dx e^{-i \frac{n\pi}{a} x} f(x) \\ &= \sum_{m=-\infty}^{+\infty} C_m \int_{-a}^a dx e^{-i \frac{n\pi}{a} x} e^{i \frac{m\pi}{a} x} \\ &= \sum_{m=-\infty}^{+\infty} C_m \int_{-a}^a dx e^{-i \frac{\pi}{a} (n-m) x} \end{aligned}$$

FOR  $n=m \Rightarrow +2a$ .

$$\text{FOR } n \neq m \quad - \frac{a}{i\pi} \frac{1}{(n-m)} \left. e^{-i \frac{\pi}{a} (n-m)x} \right|_{-a}^a$$

$$= \frac{ia}{\pi} \frac{1}{(n-m)} \left\{ e^{-i\pi(n-m)} - e^{i\pi(n-m)} \right\}$$

$$= -2i \sin((n-m)\pi)$$

||

0

FOR  $n \neq m$ 

$$= \sum_{m=-\infty}^{+\infty} C_m \cdot \delta_{nm} (2a)$$

$$= 2a C_n$$

■

QED

↳ EXTEND INTERVAL TO  $[-\infty, +\infty]$

$a \rightarrow \infty$

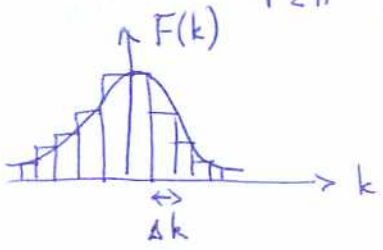
DEFINE  $k_m = \frac{n\pi}{a}$

$\Delta k = \frac{\pi}{a}$  INCREMENT IN  $k$

$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{ik_n x}$

$= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \frac{\sqrt{2\pi} a}{\pi} c_n \cdot \Delta k \cdot e^{ik_n x}$

$= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \left( \sqrt{\frac{2}{\pi}} a c_n \right) \cdot \Delta k \cdot e^{ik_n x}$



$F(k_m)$

$\Delta k \rightarrow 0 \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk F(k) e^{ikx}$

$a \rightarrow \infty$

DEFINITION OF INTEGRAL.

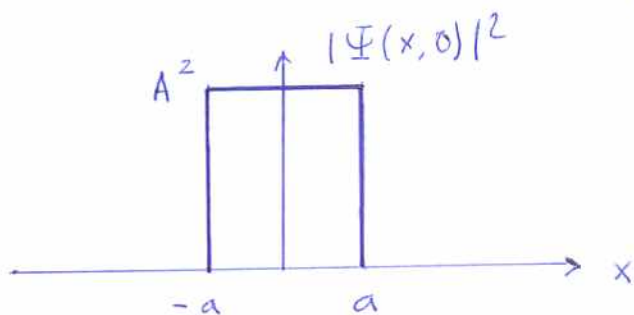
$F(k) = \sqrt{\frac{2}{\pi}} a c_n = \sqrt{\frac{2}{\pi}} a \cdot \frac{1}{2a} \int_{-a}^a dx f(x) e^{-ikx}$

$\rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{-ikx} f(x)$  QED

• EXAMPLE

FREE PARTICLE INITIALLY LOCALIZED IN INTERVAL  $x \in [-a, a]$   
( $t=0$ )

$$\Psi(x, 0) = \begin{cases} A & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$



WHAT IS  $\underline{\Psi}(x, t)$  ?

↳ NORMALIZATION  $\int_{-\infty}^{+\infty} dx |\Psi(x, 0)|^2 = 1$

$$\downarrow$$

$$2a A^2 = 1 \Rightarrow$$

$$A = \frac{1}{\sqrt{2a}}$$

↳ 
$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \Phi(k) e^{ikx} e^{-\frac{i\hbar k^2}{2m} t}$$

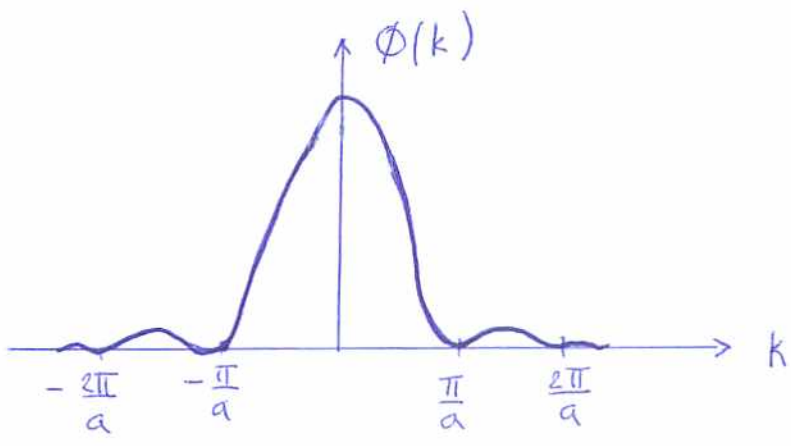
$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{-ikx} \Psi(x, 0)$$

$$\begin{aligned} \Phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a dx \frac{1}{\sqrt{2a}} e^{-ikx} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \frac{1}{(-ik)} e^{-ikx} \Big|_{-a}^a \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \frac{1}{k} \cdot \frac{e^{ika} - e^{-ika}}{i} \\ &\qquad\qquad\qquad \underbrace{\hspace{10em}}_{2 \sin ka} \end{aligned}$$

$$\boxed{\Phi(k) = \frac{1}{\sqrt{\pi a}} \frac{\sin(ka)}{k}}$$

$\frac{\sin z}{z}$  HAS MAXIMUM AT  $z = 0$

HAS ZEROS AT  $z = \pm \pi, \pm 2\pi, \dots \Rightarrow k = \pm \frac{\pi}{a}, \pm \frac{2\pi}{a}, \dots$

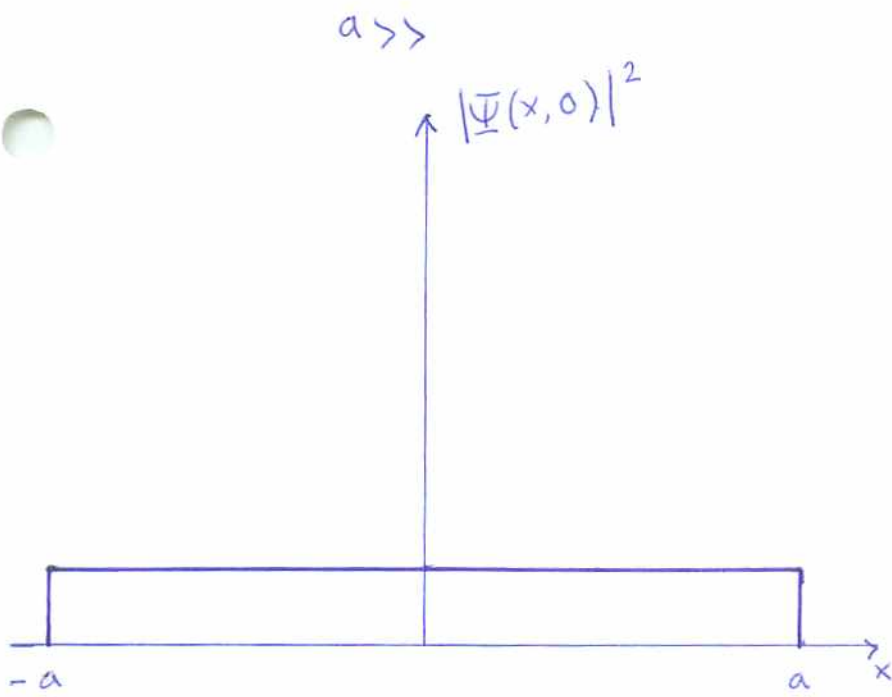


FOR  $z \rightarrow 0$

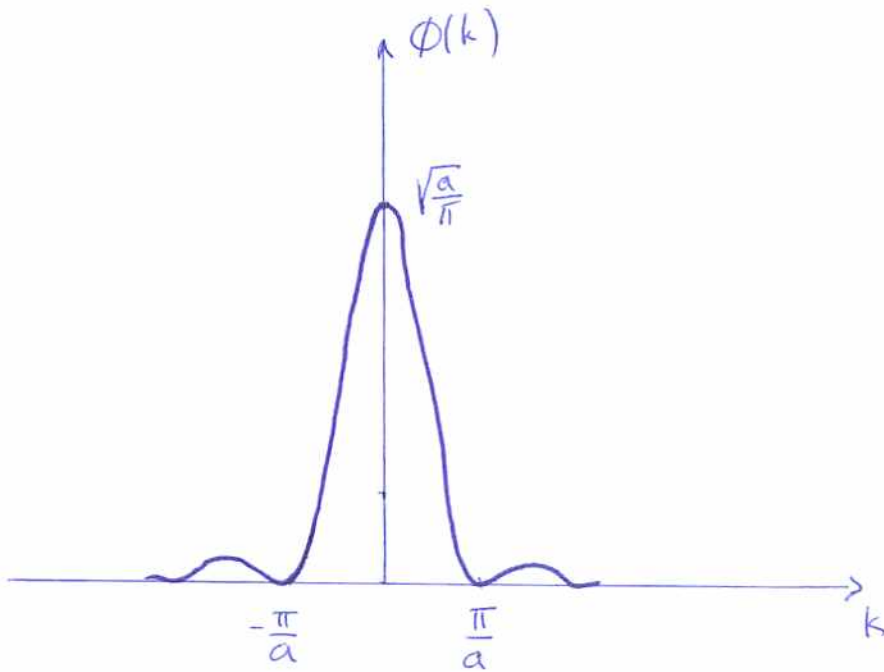
$$\frac{\sin z}{z} \rightarrow 1$$

$\Downarrow$

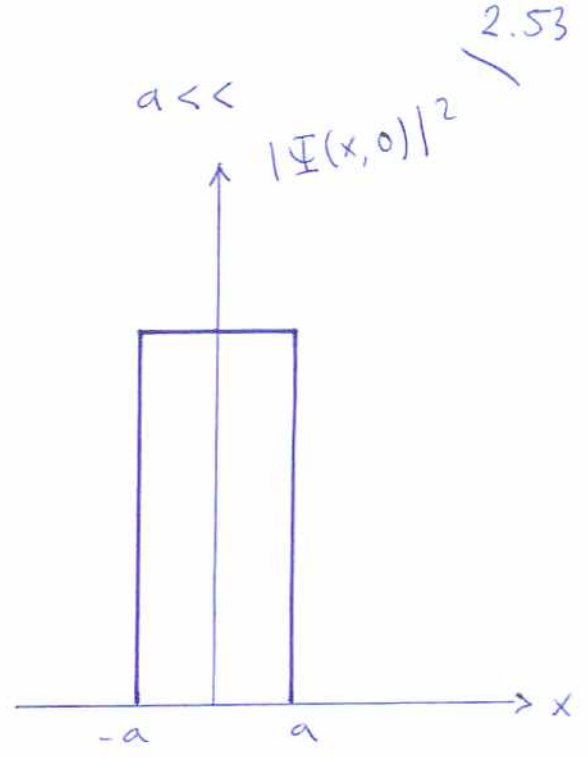
$$\Phi(k=0) = \sqrt{\frac{a}{\pi}}$$



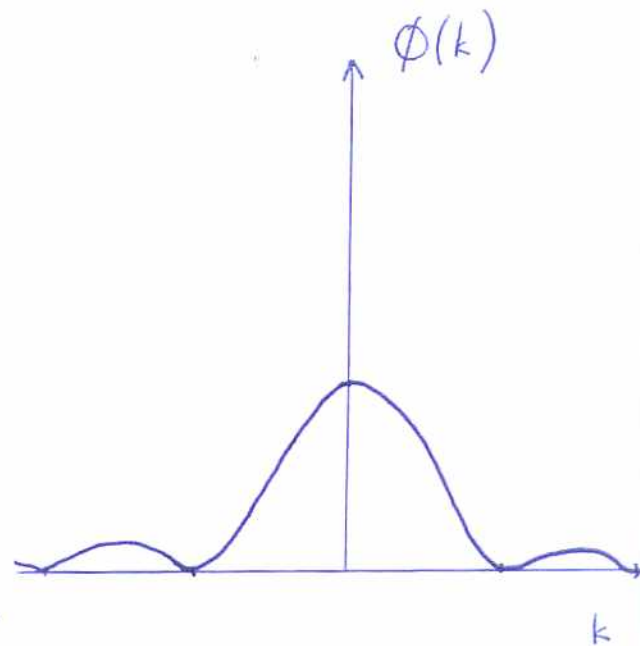
NOT WELL LOCALIZED  
IN POSITION



WELL LOCALIZED  
IN MOMENTUM



WELL LOCALIZED  
IN POSITION



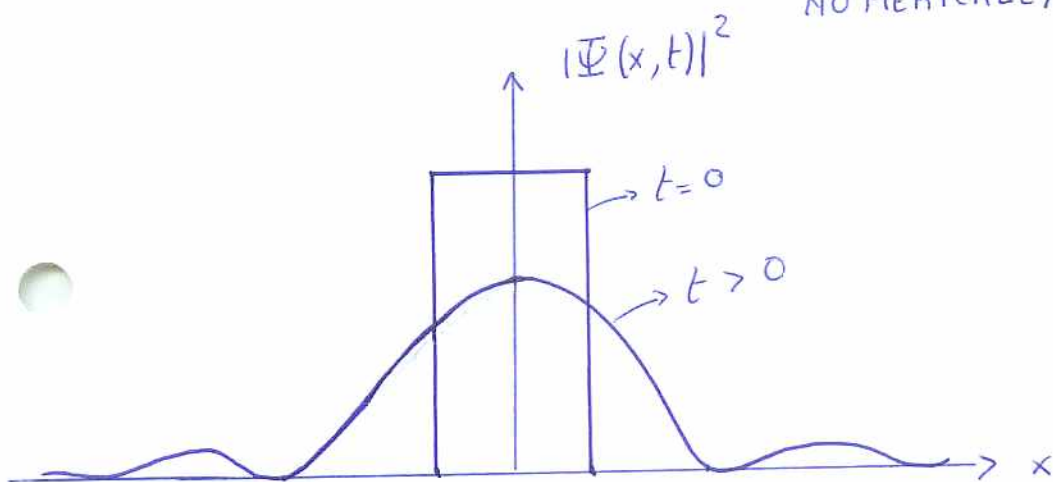
NOT WELL LOCALIZED  
IN MOMENTUM

↳ PLUG  $\Phi(k)$  INTO  $\Psi(x, t)$

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \Phi(k) e^{ikx} e^{-i\frac{\hbar k^2}{2m}t}$$

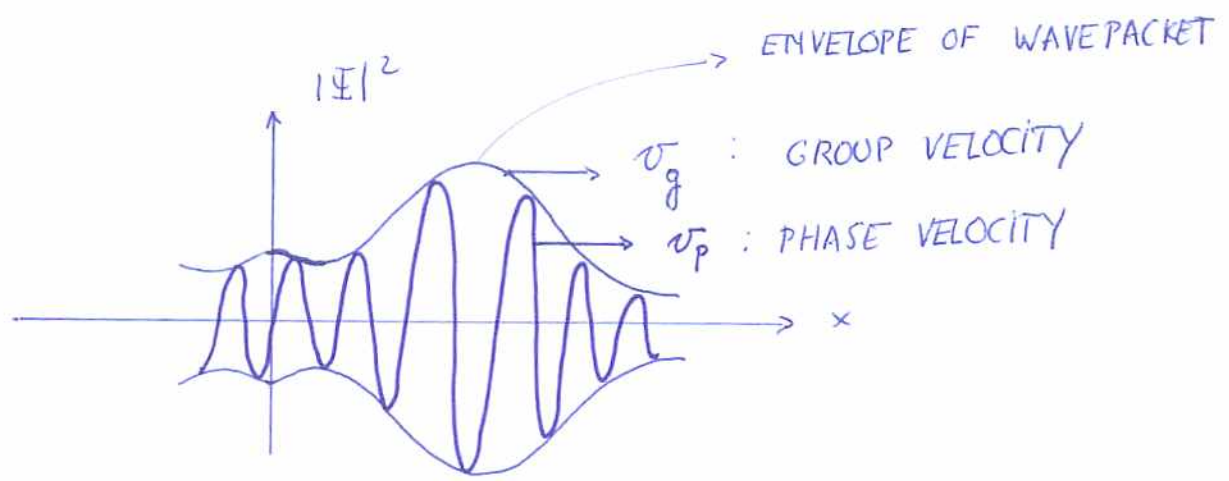
$$= \frac{1}{\sqrt{2a}} \cdot \frac{1}{\pi} \int_{-\infty}^{+\infty} dk \frac{\sin(ka)}{k} e^{ikx} e^{-i\frac{\hbar k^2}{2m}t}$$

↓  
INTEGRAL CAN ONLY BE SOLVED  
NUMERICALLY



FOR  $t > 0$  : WAVE PACKET SPREADS OUT

GROUP VELOCITY ↔ PHASE VELOCITY



$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \phi(k) e^{i(kx - \omega t)}$$

FOR FREE PARTICLE  $\omega(k) = \frac{\hbar k^2}{2m}$

ASSUME  $\phi(k)$  NARROWLY DISTRIBUTED AROUND  $k_0$

TAYLOR EXPAND  $\omega(k) \approx \overbrace{\omega(k_0)}^{\omega_0} + \omega'_0 (k - k_0)$

+ CHANGE OF VARIABLES  $\underline{k' = k - k_0}$

$$\begin{aligned} \Psi(x, t) &\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk' \phi(k' + k_0) e^{i\left[ (k_0 + k')x - (\omega_0 + \omega'_0 k')t \right]} \\ &= e^{i(-\omega_0 t + \omega'_0 k_0 t)} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk' \phi(k' + k_0) e^{i(k_0 + k')(x - \omega'_0 t)} \end{aligned}$$

$$\underline{\Psi}(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk' \phi(k' + k_0) e^{i(k_0 + k')x}$$

SHIFT FROM  $x \rightarrow x - \omega'_0 t$

$$\underline{\Psi}(x, t) = e^{-i(\omega_0 - \omega'_0 k_0)t} \underline{\Psi}(x - \omega'_0 t, 0)$$

↳ WAVE PACKET MOVES AT SPEED

$$v_g = \frac{d\omega}{dk} = \omega'_0 \quad : \text{ GROUP VELOCITY}$$

↳ PHASE VELOCITY

$$v_p = \frac{\omega}{k}$$

FOR Q.M FREE PARTICLE  $v_g = 2v_p$

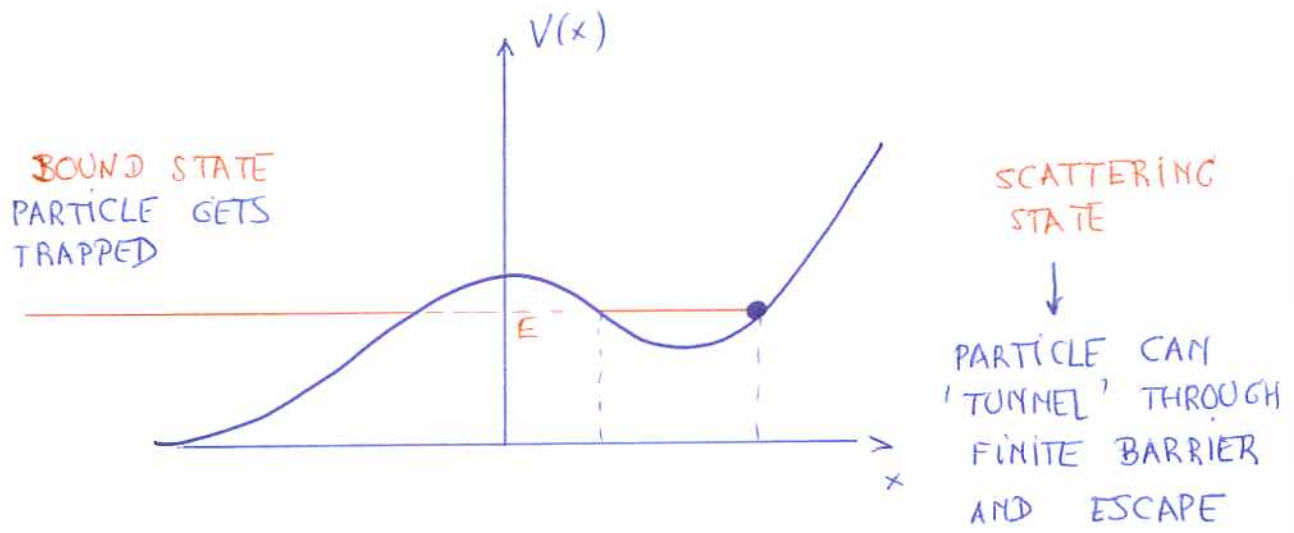
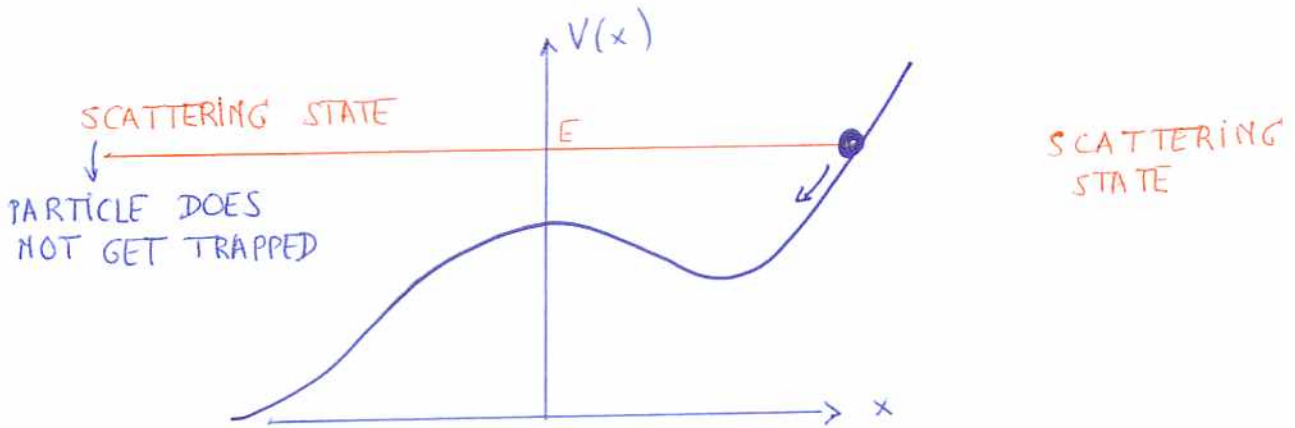
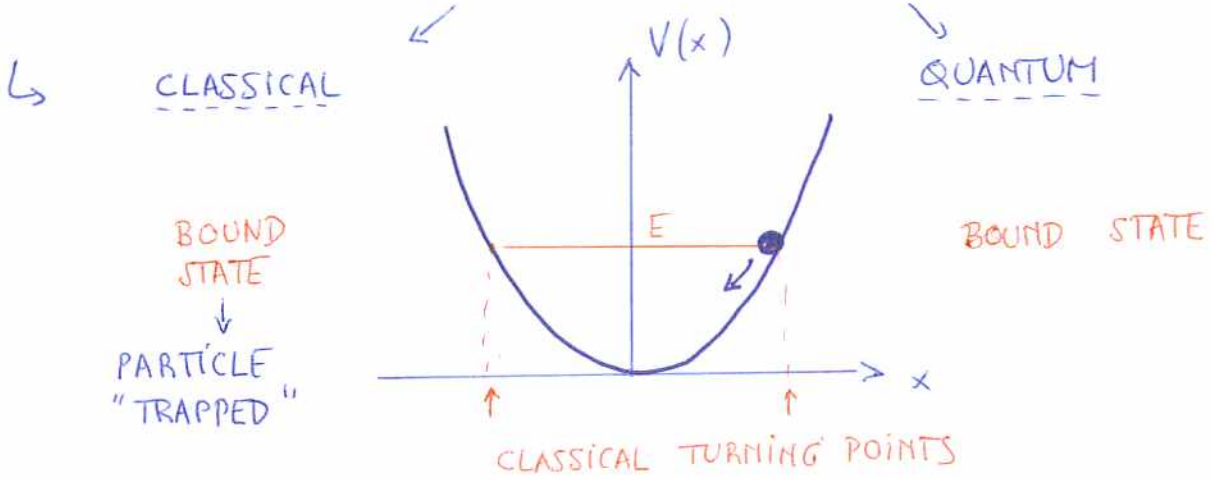
↑

CLASSICAL  
VELOCITY



# ⇒ 2.5 DELTA - FUNCTION POTENTIAL

## ● BOUND STATES VS SCATTERING STATES



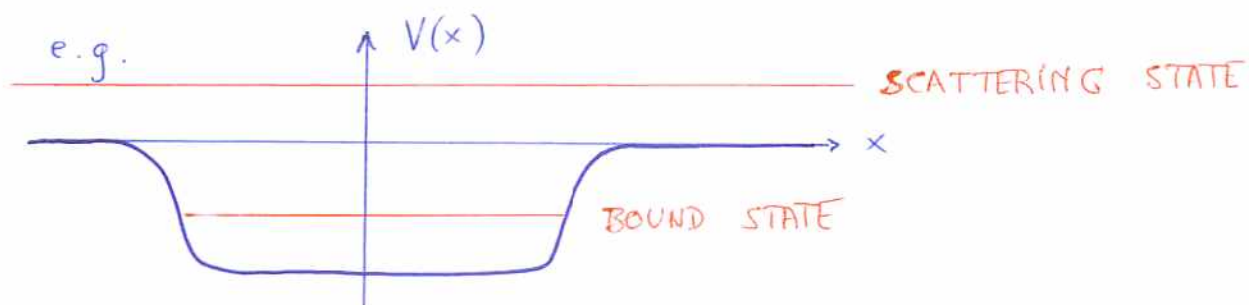
↳ QUANTUM :

BOUND STATE  $E < [V(x = +\infty) \& V(x = -\infty)]$

SCATTERING STATE  $E > [V(x = +\infty) \text{ OR } V(x = -\infty)]$

IN PHYSICAL APPLICATIONS :

OFTEN  $V(x = \pm\infty) = 0$



POTENTIAL FELT BY NUCLEON (PROTON OR NEUTRON)  
IN THE ATOMIC NUCLEUS.

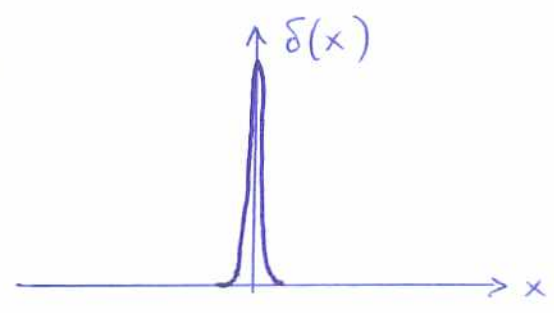


$E < 0$  : BOUND STATE  $\Rightarrow$  CHARACTERIZED BY DISCRETE INDEX  $n$

$E > 0$  : SCATTERING STATE  $\Rightarrow$  CONTINUOUS VARIABLE  $k$

• δ - FUNCTION WELL

↳ SPIKE



$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

WITH SURFACE = 1

$$\int_{-\infty}^{+\infty} dx \delta(x) = 1.$$

↳ IN MATHEMATICS :

δ(x) IS NOT A FUNCTION BUT A 'DISTRIBUTION'

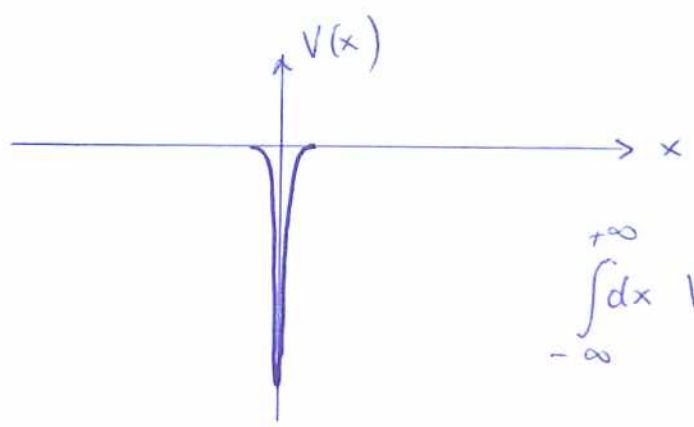
↳ δ(x-a) SPIKE POSITIONED AT x = a

↳ PROPERTIES

$$f(x) \cdot \delta(x-a) = f(a) \delta(x-a)$$

$$\int_{-\infty}^{+\infty} dx f(x) \delta(x-a) = f(a) \int_{-\infty}^{+\infty} dx \delta(x-a) = f(a)$$

$V(x) = -\alpha \delta(x)$  WITH  $\alpha > 0$



$$\int_{-\infty}^{+\infty} dx V(x) = -\alpha.$$

↳ TIME-INDEPENDENT SCHRÖDINGER EQ.

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} - \alpha \delta(x) \psi = E \psi$$

SOLUTIONS WITH  $E < 0$  : BOUND STATES

SOLUTIONS WITH  $E > 0$  : SCATTERING STATES.

BOUND STATE SOLUTIONS :  $E < 0$

↳ INTRODUCE  $E = -\frac{\hbar^2 k^2}{2m} \Rightarrow \boxed{k \equiv \frac{\sqrt{-2mE}}{\hbar}}$   
 $k > 0$

$$\frac{d^2 \psi}{dx^2} + \frac{2m\alpha}{\hbar^2} \delta(x) \psi = k^2 \psi$$

↳ FOR  $x < 0$  :  $\frac{d^2 \psi}{dx^2} = k^2 \psi$

↳ SOLUTION  $\psi(x) = A e^{-kx} + B e^{+kx}$

$\psi(x \rightarrow -\infty) = 0 \Rightarrow A = 0$

∴  $\psi(x) = B e^{kx}, x < 0$

↳ FOR  $x > 0$  :  $\frac{d^2\psi}{dx^2} = k^2 \psi$

↳ SOLUTION  $\psi(x) = F e^{-kx} + G e^{+kx}$

$\psi(x \rightarrow +\infty) = 0 \Rightarrow G = 0$

∴  $\psi(x) = F e^{-kx}, x > 0$

↳ FOR ANY  $x$  : ①  $\psi$  SHOULD BE CONTINUOUS

②  $\frac{d\psi}{dx}$  IS CONTINUOUS EXCEPT WHERE  $V = \infty$

$\psi(x \rightarrow 0^-) = \psi(x \rightarrow 0^+)$

↑  
APPROACHING  
ZERO FROM  
BELOW

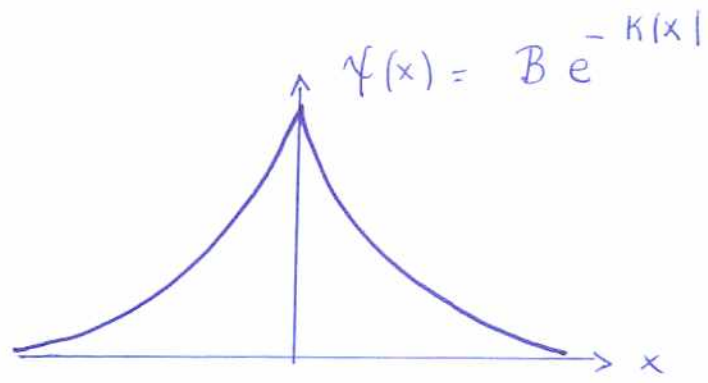
↑  
APPROACHING  
ZERO FROM ABOVE

$\psi(x \rightarrow 0^-) = B$

$\psi(x \rightarrow 0^+) = F$

∴ CONTINUITY IMPLIES  $B = F$

$\psi(x) = \begin{cases} B e^{kx} & , x \leq 0 \\ B e^{-kx} & , x \geq 0 \end{cases}$



↳ NORMALIZATION  $\int_{-\infty}^{+\infty} dx B^2 e^{-2K|x|} = 1$

$$2B^2 \int_0^{\infty} dx e^{-2Kx} = 1$$

$$\frac{2B^2}{(-2K)} e^{-2Kx} \Big|_0^{\infty} = 1$$

$$B^2 = K \Rightarrow \boxed{B = \sqrt{K}}$$

↳  $\frac{d\psi}{dx}$  is DISCONTINUOUS AT  $x=0$  WHERE  $V(x) = -\infty$

INTEGRATE SCHRÖDINGER EQ. FROM  $-\epsilon$  TO  $+\epsilon$  ( $\epsilon > 0$ )

AND LET  $\epsilon \rightarrow 0$

$$-\frac{\hbar^2}{2m} \int_{-\epsilon}^{\epsilon} dx \frac{d^2\psi}{dx^2} + \int_{-\epsilon}^{\epsilon} dx V(x) \psi(x) = E \int_{-\epsilon}^{\epsilon} dx \psi(x)$$

$$-\frac{\hbar^2}{2m} \left( \frac{d\psi}{dx} \Big|_{x=+\epsilon} - \frac{d\psi}{dx} \Big|_{x=-\epsilon} \right) + \int_{-\epsilon}^{\epsilon} dx V(x) \psi(x)$$

$$= E \psi(0) \cdot 2\epsilon$$

$$\xrightarrow{\epsilon \rightarrow 0} 0$$

# JUMP (DISCONTINUITY) IN DERIVATIVE

$$\Delta \left( \frac{d\psi}{dx} \right)_{x=0} \equiv \left. \frac{d\psi}{dx} \right|_{x=+\epsilon} - \left. \frac{d\psi}{dx} \right|_{x=-\epsilon} \quad (\epsilon \rightarrow 0)$$

$$- \frac{\hbar^2}{2m} \Delta \left( \frac{d\psi}{dx} \right)_{x=0} = - \int_{-\epsilon}^{+\epsilon} dx V(x) \psi(x)$$

**CASE 1** :  $V(x)$  IS FINITE AT  $x=0$

$$\text{RHS} \int_{-\epsilon}^{+\epsilon} dx V(x) \psi(x) = 2\epsilon V(0) \psi(0)$$

$$\Downarrow \qquad \qquad \qquad \xrightarrow{\epsilon \rightarrow 0} 0$$

$$\Delta \left( \frac{d\psi}{dx} \right)_{x=0} = 0 \quad ; \quad \frac{d\psi}{dx} \text{ IS CONTINUOUS}$$

**CASE 2** :  $V(x)$  IS INFINITE AT  $x=0$

e.g.  $V(x) = -\alpha \delta(x)$

$$\text{RHS} \int_{-\epsilon}^{+\epsilon} dx V(x) \psi(x) = -\alpha \int_{-\epsilon}^{+\epsilon} dx \delta(x) \psi(x)$$

$$\Downarrow \qquad \qquad \qquad = -\alpha \psi(0)$$

$$\Delta \left( \frac{d\psi}{dx} \right)_{x=0} = - \frac{2m \alpha}{\hbar^2} \psi(0)$$

↳ APPLY TO OUR SOLUTION

$$\Psi(x) = \begin{cases} \sqrt{K} e^{Kx} & , x \leq 0 \\ \sqrt{K} e^{-Kx} & , x \geq 0 \end{cases}$$

$$\left. \frac{d\Psi}{dx} \right|_{x=-\epsilon} = K \sqrt{K} e^{-K\epsilon} \xrightarrow{\epsilon \rightarrow 0} K^{3/2}$$

$$\left. \frac{d\Psi}{dx} \right|_{x=+\epsilon} = -K \sqrt{K} e^{-K\epsilon} \xrightarrow{\epsilon \rightarrow 0} -K^{3/2}$$

$$\Delta \left( \frac{d\Psi}{dx} \right)_{x=0} = -2K^{3/2}$$

$$\therefore -2K^{3/2} = -\frac{2m\alpha}{\hbar^2} \cdot \sqrt{K} \quad \leftarrow \Psi(0)$$

$$\boxed{K = \frac{m\alpha}{\hbar^2}}$$

BOUND STATE ENERGY

$$\boxed{E = -\frac{\hbar^2 K^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}}$$

δ - FUNCTION POTENTIAL HAS (1) BOUND STATE

$$\Psi(x) = \sqrt{K} e^{-K|x|}$$



● SCATTERING SOLUTIONS :  $E > 0$

↳ INTRODUCE  $E = \frac{\hbar^2 k^2}{2m} \Rightarrow$   $k \equiv \frac{\sqrt{2mE}}{\hbar}$

$$\frac{d^2\psi}{dx^2} + \frac{2m\alpha}{\hbar^2} \delta(x) \psi = -k^2 \psi$$

↳ FOR  $x < 0$  :  $\frac{d^2\psi}{dx^2} = -k^2 \psi$

↳ SOLUTION  $\psi(x) = A e^{ikx} + B e^{-ikx}$

NONE OF THE TERMS BLOWS UP FOR  $x \rightarrow -\infty$

↳ FOR  $x > 0$  :  $\psi(x) = F e^{ikx} + G e^{-ikx}$

↳ CONTINUITY AT  $x = 0$

$\psi(0^-) = A + B = \psi(0^+) = F + G$

↳ JUMP IN DERIVATIVE

$$\left. \frac{d\psi}{dx} \right|_{x=0^-} = ik(A - B)$$

$$\left. \frac{d\psi}{dx} \right|_{x=0^+} = ik(F - G)$$

$$\Delta \left( \frac{d\psi}{dx} \right)_{x=0} = - \frac{2m\alpha}{\hbar^2} \psi(0)$$

⇓

$$ik (F - G - A + B) = - \frac{2m\alpha}{\hbar^2} (A + B)$$

$$\hookrightarrow \underline{\underline{F - G = A(1 + 2i\beta) - B(1 - 2i\beta)}}$$

$$\boxed{\beta \equiv \frac{m\alpha}{\hbar^2 k}}$$

↳ PROBLEM : 4 UNKNOWN S A, B, F, G  
 ONLY 2 EQUATIONS.

(FOR BOUND STATES 2 UNKNOWN S WERE ELIMINATED FROM ASYMPTOTIC CONDITIONS)

↳

$e^{ikx} \cdot e^{-\frac{i}{\hbar}Et}$  : WAVE TRAVELING  $\longrightarrow$

$e^{-ikx} \cdot e^{-\frac{i}{\hbar}Et}$  : WAVE TRAVELING  $\longleftarrow$

IN AN EXPERIMENT A WAVE IS TRAVELING FROM ONE SIDE TO THE OTHER.

SAY WE CONSIDER WAVE TRAVELING  $\longrightarrow$ . STARTING FROM  $x = -\infty$

$\circ \circ \quad \underline{\underline{G = 0}}$

NO WAVE IS COMING IN FROM  $x = +\infty$

(BOUNDARY CONDITIONS)

$$\begin{cases} F = A + B \\ F = A(1 + 2i\beta) - B(1 - 2i\beta) \end{cases}$$

$$\hookrightarrow A + B = A(1 + 2i\beta) - B(1 - 2i\beta)$$

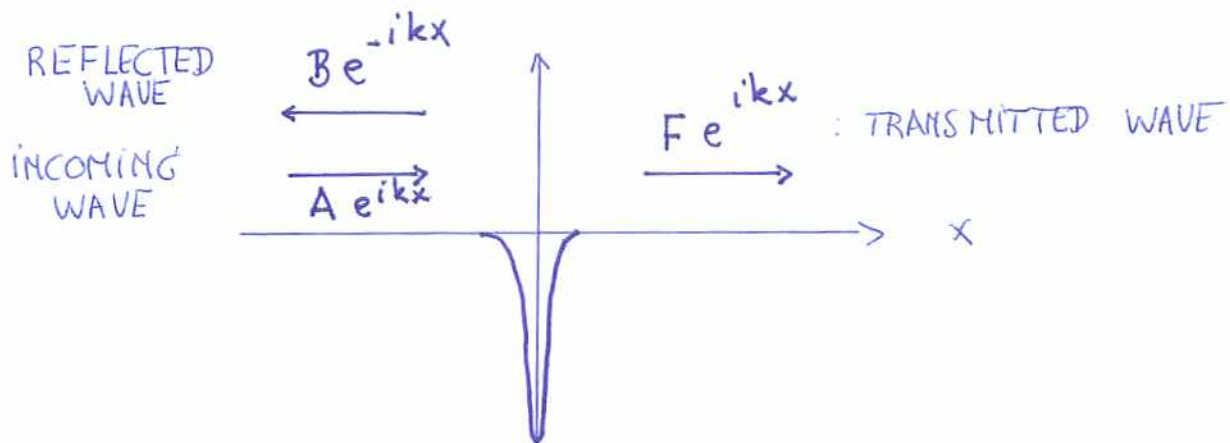
$$\Downarrow$$

$$(1 - i\beta)2B = 2i\beta A$$

$$B = \frac{i\beta}{1 - i\beta} A$$

$$\hookrightarrow F = A + B = \frac{1 - i\beta + i\beta}{1 - i\beta} A$$

$$F = \frac{1}{1 - i\beta} A$$



↳ PROBABILITY OF REFLECTION  $|F|^2$

$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1 + \beta^2}$$

↳ PROBABILITY OF TRANSMISSION

$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1 + \beta^2}$$

↳  $R + T = 1$

↳ IN TERMS OF  $\alpha$  :

$$\left\{ \begin{array}{l} R = \frac{1}{1 + 1/\beta^2} = \frac{1}{1 + \frac{\hbar^4 k^2}{m^2 \alpha^2}} = \frac{1}{1 + \frac{2\hbar^2 E}{m \alpha^2}} \\ T = \frac{1}{1 + \beta^2} = \frac{1}{1 + \frac{m \alpha^2}{2\hbar^2 E}} \end{array} \right.$$

$E \gg$  (VERY LARGE)

$R \rightarrow 0, T \rightarrow 1$

PERFECT TRANSMISSION

$E \ll$  (VERY LOW)

$R \rightarrow 1, T \rightarrow 0$

PERFECT REFLECTION

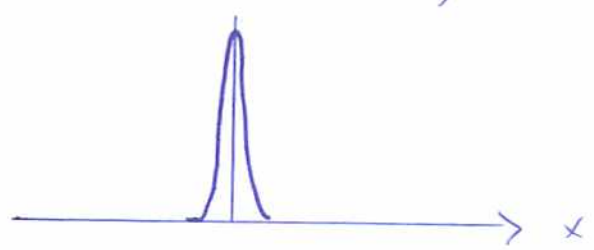
↳ NORMALIZATION

SCATTERING STATES OF DEFINITE  $k$  (PLANE WAVES) ARE NOT NORMALIZABLE

↳ PHYSICAL PARTICLES ARE DESCRIBED BY LINEAR SUPERPOSITIONS OF PLANE WAVES  $\Rightarrow$  WAVE PACKETS

•  $\delta$ -FUNCTION BARRIER

$$V(x) = + \alpha \delta(x)$$



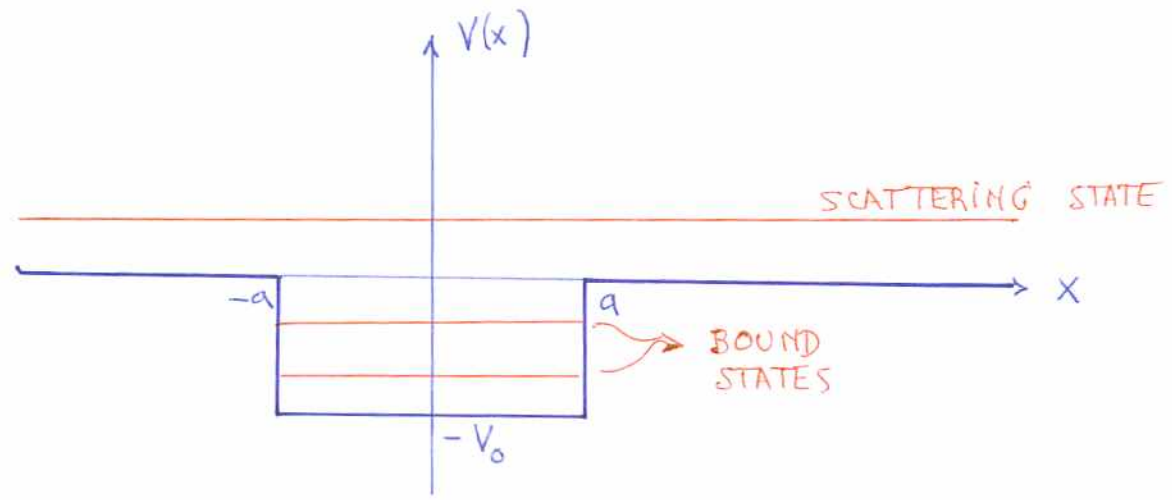
PUT  $\alpha \rightarrow -\alpha$  IN PREVIOUS

↳ NO BOUND STATE

↳ EXACTLY SAME SCATTERING STATES.

$R$  &  $T$  ONLY DEPEND ON  $\alpha^2$  !

# ⇒ FINITE SQUARE WELL



$$V(x) = \begin{cases} -V_0 & -a < x < a \\ 0 & |x| > a \end{cases} \quad (V_0 > 0)$$

↳ POTENTIAL ALLOWS FOR BOTH ↗ BOUND STATES (\$E < 0\$)  
 ↘ SCATTERING STATES (\$E > 0\$)

- BOUND STATES (\$E < 0\$)

↳ \$x < -a\$

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi$$

$$E = -\frac{\hbar^2 k^2}{2m}$$

$$k = \frac{\sqrt{-2mE}}{\hbar}$$

$$\psi(x) = \cancel{Ae^{-kx}} + Be^{kx}$$

$$\psi(x \rightarrow -\infty) = 0$$

↳ \$x > a\$ ANALOGOUSLY

$$\psi(x) = Fe^{-kx} + \cancel{Ge^{kx}}$$

$$\psi(x \rightarrow +\infty) = 0$$

↳ -a < x < a : V(x) = -V<sub>0</sub>

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - V_0 \psi = E \psi$$

⇓

$$l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$$\frac{d^2\psi}{dx^2} = -l^2 \psi$$

E < 0  
BUT V<sub>0</sub> + E > 0  
(E > -V<sub>0</sub>)

⇓

$$\psi(x) = C \sin(lx) + D \cos(lx)$$

↳ BOUNDARY CONDITIONS

ψ &  $\frac{d\psi}{dx}$  CONTINUOUS AT BOUNDARIES x = ±a

SYMMETRIC POTENTIAL : SOLUTIONS EITHER EVEN OR ODD  
$$\psi(-x) = \pm \psi(x)$$

• **EVEN SOLUTIONS** ψ(-x) = ψ(x)

BOUNDARY CONDITION AT x = +a

$$\psi(x) = \begin{cases} D \cos(lx), & 0 < x < a \\ F e^{-Kx}, & x > a \end{cases}$$

⇒ CONTINUITY OF ψ AT x = a

⇓

$$F e^{-Ka} = D \cos(la) \quad (*)$$

2.12

⇒ CONTINUITY OF  $\frac{d\psi}{dx}$  AT  $x = a$ .

↓

$$-FK e^{-Kx} = -Dl \sin(la) \quad (**)$$

$$\frac{(**)}{(*)} \Rightarrow \boxed{+l \tan(la) = K}$$

↳ TRANSCENDENTAL EQ.  
TO BE SOLVED TO DETERMINE  $E$

$$\text{DEFINE} \left\{ \begin{array}{l} z \equiv la = \frac{a}{\hbar} \sqrt{2m(E+V_0)} \\ z_0 \equiv \frac{a}{\hbar} \sqrt{2mV_0} \end{array} \right.$$

$$z_0^2 - z^2 = \frac{a^2}{\hbar^2} (-2mE) = a^2 K^2$$

$$aK = \sqrt{z_0^2 - z^2}$$

$$\therefore al \tan(la) = aK$$

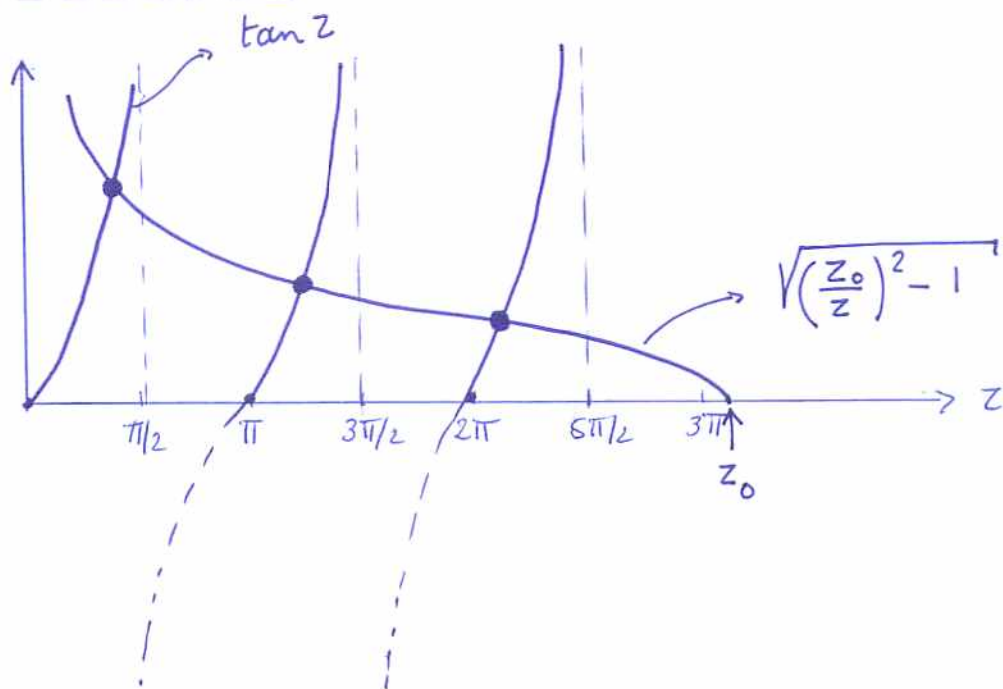


$$z \tan z = \sqrt{z_0^2 - z^2}$$

$$\boxed{\tan z = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}}$$

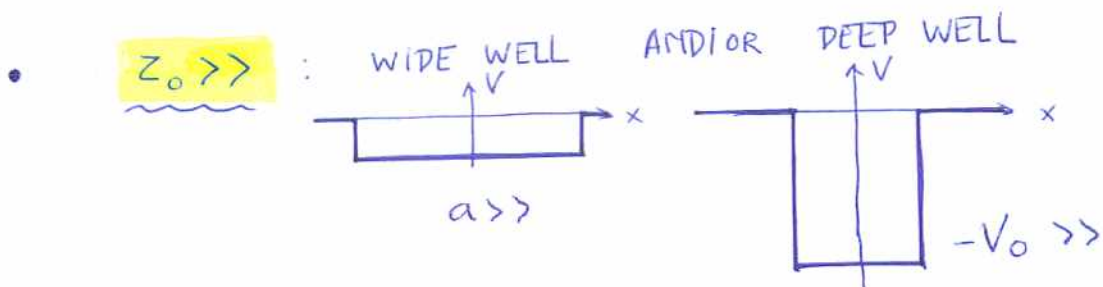


⇒ GRAPHICAL SOLUTION



INTERSECTIONS ARE SOLUTIONS.

⇒ SPECIAL LIMITING CASES.



SOLUTIONS  $z \approx z_n = n \frac{\pi}{2} \quad (n \text{ ODD})$

$$E_n + V_0 \approx \frac{\hbar^2}{2m} \frac{1}{a^2} \cdot \left( n \frac{\pi}{2} \right)^2$$

ENERGY  
ABOVE  
BOTTOM OF  
WELL

$$E_n = -V_0 + \frac{\hbar^2 \pi^2}{2m (2a)^2} \cdot n^2$$

(n ODD)

FOR  $V_0$  FINITE  $\Rightarrow$  ONLY FINITE # BOUND STATES

FOR  $V_0 \rightarrow +\infty \Rightarrow \infty$  # BOUND STATES

THIS IS INFINITE SQUARE WELL OF WIDTH  $(2a)$  FOR  $n$  ODD

$$E_1, E_3, E_5, \dots$$

(EVEN  $n$  CORRESPOND WITH ODD WAVE FUNCTIONS  $\psi(-x) = -\psi(x)$ )

- $z_0 \ll 1$  NARROW WELL AND/OR SHALLOW WELL  
 $a \ll 1$   $-V_0 \ll 1$

$z_0 \downarrow$  LESS  $\rightarrow$  LESS BOUND STATES.

FOR  $z_0 < \frac{\pi}{2} \Rightarrow$  ONLY 1 BOUND STATE REMAINS

☉ THERE IS ALWAYS 1 BOUND STATE  
EVEN FOR SUPERWEAK POTENTIAL  
OR VERY NARROW WELL



- **ODD SOLUTIONS**  $\psi(-x) = -\psi(x)$

BOUNDARY CONDITION AT  $x = +a$

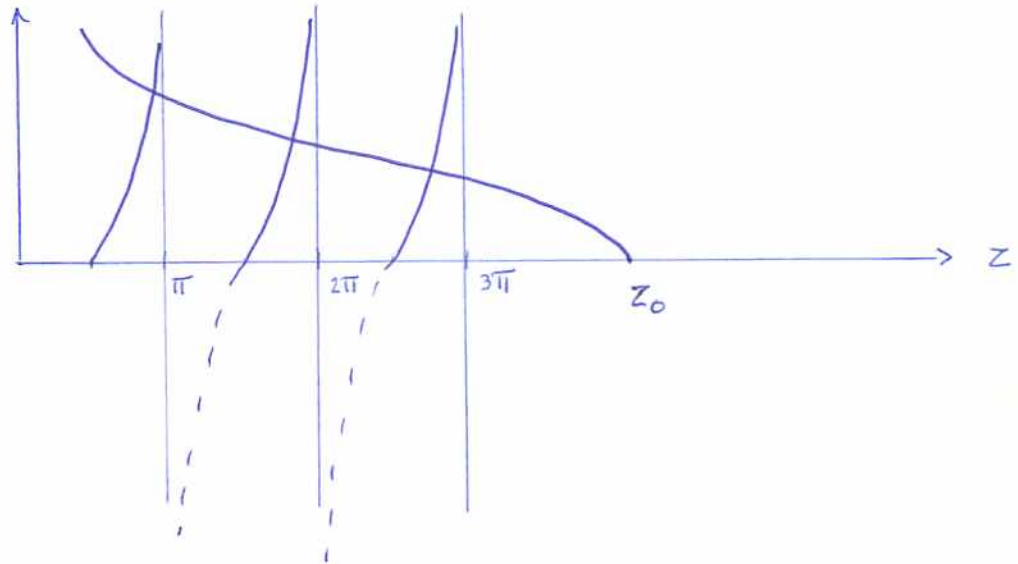
$$\psi(x) = \begin{cases} C \sin lx & , 0 < x < a \\ F e^{-Kx} & , x > a \end{cases}$$

$$\Rightarrow F e^{-Ka} = C \sin la$$

$$\Rightarrow -Fk e^{-Ka} = Cl \cos la$$

$$\circ \circ \quad l \cot(la) = -K$$

$$-\cot z = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$



SOLUTIONS

$$z_n \approx n\pi \quad n = 1, 2, 3, \dots \\ = (2n) \frac{\pi}{2}$$

NO BOUND STATE IF

$$z_0 < \frac{\pi}{2} \quad \Leftrightarrow V_0 < \frac{\pi^2 \hbar^2}{8ma^2}$$

● SCATTERING STATES (E > 0)

\* ASSUMING: INCIDENT WAVE MOVING FROM x = -∞ TO RIGHT

↳ x < -a V(x) = 0

$k \equiv \frac{\sqrt{2mE}}{\hbar}$

$\psi(x) = A e^{ikx} + B e^{-ikx}$

↳ INCIDENT AMPLITUDE

↳ -a < x < a V(x) = -V<sub>0</sub>

$\psi(x) = C \sin(\ell x) + D \cos(\ell x)$

$\ell = \frac{1}{\hbar} \sqrt{2m(E+V_0)}$  AS BEFORE

↳ x > a V(x) = 0

$\psi(x) = F e^{ikx}$

(TERM ~ e<sup>-ikx</sup> ABSENT FOR INCIDENT WAVE MOVING FROM x = -∞ TO RIGHT)

A: INCIDENT AMPLITUDE

B: REFLECTED AMPLITUDE

F: TRANSMITTED AMPLITUDE

↳ BOUNDARY CONDITIONS

CONTINUITY OF  $\Psi$  AND  $\frac{d\Psi}{dx}$  AT  $x = -a$  AND  $x = +a$

(1) •  $\Psi(-a): Ae^{-ika} + Be^{+ika} = -C \sin(la) + D \cos(la)$

(2) •  $\frac{d\Psi}{dx}(-a): ik [Ae^{-ika} - Be^{+ika}] = l [+C \cos(la) + D \sin(la)]$

(3) •  $\Psi(+a): Fe^{ika} = C \sin(la) + D \cos(la)$

(4) •  $\frac{d\Psi}{dx}(+a): ik Fe^{ika} = l [C \cos(la) - D \sin(la)]$

(3)  $l \sin(la) + (4) \cos(la)$

↳  $l C = Fe^{ika} [l \sin(la) + ik \cos(la)]$

$C = Fe^{ika} \left[ \sin(la) + \frac{ik}{l} \cos(la) \right]$

(3)  $l \cos(la) - (4) \sin(la)$

↳  $D = Fe^{ika} \left[ \cos(la) - i \frac{k}{l} \sin(la) \right]$

PLUG C, D INTO (1) & (2)

$$\begin{aligned}
 (*) \quad & A e^{-ika} + B e^{ika} \\
 & = F e^{ika} \left\{ -\sin^2(la) - i \frac{k}{l} \sin(la) \cos(la) \right. \\
 & \quad \left. + \cos^2(la) - i \frac{k}{l} \sin(la) \cos(la) \right\} \\
 & = F e^{ika} \left\{ \cos(2la) - i \frac{k}{l} \sin(2la) \right\}
 \end{aligned}$$

$$\begin{aligned}
 (**) \quad & ik [A e^{-ika} - B e^{ika}] \\
 & = F e^{ika} l \left\{ + \sin(la) \cos(la) + \frac{ik}{l} \cos^2(la) \right. \\
 & \quad \left. + \sin(la) \cos(la) - \frac{ik}{l} \sin^2(la) \right\} \\
 & = F e^{ika} l \left\{ + \sin(2la) + \frac{ik}{l} \cos(2la) \right\}
 \end{aligned}$$

↳  $ik(*) - (**)$

$$\begin{aligned}
 B e^{ika} \cdot 2ik & = F e^{ika} \left\{ \cancel{ik \cos(2la)} + \frac{k^2}{l} \sin(2la) \right. \\
 & \quad \left. - l \sin(2la) - \cancel{ik \cos(2la)} \right\} \\
 & = F e^{ika} \frac{k^2 - l^2}{l} \sin(2la)
 \end{aligned}$$

$$\boxed{B = i \frac{\sin(2la)}{2kl} (l^2 - k^2) F}$$

↳ ik (\*) + (\*\*)

$$A e^{-ika} \cdot 2ik = F e^{ika} \left\{ ik \cos(2la) + \frac{k^2}{e} \sin(2la) + l \sin(2la) + ik \cos(2la) \right\}$$

$$= F e^{ika} \left\{ 2ik \cos(2la) + \frac{k^2 + l^2}{e} \sin(2la) \right\}$$

↓

$$F = A \cdot \frac{e^{-2ika}}{\cos(2la) - i \frac{(k^2 + l^2)}{2kl} \sin(2la)}$$

↳ R & T

$$R = \frac{|B|^2}{|A|^2}, \quad T = \frac{|F|^2}{|A|^2}, \quad R + T = 1$$

$$T^{-1} = \cos^2(2la) + \frac{(k^2 + l^2)^2}{4k^2 l^2} \sin^2(2la)$$

$$= 1 + \frac{(k^2 - l^2)^2}{4k^2 l^2} \sin^2(2la)$$

$$l^2 - k^2 = \frac{1}{\hbar^2} 2mV_0$$

$$\Downarrow$$

$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2 \left( \frac{2a}{\hbar} \sqrt{2m(E+V_0)} \right)$$

$$T \leq 1$$

$\Rightarrow$  REACHES 1 FOR ZEROES OF  $\sin^2(\quad)$

$$\Downarrow$$

$$T=1 \Leftrightarrow \frac{2a}{\hbar} \sqrt{2m(E_n+V_0)} = n\pi \quad n \text{ INTEGER}$$

$\hookrightarrow$  FOR THESE ENERGY: WELL BECOMES PERFECTLY TRANSPARENT

$$\Updownarrow$$

$$2la = n\pi$$

$e^{i2la} \Leftrightarrow$  WAVE WITH WAVELENGTH  $\lambda = \frac{2\pi}{l}$

PERFECT TRANSPARENT FOR  $2 \cdot \frac{2\pi}{\lambda} \cdot a = n\pi$

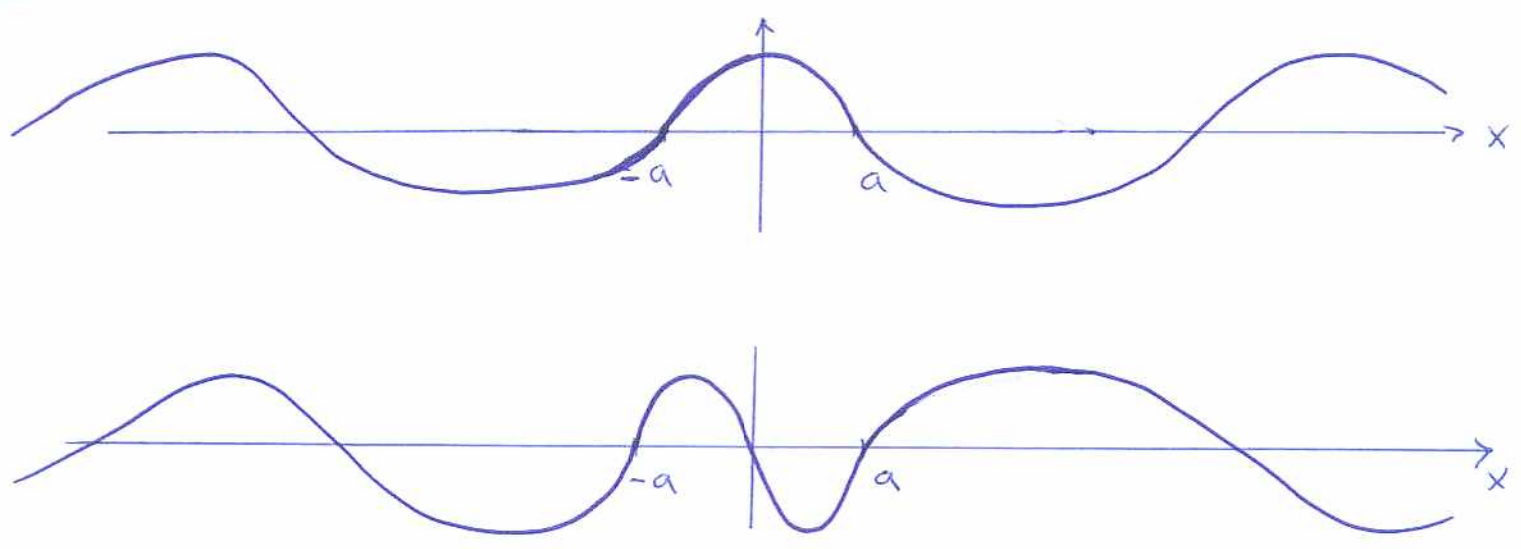
$$\Updownarrow$$

$$T=1 \Leftrightarrow$$

$$(2a) = n \frac{\lambda}{2}$$



▽ CONDITION FOR PERFECT TRANSPARANCY ( $T=1$ )  
 0 WHEN WIDTH OF WELL FITS AN INTEGER TIMES  $\lambda/2$

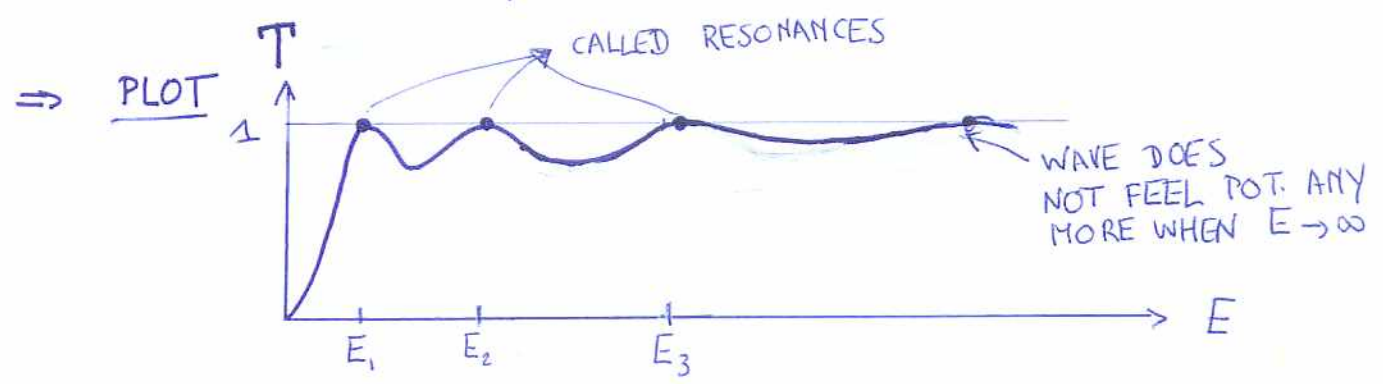


ENERGIES FOR PERFECT TRANSMISSION

$$E_m = -V_0 + m^2 \frac{\pi^2 \hbar^2}{2m (2a)^2}, \quad m \text{ INTEGER}$$

$m = 1, 2, \dots$

(cf. ENERGIES OF INFINITE SQUARE WELL)



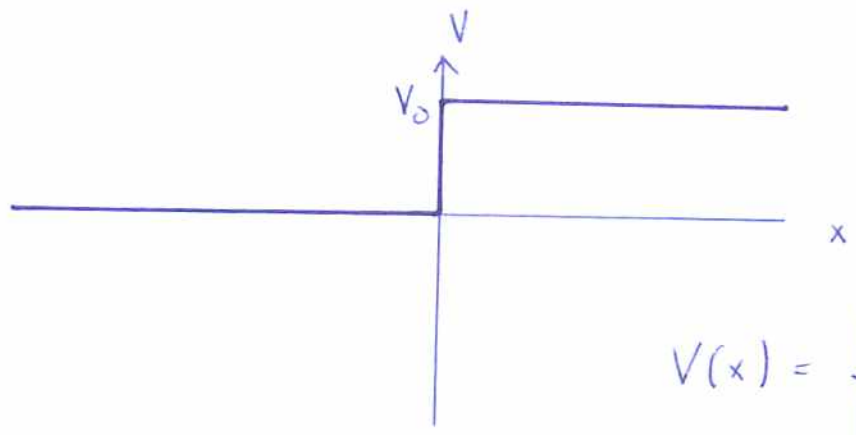
T HAS MINIMA FOR  $\sin(2al) = 1$

$$2a \cdot \frac{2\pi}{\lambda} = n' \frac{\pi}{2} \quad (n' = 1, 3, 5, \dots)$$

$2a = n' \cdot \frac{\lambda}{4}$

$(n' = 1, 3, 5, \dots)$

# ⇒ FURTHER EXAMPLE : STEP POTENTIAL



$$V(x) = \begin{cases} 0, & x \leq 0 \\ V_0, & x > 0 \end{cases}$$

- $E < V_0$

↳  $x < 0$  : SOLUTION

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

↳  $x > 0$   $-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V_0\psi = E\psi$

$$\frac{d^2\psi}{dx^2} = + \frac{2m}{\hbar^2} (V_0 - E)\psi$$

↓  $V_0 > E$

$$K = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$\psi(x) = C e^{-Kx}$$

↳ BOUNDARY CONDITION  $x=0$

$\psi$  CONTINUOUS :  $A + B = C$

$\frac{d\psi}{dx}$  CONTINUOUS :  $ik(A - B) = -CK$

⇓

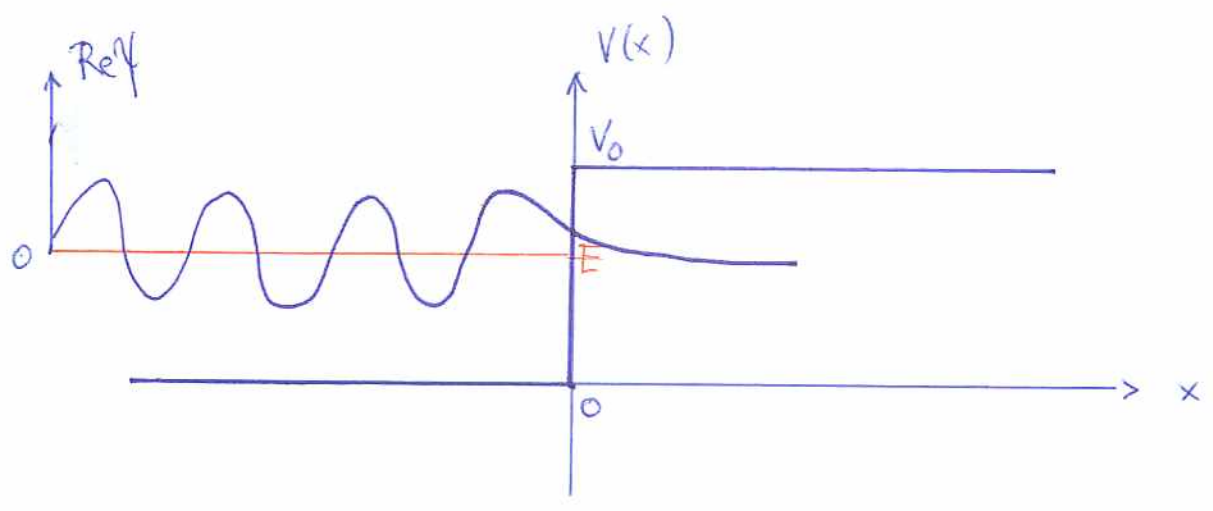
$\frac{A - B}{A + B} = + \frac{i}{k} K$

$B = A \cdot \frac{1 - \frac{i}{k} K}{1 + \frac{i}{k} K}$

REFLECTION COEFF.

$R = \frac{|B|^2}{|A|^2} = 1$

TOTAL REFLECTION



$E = 0 \Rightarrow k = 0 \Rightarrow B = -A$

$E = V_0 \Rightarrow k = 0 \Rightarrow B = +A$

•  $E > V_0$

↳  $x < 0$ :  $\psi(x) = A e^{ikx} + B e^{-ikx}$

↳  $x > 0$ :  $\psi(x) = D e^{ilx}$  FOR INCOMING WAVE FROM  $x = -\infty \rightarrow$

$$l \equiv \frac{1}{\hbar} \sqrt{2m(E - V_0)}$$

↳ BOUNDARY CONDITION  $x = 0$

$\psi$ :  $A + B = D$

$\frac{d\psi}{dx}$ :  $ik(A - B) = ilD$

⇓

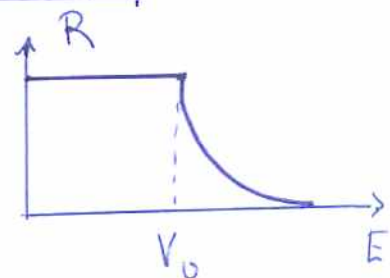
$$k(A - B) = l(A + B)$$

$$B = A \cdot \frac{(k - l)}{(k + l)} \Rightarrow D = A + B = A \frac{2k}{(k + l)}$$

↳ REFLECTION COEFF.

$$R = \frac{|B|^2}{|A|^2} = \frac{(k - l)^2}{(k + l)^2}$$

$$R = \frac{(\sqrt{E} - \sqrt{E - V_0})^2}{(\sqrt{E} + \sqrt{E - V_0})^2} = \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}$$



↳ TRANSMISSION COEFF.  $T = 1 - R$

$$T = 1 - \frac{(k-l)^2}{(k+l)^2}$$

$$= \frac{(k^2 + 2kl + l^2 - k^2 + 2kl - l^2)}{(k+l)^2}$$

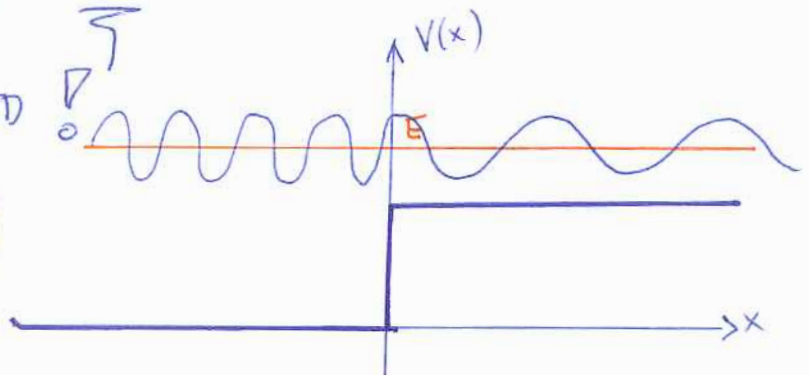
$$= \frac{4kl}{(k+l)^2}$$

$$= \frac{l}{k} \cdot \underbrace{\left( \frac{4k^2}{(k+l)^2} \right)}_{\frac{|D|^2}{|A|^2}}$$

$$\boxed{T = \frac{l}{k} \cdot \frac{|D|^2}{|A|^2} = \frac{\sqrt{E-V_0}}{\sqrt{E}} \cdot \frac{|D|^2}{|A|^2}}$$

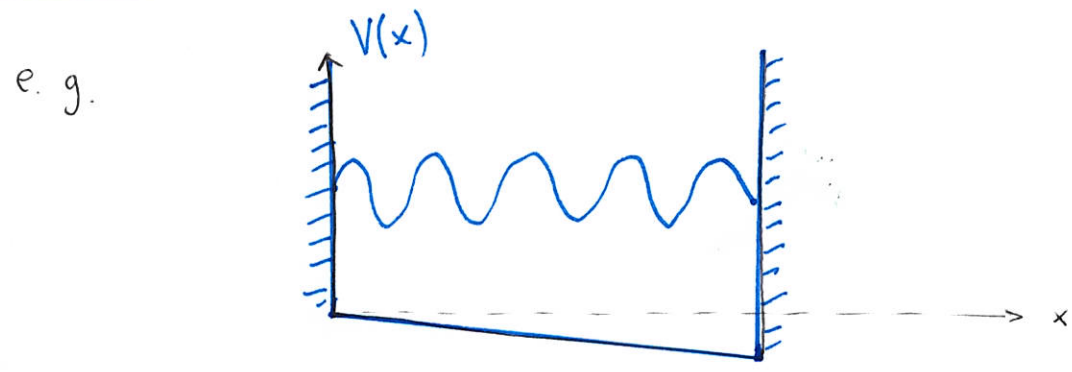
↳ NOTE T COEFF. IS ONLY OBTAINED AS RATIO OF  $|D|^2$  IF BOTH WAVES TRAVEL AT SAME SPEED

(SAME FOR R)



# ⇒ 2.7 WKB APPROXIMATION

- SEMI-CLASSICAL METHOD BY WENTZEL, KRAMERS, BRILLOUIN (WKB) TO TIME-INDEPENDENT SCHRÖDINGER EQ. WHEN  $V(x)$  VARIES VERY SLOWLY IN COMPARISON WITH  $\lambda$  (WAVELENGTH)



$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{1}{\hbar^2} 2m (E - V(x)) \psi$$

$$k(x) \equiv \frac{1}{\hbar} \sqrt{2m(E - V(x))}$$

FOR  $E > V(x)$

$$\psi(x) = A(x) e^{i\phi(x)}$$

$A(x)$  &  $\phi(x)$  ARE REAL FUNCTIONS

$$\Psi' = (A' + i\phi' A) e^{i\phi}$$

$$\Psi'' = (A'' + 2i\phi' A' + iA\phi'' - A(\phi')^2) e^{i\phi}$$

SCHRÖDINGER EQ.

$$A'' + 2i\phi' A' + iA\phi'' - A(\phi')^2 = -k^2(x) A$$

→ REAL PART:  $A'' - A(\phi')^2 = -k^2(x) A$

↘ IM. PART  $2A\phi' + A\phi'' = 0$

⇓

$$\frac{d}{dx} (A^2 \phi') = 0$$

$$A(x) = \frac{C}{\sqrt{\phi'(x)}} \quad \text{CONSTANT}$$

APPROXIMATION FOR REAL PART

VARIATION OF  $A(x)$  SLOW COMPARED TO  $\lambda$

⇓

$$A''(x) \approx 0$$

$$\phi'^2 = k^2(x)$$

$$\frac{d\phi}{dx} = \pm k(x)$$

$$\phi(x) = \pm \int dx \quad k(x)$$

$$\psi(x) \approx \frac{C}{\sqrt{k(x)}} \exp\left\{\pm i \int dx \quad k(x)\right\}$$

$$|\psi(x)|^2 \approx \frac{|C|^2}{k(x)}$$

PROBABILITY INVERSELY PROP. TO MOMENTUM (OR VELOCITY)  
AT A POINT  $x \rightsquigarrow$  cf. CLASSICAL RESULT

• EXAMPLE : INFINITE WELL WITH SLOWLY VARYING BOTTOM

$$\psi(x) \approx \frac{1}{\sqrt{k(x)}} \left\{ C_+ e^{i\phi(x)} + C_- e^{-i\phi(x)} \right\}$$

OR EQUIVALENTLY

$$\approx \frac{1}{\sqrt{k(x)}} \left\{ C_1 \sin \phi(x) + C_2 \cos \phi(x) \right\}$$

BOUNDARY CONDITIONS :

$$\psi(0) = 0$$

$$\psi(a) = 0$$



$$\Phi(x) = \int_0^x dx \quad k(x)$$

↳ SINCE  $\Phi(0) = 0 \implies \Psi(0) = 0 \iff C_2 = 0$

↳  $\Psi(a) = 0 \implies \Phi(a) = m\pi \quad (m = 1, 2, \dots)$

$$\int_0^a dx \quad k(x) = m\pi$$

↳ SPECIAL CASE: FOR  $V(x) = 0$

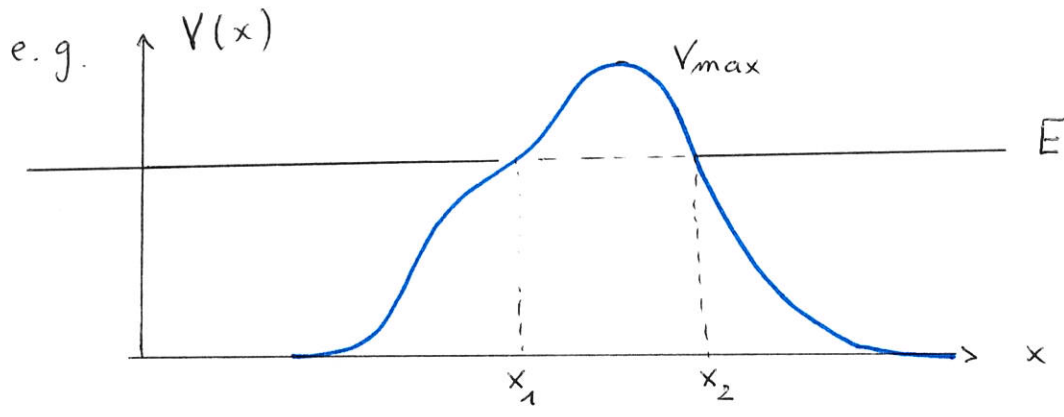


$$k(x) = k = \frac{1}{\hbar} \sqrt{2mE} \quad \text{CONSTANT}$$

$$k \cdot a = m\pi$$

$$k = \frac{m\pi}{a} \quad (\text{RESULT OF BEFORE})$$

# TUNNELING THROUGH A BARRIER



FOR  $E < V_{max}$

$$\frac{d^2 \Psi}{dx^2} = -\frac{1}{\hbar^2} 2m [V(x) - E] \Psi$$

FOR  $x_1 < x < x_2$  : CLASSICALLY FORBIDDEN REGION

$$K(x) \equiv \frac{1}{\hbar} \sqrt{2m (V(x) - E)}$$

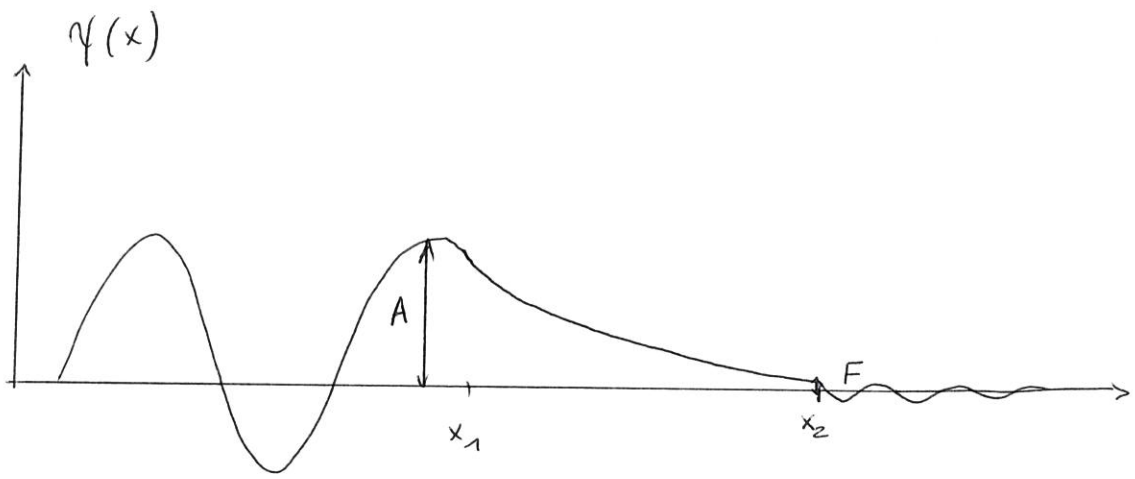
$$\Psi(x) \approx \frac{C_1}{\sqrt{K(x)}} \exp \left\{ \int dx K(x) \right\}$$

$$+ \frac{C_2}{\sqrt{K(x)}} \exp \left\{ - \int dx K(x) \right\}$$

↓ FOR  $x_2 - x_1 \gg \lambda$   
VERY THICK WELL

$C_1 \approx 0$   
(OTHERWISE EXP. INCREASING)

$$\approx \frac{C_2}{\sqrt{K(x)}} \exp \left\{ - \int dx K(x) \right\}$$



INCIDENT WAVE  $A e^{ikx}$   
 TRANSMITTED WAVE  $F e^{ikx}$

$$\frac{|F|}{|A|} \sim \exp \left\{ - \int_{x_1}^{x_2} dx K(x) \right\}$$

$$T \sim \exp \{ - 2\gamma \}$$

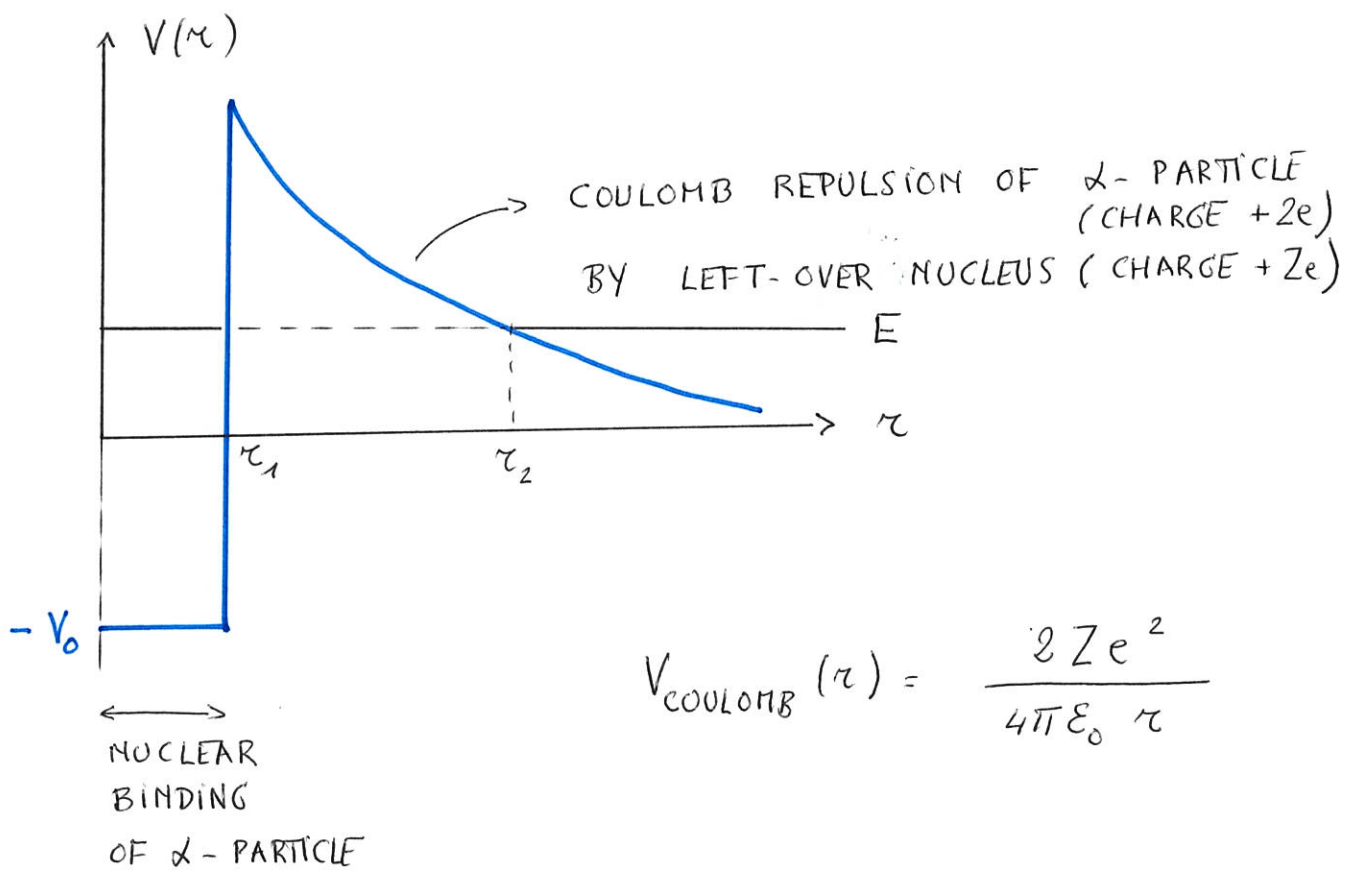
WITH

$$\gamma \equiv \int_{x_1}^{x_2} dx K(x)$$

EXAMPLE : GAMOW'S THEORY OF ALPHA DECAY

SPONTANEOUS EMISSION OF  $\alpha$ -PARTICLE ( 2 PROTONS + 2 NEUTRONS ) BY CERTAIN RADIOACTIVE NUCLEI

MODEL POTENTIAL



$\hookrightarrow$  E : ENERGY OF EMITTED  $\alpha$ -PARTICLE :  $E = \frac{2Ze^2}{4\pi\epsilon_0 r_2}$

$$\begin{aligned} \hookrightarrow \gamma &= \int_{r_1}^{r_2} dr \frac{1}{\hbar} \sqrt{2m [V(r) - E]} \\ &= \frac{\sqrt{2m}}{\hbar} \int_{r_1}^{r_2} dr \sqrt{\frac{2Ze^2}{4\pi\epsilon_0 r} - E} \end{aligned}$$

$$\gamma = \frac{\sqrt{2m}}{\hbar} \int_{r_1}^{r_2} dr \sqrt{\frac{2Ze^2}{4\pi\epsilon_0 r} \frac{r_2}{r} - E}$$

$$= \frac{\sqrt{2mE}}{\hbar} \int_{r_1}^{r_2} dr \sqrt{\frac{r_2}{r} - 1}$$

SUBSTITUTION  $r = r_2 \sin^2 \theta$

$$\sqrt{\frac{r_2}{r} - 1} = \frac{\cos \theta}{\sin \theta}$$

$$dr = 2r_2 \sin \theta \cos \theta d\theta$$

$$\gamma = \frac{\sqrt{2mE}}{\hbar} 2r_2 \int_{\sin^{-1}\sqrt{\frac{r_1}{r_2}}}^{\pi/2} d\theta \underbrace{\cos^2 \theta}_{\frac{1}{2}(\cos 2\theta + 1)}$$

$$= \frac{\sqrt{2mE}}{\hbar} r_2 \left[ \underbrace{\frac{1}{2} \sin 2\theta}_{\sin \theta \cos \theta} + \theta \right]_{\sin^{-1}\sqrt{\frac{r_1}{r_2}}}^{\pi/2}$$

$$= \frac{\sqrt{2mE}}{\hbar} r_2 \left[ 0 - \sqrt{\frac{r_1}{r_2}} \sqrt{1 - \frac{r_1}{r_2}} + \frac{\pi}{2} - \sin^{-1} \sqrt{\frac{r_1}{r_2}} \right]$$

$$= \frac{\sqrt{2mE}}{\hbar} \left[ r_2 \left( \frac{\pi}{2} - \sin^{-1} \sqrt{\frac{r_1}{r_2}} \right) - \sqrt{r_1(r_2 - r_1)} \right]$$

FOR  $\tau_2 \gg \tau_1$  (TYPICAL CASE)

$$\sin^{-1} \sqrt{\frac{\tau_1}{\tau_2}} \approx \sqrt{\frac{\tau_1}{\tau_2}}$$

$$\gamma \approx \frac{\sqrt{2mE}}{\hbar} \left[ \tau_2 \frac{\pi}{2} - 2 \sqrt{\tau_1 \tau_2} \right]$$

$$E \sim \frac{Z}{\tau_2} \Rightarrow \tau_2 \sim \frac{Z}{E}$$

$$\gamma = K_1 \frac{Z}{\sqrt{E}} - K_2 \sqrt{Z \tau_1}$$

WITH  $K_1, K_2$  CONSTANTS

$$K_1 \equiv \left( \frac{e^2}{4\pi \epsilon_0} \right) \frac{\pi \sqrt{2m}}{\hbar}$$

$$K_2 \equiv \left( \frac{e^2}{4\pi \epsilon_0} \right)^{1/2} \frac{4 \sqrt{m}}{\hbar}$$

$\hookrightarrow$  PROBABILITY OF ESCAPE OF  $\alpha$ -PARTICLE  $\sim e^{-2\gamma}$   
 TIME BETWEEN 2 COLLISIONS WITH WELL  $\sim \frac{2\tau_1}{v}$   
 $v$  = VELOCITY OF  $\alpha$ -PARTICLE

PROBABILITY OF ESCAPE / PER UNIT TIME

$$\sim \left( \frac{v}{2\kappa_1} \right) e^{-2\gamma}$$

LIFETIME  $\tau \approx \left( \frac{2\kappa_1}{v} \right) e^{2\gamma}$

ON LOG PLOT

$$\ln \tau \sim 2\gamma \sim \frac{1}{\sqrt{E}}$$

↳ e.g. FOR  $^{238}\text{U} \Rightarrow \tau \approx 4.5 \cdot 10^9 \text{ YEARS} \nabla$