Exercise sheet 9
Theoretical Physics 3 : QM SS2017
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16.06.2017

Exercise 1 – Algebraic method for the hydrogen atom. (85 points)

We consider the radial dimensionless equation for the Coulomb problem of the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{l(l+1)}{\rho^2} + \frac{2}{\rho}\right) u_{n,l}(\rho) = \epsilon\, u_{n,l}(\rho),$$

where $\rho = r/a$ (and a is the Bohr radius), and $u_{n,l}(\rho) = \rho R_{n,l}(\rho)$ is the reduced wave function which satisfies the conditions $\int_0^\infty |u_{n,l}(\rho)|^2 \mathrm{d}\rho = 1$ and $u_{n,l}(0) = 0$. We introduce the operators:

$$A_l^- = \frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{l+1}{\rho} - \frac{1}{l+1} \qquad A_l^+ = \frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{l+1}{\rho} + \frac{1}{l+1}.$$

a) *(10 p.)* Calculate $A_l^- A_l^+$. Show that the dimensionless radial equation can be written as

$$\left(A_l^- A_l^+\right) u_l = \left(\epsilon - \frac{1}{(l+1)^2}\right) u_l.$$

b) *(15 p.)* Show that

$$A_l^+ A_l^- = A_{l+1}^- A_{l+1}^+ - \frac{1}{(l+2)^2} + \frac{1}{(l+1)^2}.$$

By multiplying the equation from a) by A_l^+, show that $A_l^+ u_l(\rho)$ satisfies the radial equation with the same eigenvalue ϵ but for an angular momentum $l' = l + 1$.

c) *(15 p.)* Similarly, show that $A_{l-1}^- u_l(\rho)$ satisfies the radial equation with the same eigenvalue ϵ but for an angular momentum $l' = l - 1$.

d) *(15 p.)* Calculate the expectation value of $A_l^- A_l^+$ with $u_l(\rho)$, and show that $\epsilon \leq \frac{1}{(l+1)^2}$.

e) *(15 p.)* Show that, for a given value of ϵ, there exists a maximum value $l_{\max}$ of the angular momentum such that $\epsilon = 1/n^2$, where we have set $n = l_{\max} + 1$. Show that the corresponding radial wave function $u_{l_{\max}}$ satisfies the differential equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{n}{\rho} + \frac{1}{n}\right) u_{l_{\max}}(\rho) = 0.$$

f) *(15 p.)* Deduce from these results the energy levels and the corresponding wave functions of the hydrogen atom.

Exercise 2 – Hydrogenic atom. (15 points)

A hydrogenic atom consists of a single electron orbiting a nucleus with Z protons. Determine the Bohr energies $E_n(Z)$, the binding energy $E_1(Z)$, the Bohr radius $a(Z)$, and the Rydberg constant $R(Z)$ for a hydrogenic atom. Express your answers as appropriate multiples of the hydrogen values. Are these expressions still accurate for large Z? Why?