Exercise sheet 8
Theoretical Physics 3 : QM SS2017
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Exercise 1. (20 points)

Show that the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in three dimensions in spherical coordinates takes the form

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Exercise 2. (40 points)

To solve the radial equation for an infinite spherical well, we introduce spherical Bessel functions $j_l(x)$ of order l defined as:

$$j_l(x) \equiv (-x)^l\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^l\frac{\sin x}{x}.$$

A spherical Bessel functions is a particular case of a Bessel function $J_\alpha(x)$ defined as:

$$J_\alpha = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{x}{2}\right)^{2n+\alpha},$$

for α half integer, so $J_{n+1/2} = \sqrt{\frac{2x}{\pi}}\, j_n(x)$.

a) *(20 p.)* Using the definition of the Bessel function, compute $J_{1/2}$ and $J_{3/2}$ and check that, indeed, the relation between $J_{l+1/2}$ and j_l is correct.

The spherical Bessel functions have well defined $x \to 0$ and $x \to \infty$ limits:

$$j_l(x) \to \frac{2^l l!}{(2l+1)!}x^l \qquad \text{for } x << 1$$

$$j_l(x) \to \frac{1}{x}\cos\left(x - \frac{(l+1)\pi}{2}\right) \qquad \text{for } x >> 1$$

b) *(10 p.)* Compute both limits for j_0, j_1 and j_2.

c) *(10 p.)* Compute j_3 using the definition given at the beginning.

Math hints:

$$\sin(x) = \sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!}$$

$$\Gamma(m+1/2) = \frac{1\cdot 3\cdot 5\cdot (2m-1)}{2^m}\sqrt{\pi}$$

$$\int_0^\pi \sin^n(x)\mathrm{d}x = \frac{\Gamma(1/2+n/2)}{\Gamma(1+n/2)}\sqrt{\pi}$$

Exercise 3. (20 + 20 points)

Consider a particle confined in a three-dimensional square box with side a with *periodic* boundary conditions:

$$\begin{cases} \Psi(r_i = a,\, t) = \Psi(r_i = 0,\, t), \\ \frac{\partial}{\partial r_i}\Psi(r_i = a,\, t) = \frac{\partial}{\partial r_i}\Psi(r_i = 0,\, t). \end{cases}$$

a) *(20 p.)* Solve the Schrödinger equation in Cartesian coordinates.
Hint: Separate variables to reduce the problem to one-dimensional and solve it.

b) *(10 p.) (bonus)* Write down the energy spectrum.
What are the degrees of degeneracy of the four lowest energy states?

c) *(10 p.) (bonus)* Consider (fictitious) generalisation of the problem to *four* spatial dimensions.
How would the spectrum be compared to the spectrum of the isotropic harmonic oscillator?
Hint: There is the Lagrange's four-square theorem, which states that every natural number can be represented as the sum of four integer squares:

$$\forall n \in \mathbb{N} :\ \exists\, i, j, k, l \in \mathbb{Z} :\ n = i^2 + j^2 + k^2 + l^2.$$

Exercise 4. (20 points)

Using the three-dimensional Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\Psi(\vec{r}, t) = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\Psi(\vec{r}, t),$$

show that the probability density $\rho = \Psi^*\Psi$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0,$$

where the three-current $\vec{j} = \frac{\hbar}{2mi}\left[\Psi^*(\vec{\nabla}\Psi) - (\vec{\nabla}\Psi^*)\Psi\right]$ can be interpreted as the flow of probability through a unit surface.