Exercise sheet 6
Theoretical Physics 3 : QM SS2017
Lecturer : Prof. M. Vanderhaeghen

26.05.2017

Exercise 1. (35 + 20 points)

Consider Schrödinger equation with the following potential:

$$V(x) = \begin{cases} 0 & x < 0 \\ -V_0 & 0 < x < a \\ V_1 & x > a. \end{cases}$$

a) *(10 p.)* Consider bound states of the system ($E < 0$). Derive the transcendental equation(s) for the energy quantum number. Notice in the case $V_1 = 0$ one should recover expressions for the finite well problem considered in the lecture.

b) *(10 p.)* Write down the eigenfunctions of the Hamiltonian for the case $0 < E < V_1$. Sketch an eigenfunction for some intermediate value of E.

c) *(10 p.)* Consider scattering states for the case $E > V_1$. Derive expressions for the reflection and transmission coefficients.

d) *(5 p.)* Show that in the limit $V_1 \to +\infty$ the eigenfunctions of the Hamiltonian vanish for $x > a$. Is it true that not only the eigenfunctions of the Hamiltonian, but *all* the wave functions must vanish for $x > a$?

e) *(10 p.) (Bonus)* Consider the limit $V_1 \to +\infty$. Prove that the system admits no bound states if and only if $V_0 \leq \pi^2\hbar^2/(8ma^2)$.

f) *(10 p.) (Bonus)* Assume that in the limit $V_1 \to +\infty$ the system does not admit bound states ($\sqrt{2mV_0a^2/\hbar^2} < \pi/2$), whereas it is known that for $V_1 = 0$ the system always admits at least one bound state. Determine $\bar{V_1}$ such that for $V_1 < \bar{V_1}$ the system admits at least one bound state.

Exercise 2. – Hermitian operators (30 points)

Given that the two hermitian operators $\hat{A}$ and $\hat{B}$ commute $[\hat{A}, \hat{B}] = 0$, prove that

a) *(5 p.)* if $|\psi\rangle$ is an eigenvector of $\hat{A}$, then $\hat{B}\,|\psi\rangle$ is also an eigenvector of $\hat{A}$ with the same eigenvalue;

b) *(5 p.)* if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two eigenvectors of $\hat{A}$ with different eigenvalues, then $\langle\psi_1|\,\hat{B}\,|\psi_2\rangle = 0$;

c) *(15 p.)* one can construct an orthonormal basis made of eigenvectors common to both $\hat{A}$ and $\hat{B}$ (consider only the case of discrete spectra).

d) *(5 p.)* Prove the reverse of the last statement.

Exercise 3. – Operators and Dirac notation (35 points)

a) *(5 p.)* Recall the ladder operators for the quantum harmonic oscillator problem. Show that

$$\hat{a}_+ = (\hat{a}_-)^\dagger.$$

b) *(5 p.)* Show that for any observable $\hat{q}$ with *nondegenerate* spectrum

$$\hat{q} = \sum_q q \left|q\right\rangle \left\langle q\right| ,$$

where $\hat{q}\left|q\right\rangle = q\left|q\right\rangle$, and in the case of continuous spectrum $\sum_q \to \int \mathrm{d}q$.
Hint: since the set of eigenfunctions is complete and orthonormal, one can use

$$\sum_q \left|q\right\rangle \left\langle q\right| = \hat{1}.$$

c) *(5 p.)* We already know from the lecture that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}px}.$$

Show that

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle .$$

d) *(5 p.)* Show that

$$\langle \psi|\hat{x}|\psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}p\, \phi^*(p) \left[i\hbar\frac{\partial}{\partial p}\right]\phi(p),$$

where

$$\phi(p) \equiv \langle p|\psi\rangle$$

e) *(5 p.)* Show that

$$\left\langle x\middle|\hat{p}\middle|x'\right\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x-x').$$

f) *(10 p.)* Recall the infinite square well stationary state wave function

$$\langle x|n\rangle = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right) & 0 < x < a, \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\langle p|n\rangle$.