Exercise sheet 5
Theoretical Physics 3 : QM SS2017
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Exercise 1 – Free particle. (40 points)

Consider a free particle that moves in one dimension. Its wave function at $t = 0$ is given by two gaussians

$$\Psi(x,0) = A\left(e^{ik_0x}\, e^{-\frac{\alpha}{2}\,(x+x_0)^2} + e^{-ik_0x}\, e^{-\frac{\alpha}{2}\,(x-x_0)^2}\right)$$

where α and k_0 are real parameters.

a) *(10 p.)* Determine A such that the wave function is normalized to 1.

b) *(10 p.)* Compute the Fourier transform $\phi(k)$ of $\Psi(x,0)$.

c) *(15 p.)* Show that the wave function for any time t is given by

$$\Psi(x,t) = \frac{A}{\sqrt{z}}\left(e^{ik_0x}\, e^{-\frac{\alpha}{2z}\,(x+x_0-v_0t)^2} + e^{-ik_0x}\, e^{-\frac{\alpha}{2z}\,(x-x_0+v_0t)^2}\right),$$

where we have introduced $z \equiv 1 + i\frac{\alpha\hbar t}{m}$ and $v_0 = \frac{\hbar k_0}{m}$.

d) *(5 p.)* Write down the position probability density and interpret it.

Exercise 2 – Matrices: eigenvalues and eigenvectors. (20 points)

a) *(5 p.)* Find the eigenvalues and eigenvectors of the 2D rotation and hyperbolic rotation matrices, correspondingly:

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}; \qquad \begin{bmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{bmatrix}.$$

b) *(5 p.)* Find the eigenvalues and eigenvectors of the following 3×3 matrices:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}; \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}.$$

c) *(10 p.)* Diagonalise the following matrices:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}; \qquad \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}.$$

Exercise 3 – Double δ-potential. (40 points)

Consider a one-dimensional model for a molecule in which the levels are doubly degenerate following the potential:

$$V(x) = -V_0 a \left(\delta(x-a) + \delta(x+a)\right),$$

where V_0 and a are real parameters.

a) *(15 p.)* Find the reflection and transmission coefficients for a beam of particles incident on that potential.

b) *(10 p.)* Using the momentum space representation, find the bound state of the system. How many bound states does the system have?

c) *(10 p.)* For $V_0 a = \frac{\hbar^2}{ma}$, find the allowed energies for the bound states and sketch the corresponding wave functions.

d) *(5 p.)* Discuss finally the role of the parameter a on the wave func- tions, that is, after finding the even and odd wave functions of the system, what is the a dependence on the solutions.

(Bonus) Exercise 4 – Finite square well potential. (20 points)

A particle of mass m is moving in a finite square potential well

$$V(x) = \begin{cases} -\frac{\alpha}{2a} & \text{for } -a \leq x \leq a \\ 0 & \text{for } x > |a| \end{cases}.$$

The energy levels are determined by the condition

$$z \tan z = \sqrt{z_0^2 - z^2}$$

where

$$z = \frac{a}{\hbar}\sqrt{2m\left(E + \frac{\alpha}{2a}\right)}, \qquad z_0 = \frac{a}{\hbar}\sqrt{\frac{m\alpha}{a}}.$$

a) *(10 p.)* Considering the limit $a \to 0$ and assuming that E is finite in this limit, show that you recover the unique bound state of the δ potential well $E = -\frac{m\alpha^2}{2\hbar^2}$.

b) *(5 p.)* What should be the value of $m\alpha a/\hbar^2$ in order for the system to have precisely n bound states?