Exercise sheet 4
Theoretical Physics 3 : QM SS2017
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Exercise 1 – 1D Fourier transform. (30 points)

We define the (spatial) Fourier transform of a wave function $\Psi(x,t)$, and its corresponding inverse transform as

$$\Phi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,t) e^{-ikx} \mathrm{d}x,$$
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k,t) e^{ikx} \mathrm{d}k.$$

a) *(10 p.)* Apply the Fourier transform to write down the Schrödinger equation representation for the quantum harmonic oscillator *in the k-domain.*
Hint: Use $\int_{-\infty}^{\infty} x e^{ax} \mathrm{d}x = \frac{\partial}{\partial a} \int_{-\infty}^{\infty} e^{ax} \mathrm{d}x$

b) *(10 p.)* Write down the Schrödinger equation with an arbitrary potential *in the k-domain.*
Assume the potential is expandable in the power series $V(x) = \sum_n a_n x^n$.

c) *(10 p.)* Solve the one-dimensional *free particle* Schrödinger equation *in the k-domain.*
Then apply a reverse Fourier transform to write down the general solution in the spatial domain.

Exercise 2 – Inner product space. (20 points)

a) *(10 p.)* Prove the Cauchy-Bunyakovsky-Schwarz inequality:

$$\forall\ |\alpha\rangle, |\beta\rangle : \quad |\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle \langle\beta|\beta\rangle.$$

Hint: Consider the norm of the vector $|\psi\rangle = |\alpha\rangle - \lambda|\beta\rangle$ with $\lambda = \langle\beta|\alpha\rangle / \langle\beta|\beta\rangle$.

b) *(10 p.)* Show that there is a strict equality $|\langle\alpha|\beta\rangle|^2 = \langle\alpha|\alpha\rangle \langle\beta|\beta\rangle$ *if and only if* $|\alpha\rangle$ and $|\beta\rangle$ are proportional.

Exercise 3 – Matrices. (60 points)

Recall the matrix multiplication is a non-commutative operation. We define the commutator of two matrices A and B as $[A,B] = AB - BA$.

a) *(10 p.)* Consider three Pauli matrices:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Compute σ_x^2, σ_y^2, σ_z^2, and the commutators $[\sigma_x, \sigma_y]$, $[\sigma_y, \sigma_z]$, $[\sigma_z, \sigma_x]$.

b) *(10 p.)* Prove the following identities for arbitrary matrices A, B and C:

$$[A, BC] = [A, B]C + B[A, C],$$
$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

c) *(10 p.)* We define a function of a matrix variable $f(A)$ through the Maclaurin series expansion (assuming it's possible):

$$f(A) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} A^n.$$

Consider the case $[A, B] = 0$. Prove that

$$e^A \, e^B = e^{A+B}.$$

d) *(15 p.)* Consider *another* specific case when $[A, [A, B]] = [B, [A, B]] = 0$.
Prove by induction, that

$$[A, B^n] = [A, B] n B^{n-1}.$$

Then show that

$$[A, F(B)] = [A, B] F'(B),$$

where F' is the function obtained by differentiation of F.

e) *(15 p.)* Consider the same case $[A, [A, B]] = [B, [A, B]] = 0$, and prove the Glauber formula

$$e^A \, e^B = e^{A+B} \, e^{\frac{1}{2}[A,B]}.$$

Hint: Consider the function $F(t) = e^{tA} \, e^{tB}$. Show that it has to satisfy the differential equation $\frac{\mathrm{d}F(t)}{\mathrm{d}t} = (A+B+t[A, B])F(t)$. Then solve the equation by noting that $(A+B)$ and $[A, B]$ commute, and, hence, can be treated as mere numbers.

(Bonus) Exercise 4. (20 points)

The relativistic form of the energy for a free particle with momentum $p = \hbar k$ is given by

$$E(p) = \sqrt{p^2 c^2 + m^2 c^4},$$

where c is the velocity of light in the vacuum and m the mass of the free particle. The angular frequency is then defined as follows

$$\omega(k) = \frac{E(\hbar k) - E(0)}{\hbar}$$

Compute the relativistic expression for the group and phase velocities. Check that they cannot be larger than c. The non-relativistic limit corresponds formally to $c \to \infty$. Check that you recover the non-relativistic expression for the group and phase velocities in that limit.