

Exercise sheet 3
Theoretical Physics 3 : QM SS2017
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Exercise 1. (35 points)

Consider a quantum harmonic oscillator, the time-independent ground state wave function of which is given by

$$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega x^2}{2\hbar}} \equiv \alpha e^{-\frac{y^2}{2}},$$

where, for further simplicity, we have introduced $\alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$ and dimensionless variable $y = \sqrt{\frac{m\omega}{\hbar}}x$.

a) (5 p.) Using explicit definition of the raising ladder operator

$$a_+ = \frac{1}{\sqrt{2\hbar\omega m}}(-i\hat{p} + m\omega x) \equiv \frac{1}{\sqrt{2}}\left(-\frac{d}{dy} + y\right),$$

derive expression for the first excited state wave function ψ_1 and check its orthogonality to ψ_0 .

b) (20 p.) Compute $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$ and $\langle p^2 \rangle$, for the states ψ_0 and ψ_1 by explicit integration.

c) (5 p.) Check the uncertainty principle for these two states.

d) (5 p.) Compute expectation values of the kinetic energy $\langle T \rangle$ and the potential energy $\langle V \rangle$. Check these to sum up to $\langle H \rangle$.

Exercise 2. – Harmonic oscillator: power series method (65 points)

The quantum harmonic oscillator problem can be solved using the power series method. One starts with the stationary Schrödinger equation ($\psi'' \equiv d^2\psi/dx^2$)

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}m\omega^2x^2\psi(x) = E\psi(x).$$

a) (10 p.) To simplify the initial problem, rewrite the equation using the dimensionless quantities

$$y = \sqrt{\frac{m\omega}{\hbar}}x, \quad \varepsilon = E/\hbar\omega.$$

Further on, define $\varphi(y) = c\psi(x)$ and find c , such that $\varphi(y)$ is normalized.

b) (10 p.) Investigate the asymptotical behaviour of the equation for large y . Show that for $y \rightarrow \infty$

$$\varphi(y) \sim e^{-\frac{y^2}{2}}.$$

c) (10 p.) We can now explicitly isolate the asymptotic behaviour of the unknown function:

$$\varphi(y) = h(y) e^{-\frac{y^2}{2}}.$$

Derive the following equation on $h(y)$:

$$h'' - 2yh' + (2\varepsilon - 1)h = 0.$$

d) (15 p.) At this point, assume that $h(y)$ can be written as an infinite power series in y

$$h(y) = \sum_{m=0}^{\infty} a_m y^m.$$

Derive the recurrence relation between the coefficients a_m and show that there are two sets of independent solutions (*even* and *odd*).

e) (15 p.) Prove that, in order for the wave function to be finite and normalizable, one has to imply that the infinite series must be “cut off” at some finite integer n : $a_{m>n} = 0$.

(Hint: consider Maclaurin expansion of e^{y^2} and compare it to the series behaviour for large y).

f) (5 p.) Using the previous conclusion, show that the energy is quantized $E_n = (n + \frac{1}{2})\hbar\omega$.

The obtained polynomials $h_n(y)$ are proportional to *Hermite polynomials* $H_n(y)$. The orthonormal set of solutions of the initial stationary Schrödinger equation then reads:

$$\psi_n(x) = \left(2^n n! \sqrt{\frac{\pi \hbar}{m\omega}}\right)^{-\frac{1}{2}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{2\hbar}}, \quad n = 0, 1, 2, \dots$$

(Bonus) Exercise 3. – Supersymmetric QM (50 points)

We consider in this exercise a generalization of the raising and lowering operator method. For a given potential $V_-(x)$, the idea is to construct a partner potential $V_+(x)$ which has the same energy eigenvalues, except for the ground state. Without loss of generality, we can shift the potential $V_-(x)$ so that the corresponding ground state $\psi_0(x)$ has zero energy $E_0^- = 0$.

a) (10 p.) Show that the Hamiltonian $H_- = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_-$ can be written in the form

$$H_- = A^+ A$$

with

$$A^+ \equiv \frac{\hbar}{\sqrt{2m}} \left(-\frac{d}{dx} - \frac{\psi'_0}{\psi_0}\right),$$

$$A \equiv \frac{\hbar}{\sqrt{2m}} \left(\frac{d}{dx} - \frac{\psi'_0}{\psi_0}\right),$$

and $\psi'_0 \equiv \frac{d}{dx}\psi_0$.

b) (10 p.) Consider the partner Hamiltonian $H_+ = AA^+$ which can also be defined as $H_+ = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_+$. Show that the partner potential V_+ is related to V_- as follows

$$V_+ = V_- - \frac{\hbar^2}{m} \frac{d}{dx} \left(\frac{\psi'_0}{\psi_0}\right)$$

c) (15 p.) Show that H_- and H_+ have same spectrum, except for the ground state (Hint: Consider the states $A\psi_n^-$ and $A^+\psi_n^+$ with ψ_n^\pm eigenstates of H_\pm). Write the eigenstates ψ_n^+ and energies E_n^+ in terms of ψ_n^- and E_n^- .

d) (15 p.) Consider a particle in the infinite square potential well

$$V_-(x) = \begin{cases} V_0 & \text{for } 0 \leq x \leq a \\ +\infty & \text{otherwise,} \end{cases}$$

Find V_0 such that the ground state has zero energy. Derive the partner potential $V_+(x)$. Write down the properly normalized eigenstates $\psi_n^+(x)$. Explain why the existence of partner potentials may be useful.