Exercise sheet 10
Theoretical Physics 3 : QM SS2017
Lecturer : Prof. M. Vanderhaeghen

23.06.2017

Exercise 1. (40 points)

Show that the simultaneous eigenstates of the operators L^2 and L_z, where $\vec{L} = \vec{r} \times \left(\frac{\hbar}{i}\vec{\nabla}\right)$ is the orbital angular momentum operator, are the spherical harmonics $Y_{lm}(\theta, \phi)$.

Hints: Use the spherical coordinates to compute L_x, L_y and L_z as functions of θ and ϕ. Write explicitly the product L_+L_-, where $L_\pm = L_x \pm iL_y$. Express the eigenstate equations for L^2 and L_z in a differential form and show that they are exactly the equations defining spherical harmonics.

Exercise 2. (30 points)

Show that $L_\pm Y_l^m = \hbar\sqrt{l(l+1) - m(m \pm 1)}\, Y_l^{m\pm1}$.
Hint: Consider the norm of $L_\pm Y_l^m$.

Exercise 3. (30 points)

Prove that for a particle in a potential $V(\vec{r})$ the rate of change of the expectation value of the orbital angular momentum $\vec{L}$ is equal to the expectation value of the torque (rotational analog to Ehrenfest's theorem)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{L}\rangle = \langle\vec{N}\rangle \qquad \text{where} \qquad \vec{N} = \vec{r} \times (-\vec{\nabla}V).$$

Show that $\langle\vec{L}\rangle$ is constant for any spherically symmetric potential. (This is one form of the quantum statement of conservation of angular momentum.)

Exercise 4. (*Bonus* 20 points)

Confirm or invalidate the following assertions:

a) *(10 p.)* If $[\hat{H}, \hat{\vec{L}}] = \vec{0}$, the energy levels do not depend on m (i.e., on the eigenvalues of the projection of one of the components of the angular momentum $\hat{\vec{L}}$).

b) *(10 p.)* If $[\hat{H}, \hat{L}^2] = 0$, the energy levels do not depend on l.