Compton Scattering Sum Rules for Massive Vector Bosons

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Contents

Introduction

Historical Background

In the mid-1960s, S.D. Drell and A.C. Hearn, and, independently, S.B. Gerasimov, developed a relation to determine the anomalous magnetic moment from polarized Compton scattering (CS) on nuclei in the framework of dispersion theory, the *GDH sum rule* (in short, GDH) [Ger66, DH66]. Sum rules are a special kind of dispersion relations. Dispersion relations are useful as it is possible e.g. to calculate 2-loop contributions from pure one-loop cross sections (c.f. [Kni96]). In case of the GDH, the sum rule allows the exact calculation of the anomalous magnetic moment from the measured cross section, i.e. connect a low-energy constant to a dynamical spectral integral.

The GDH sum rule has later been extended to particles of arbitrary spin [Lin71]. With the advent of *quantum chromodynamics* (QCD) as a complete theory of strong interactions, however, interest in the dispersion theoretic approach has languished. In recent years, the GDH has come to new fame. New experiments, e.g. at the Mainz Mikrotron facility (MAMI) [Tho06], and theoretical predictions, e.g. by L. Tiator [Tia00, Tia02] and D. Drechsel et al. [DKT01], further solidified the validity of the sum rule.

In 1966, Hosoda and Yamamoto showed [HY66] that the sum rule can also be gained from equal-times current algebra theory. The dispersion theoretic approach however is stronger as the current algebra approach relies on some assumptions which cannot be proven generally [*ibid.*, footnote 2]. For this reason, we will focus solely on the dispersion theory approach in this work.

Purpose of this Study

A particle of arbitrary spin j has in general $2j+1$ electromagnetic moments. While there have been several studies on generalizing the GDH to higher spins [Pai67, LC75] and the extension of the GDH to arbitrary spin has proven to be valid [DHK$^+$04], there has been no rigorous determination of sum rules for higher order electromagnetic moments. Although there have also been some efforts to derive sum rules for higher order electromagnetic moments (e.g. by Pais [Pai68], Lin [LC75], and Ji et al. [JL04, CJL04]), no consistent form has been found so far which considers all contributions.

It is the purpose of this study to fill in this gap for the $j = 1$ case. In the course of this study we have derived the GDH for massive vector bosons. Additionally, we have derived a new quadrupole sum rule (QSR). These sum rules have been put to test in the framework of a quantum field theory. We find that the GDH is verified only for particular theories of massive charged spin-1 fields, pointing towards higher symmetries. For the QSR, we find that polarizabilities have to be taken into account in order to give a complete description.

Structure of this Thesis

In chapter 1, the important concepts and methods for this work are introduced. For the gauge field theory part, we follow the approaches by Chaichian and Nelipa [CN84] and Aitchison [Ait80]. For the introduction to dispersion theory, the works of Nussenzveig [Nus72], and Queen and Violini [QV74] are used for reference.

In chapter 2 an effective Lagrangian for massive vector bosons is constructed. We introduce the concept of natural values and discuss its importance for the testing of the theory.

Chapter 3 introduces Compton scattering off massive vector bosons and continues with a detailed derivation of the GDH for spins $j = 1/2$ and $j = 1$, as well as a new quadrupole sum rule for the latter case. In order to accomplish this, low-energy theorems (LETs) for the magnetic dipole and electric quadrupole structure functions are derived from the effective Lagrangian. An excellent primer for the derivation of the original GDH is the PhD thesis by R. Pantfoerder [Pan98], to which we have adhered for our analogous calculations.

Chapter 4 is dedicated to the verification of the sum rules based on the effective

Lagrangian in the limit of natural values. At tree level, the GDH can be confirmed. However, at one-loop we find that the effective Lagrangian is not complete. An additional self-interaction tadpole is needed to derive the anomalous magnetic moment. Additionally, for the quadrupole sum rule, it is found that even at tree level polarization effects might give a contribution, since the QSR yields a finite, non-zero result.
The SU(2) Yang-Mills ansatz and the resulting tadpole contribution is given in chapter 5, where the complete Yang-Mills result at one-loop is discussed. As a concluding remark, the anomalous magnetic moment for $j = 1$ is compared qualitatively to the value for spin $1/2$. The relation to the experimental results for deuteron and W bosons is commented upon.
The conclusion is followed by an appendix containing Feynman rules for both theories, the vertex diagrams, and some helpful references on the handling of loop integrals.

Tools

In the course of this work, two major tools have been used in the calculation process. Besides manual calculations, we used Nikhef's FORM algebra manipulation program [Ver00] and Wolfram's Mathematica 6 suite. FORM has primarily been used to derive the structures used in this work, i.e. the diagrams and the LETs, as well as to confirm the Ward-Takahashi-identities. In Mathematica we derived the amplitude decompositions and identified the LETs with the appropriate dispersion relations. The sum rule tree-level integrations were done in Mathematica, which we also used to plot the integrands.
For the creation of this work free software was used where possible. The thesis was written in LaTeX using Kile and Aquamacs Emacs. The document is based on a template by T. Beranek. It uses the KOMA-Script scrbook class. Also, the physics-related TeX package PhysTeX by F. Jung [Jun02] was used, with some personal extensions. All Feynman diagrams and other illustrations were either created in JaxoDraw [BT04, BCKT08] or done using PGF and TikZ [Tan06].

Notation and Conventions

Throughout this thesis, the Einstein summation convention is used, i.e. summation is implied over indices which appear twice. If not mentioned otherwise, latin letters imply

Euclidian metric and summation over $i = 1, 2, 3$, greek letters stand for Minkowski metric and summation over $0, .., 3$. We use the metric convention

$$(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1). \tag{1}$$

As is common in particle and high-energy physics, we use the system of natural units with $\hbar = c = 1$. The structure constant of the electromagnetic interaction is $\alpha = \frac{e^2}{4\pi} \approx {}^1\!/_{137}$. The electric charge e is defined such that electrons carry the charge $-e$.

For loop integrals we will often use the abbreviation

$$\mathrm{d}\tilde{k} := \frac{\mathrm{d}^D k}{(2\pi)^D},$$

where D is the number of dimensions.

When referring to the terms of different order in the perturbative expansion of the amplitudes, we use the abbreviations LO for leading-order and NLO for next-to-leading-order.

Chapter 1

Physical method

1.1 Introduction to Field Theory

The mathematical framework of particle physics is quantum field theory (QFT). One can transit from the formalism of classical mechanics to field theory by replacing the n trajectories with—here, quantized—fields:

$$q_i(t) \to \phi(\boldsymbol{x}, t). \tag{1.1}$$

While in classical mechanics the n generalized coordinates of a system represent n degrees of freedom (*d.o.f.*s), the transformation $i \to \boldsymbol{x}$ induces infinite *d.o.f.*s, with $\phi(\boldsymbol{x}, t)$ being *one d.o.f.* at a given point $\boldsymbol{x}$. In (quantum) field theory, the generalized variables are the fields (operators) and derivatives thereof, so that the Lagrange density can be written as

$$\mathcal{L} = \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right). \tag{1.2}$$

The four-dimensional spatial integral over $\mathcal{L}$ is the *action*:

$$S = \int \mathrm{d}^4 x \, \mathcal{L}, \tag{1.3}$$

assuming that the fields vanish at infinity. Since the action is a Lorentz invariant function and it is stationary under arbitrary variations of the field $\delta\phi_i(x)$ (*action principle*),

$$\delta S = \int \mathrm{d}^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))} \right\} \delta\phi_i(x) \stackrel{!}{=} 0, \tag{1.4}$$

the term in brackets has to vanish so that we obtain the Lorentz covariant equations of motion from the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0. \tag{1.5}$$

Noether's Theorem

EMMY NOETHER[1] found one of the most important theorems in physics. It connects global symmetries to conserved quantities:

Theorem 1.1.1. *To any continuous global symmetry which leaves the Lagrange density invariant, there exists a corresponding law of conservation and a conserved quantity:*

$$\delta\mathcal{L} = 0 \quad \Leftrightarrow \quad \partial_\mu J^\mu = 0, \tag{1.6}$$

where $\delta\mathcal{L}$ denotes the variation of the Lagrange density. The conserved quantity is also called Noether charge, which is a spatial integral over the charge density,

$$Q(t) = \int d^3x\, J^0(t, \boldsymbol{x}). \tag{1.7}$$

Examples for Field Theories

Particles are divided into two inherently different types depending on their spin; half-integer spin particles, called *fermions*[2], and integer spin particles, called *bosons*[3]. They differ in one decisive property: fermions obey the *Fermi-Dirac statistics theorem*, implying that no two fermions can occupy the same state following the *Pauli exclusion principle*. Bosons, on the other hand, obey *Bose-Einstein statistics*, which means that there is no boundary on the number of particles which can occupy the same state. This also implies that particles in the same state are indistinguishable, in contrast to *classical* particles. Massive spin 0- and spin-$1/2$ particles are described by the Klein-Gordon and Dirac theory, respectively [BD65]:

For a free massive, neutral scalar field ($j = 0$, $m \neq 0$) the Lagrange density is given by

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu \phi^*)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2, \tag{1.8}$$

[1] Amalie "Emmy" Noether, *1882 Erlangen, Germany; †1935 Bryn Mawr (PA), USA
[2] named after Enrico Fermi, *1901 Rome, Italy, †1954 Chicago (IL), USA
[3] named after Satyendra Nath Bose, *1894 Kolkata, India, †1974, ibid.

with the equation of motion

$$\left(\Box + m^2\right)\phi = 0, \tag{1.9}$$

known as the *Klein-Gordon equation*. The equation implies the validity of the theory of *special relativity*: any relativistic covariant free field satisfies this equation (component-wise for n-dimensional fields).
A massive, charged spin-1/2 field is described by the *Dirac equation* resulting from the Lagrangian

$$\mathcal{L}_{Dirac} = \mathrm{i}\left(\bar{\psi}\partial\!\!\!/\psi\right) - m\bar{\psi}\psi \tag{1.10}$$

$$\Rightarrow -\mathrm{i}\gamma^\mu\partial_\mu\psi + m\psi = 0. \tag{1.11}$$

1.2 Gauge Theory

Gauge theories are the very foundation of modern physics. The concept of theories with intrinsic symmetries, meaning that they are invariant under certain transformations, has led to the development of many successful theories improving our understanding of nature in many ways. This formalism connects Noether's theorem with the concept of group theory: to each conserved quantity there exists an underlying symmetry which can be expressed by a symmetry group. Hence, the theory is invariant under the action of a symmetry group G, being a representation of a Lie algebra $\mathcal{G}$. There are different types of symmetries which we will discuss in the following.

1.2.1 Internal Symmetries and Global Transformations

Transformations of a Lagrangian $\mathcal{L}(\phi_i, \partial_\mu\phi_i)$ under actions of an arbitrary symmetry group which transform only the fields while leaving the coordinates untouched,

$$\phi_i(x) \to \phi_i'(x) = \phi(x) + \delta\phi_i(x) \qquad\qquad (x \to x' = x), \tag{1.12}$$

are called *internal symmetries*. The symmetry group is defined through a corresponding *Lie algebra*

$$[T_i, T_j] = f_{ijk}T_k, \tag{1.13}$$

where the T_i are the *generators* of the group. The f_{ijk} are the structure constants of the Lie algebra. They fulfill the total antisymmetry relation and the Jacobi identity,

$$f_{ijk} = -f_{jik}, \tag{1.14a}$$

$$f_{ijk} f_{klm} + f_{jlk} f_{kim} + f_{lik} f_{kjm} = 0. \tag{1.14b}$$

An infinitesimal transformation (ϵ_k being an arbitrary global infinitesimal parameter) is then represented by

$$\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \delta\phi_i(x), \tag{1.15a}$$

$$\text{where} \qquad \delta\phi_i(x) = (T^k)_{ij} \epsilon_k \phi_j(x). \tag{1.15b}$$

The invariance condition can be derived in analogy to the Euler-Lagrange equations: For infinitesimal transformations, the invariance of the action S of a system can be written as

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) \right) = 0. \tag{1.16}$$

Since the integral has to fulfill this identity regardless of the path of integration, the integrand has to be zero,

$$\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) = 0. \tag{1.17}$$

Using the identity $\delta(\partial_\mu \phi_i) = \partial_\mu \delta\phi_i$, this can be simplified further to

$$\underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \right)}_{=0 \quad \text{Euler-Lagrange eq.s}} \delta\phi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i \right) = 0 \tag{1.18}$$

from which we obtain the Noether current $\mathcal{J}^{a,\mu}$ by using the definition of $\delta\phi_i$,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \varepsilon_a T^a \phi_i \right) = \varepsilon_a \partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} T^a \phi_i \right)}_{=:\mathcal{J}^{a,\mu}} = 0 \tag{1.19}$$

which is conserved.

1.2.2 Local Invariance and Gauge Groups

A *gauge group* is a group of *local* (or *gauge*) transformations

$$\phi_i(x) \to \phi'_i(x) = \phi_i(x) + (T^k)_{ij}\epsilon_k(x)\phi_j(x). \tag{1.20}$$

Local transformations are used to describe the dynamics of particle fields and their interaction with *gauge fields* in so-called *gauge theories.*

Despite being invariant under global transformations, the invariance of a theory can be broken locally; that is, by a transformation that is characterized by a parameter depending on the coordinate, $\epsilon = \epsilon(x)$. Local transformations can result in non-vanishing terms

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)} T^k_{ij}\phi_j(x)\partial_\mu\epsilon_k(x) \neq 0 \tag{1.21}$$

However, we can regain the invariance by introducing so-called *gauge fields* $\mathcal{A}'^a_l(x)$ such that the new Lagrangian is invariant under local transformations.

The Lagrangian of the free gauge fields is given by

$$\mathcal{L} = -\frac{1}{4}\sum_{a=1}^{n}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a,\mu\nu}, \tag{1.22}$$

where $\mathcal{F}^a_{\mu\nu}$ is the field strength tensor,

$$\mathcal{F}^a_{\mu\nu} = \partial_\mu\mathcal{A}^a_\nu - \partial_\nu\mathcal{A}^a_\mu - f_{abc}\mathcal{A}^b_\mu\mathcal{A}^c_\nu. \tag{1.23}$$

If the generators commute, the theory is called *abelian.* In this case, the generators are diagonal and the structure constants vanish. An example for such a theory is QED. If the generators are non-commuting, the theory is consequently called *non-abelian.* In the following, we will discuss a description of non-abelian gauge theories, the Yang-Mills theory.

1.2.3 Nonabelian Gauge Theory: Yang-Mills Theory

C.N. Yang and R.L. Mills [YM54] proposed in 1954 a locally invariant Lagrangian for non-abelian gauge fields which is now referred to as Yang-Mills theory. Its renormalizability was proven by Gerard ’t Hooft [’t 71]. The theory is described by the

Lagrangian

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\,\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = -\frac{1}{4}\sum_{a=1}^{N^2-1}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a,\mu\nu}, \qquad \mathcal{F}_{\mu\nu} = \mathcal{F}^a_{\mu\nu}T_a, \tag{1.24}$$

which is invariant under the gauge group SU(N). The generators of the symmetry group satisfy

$$\left[T^a, T^b\right] = \mathrm{i} f^{abc}T_c. \tag{1.25}$$

The covariant derivative for the SU(N) Yang-Mills gauge theory is defined as

$$D_\mu = \partial_\mu - \mathrm{i}g\,T_a\mathcal{A}^a_\mu, \tag{1.26}$$

where g is the SU(N) coupling constant. The field strength tensor in eq. (1.23) can be derived from the commutation relation of the covariant derivative,

$$[D_\mu, D_\nu] = -\mathrm{i}g\,T_a\mathcal{F}^a_{\mu\nu}. \tag{1.27}$$

In modern physics, Yang-Mills theories play an important role. In the standard model of particle physics, the description of strong interactions, *quantum chromodynamics* (QCD), is an SU(3) Yang-Mills theory. Also, the electroweak interaction is of Yang-Mills type, with the underlying symmetry being SU(2) $\times$ U(1).

Due to Noether's theorem (c.f. section 1.1), the Lagrangian (1.24) leads to the conserved currents

$$\mathcal{J}^a_\mu = f_{abc}\,\mathcal{A}^{b,\nu}\mathcal{F}^c_{\mu\nu} \quad \Leftrightarrow \quad D^\mu\mathcal{J}^a_\mu = 0. \tag{1.28}$$

1.3 S-Matrix Formalism and Dispersion Theory

The S-Matrix formalism is an approach to describe scattering processes commonly used in modern physics. It was first introduced by Wheeler [Whe37] based on the Heisenberg description of interactions. The scattering Matrix, or S-Matrix, can be decomposed as $S = I + \mathrm{i}\,T$, where T is the transition matrix which is defined such that

$$\langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(P_f - P_i)\mathcal{M}_{fi}. \tag{1.29}$$

The formalism is based on the assumption that the S-Matrix fulfills, besides Lorentz invariance, fundamental properties like unitarity, analyticity, and crossing symmetry. From unitarity, $S^\dagger S = 1$, we obtain

$$-\,\mathrm{i}(T - T^\dagger) = T^\dagger T. \tag{1.30}$$

From this, we can follow by virtue of *Cutkosky's rule* that the absorptive part of the transition amplitude is proportional to the sum over all kinematically allowed intermediate states [Kni96],

$$\mathrm{Abs}\, T = 2\,\mathrm{Im}\,\mathcal{M}_{fi} = (2\pi)^4 \sum_n \delta^{(4)}(P_n - P_i)\mathcal{M}^*_{nf}\mathcal{M}_{ni} \tag{1.31}$$

As a consequence of the postulate about analyticity, scattering variables can be continued into the complex plane if they are expressed as functions of specific kinematic variables. This property is on of the foundations of dispersion theory.

1.3.1 Causality and Analyticity

There is a direct connection between *analyticity* and *causality* which we will show in the following. A physical system is *analytic* if it satisfies the conditions

(i) Linearity (superposition principle),

(ii) Time-translation invariance, and

(iii) Causality.

Consider a simple scattering process of massless particles, written in the framework of the propagation of waves along one axis. The incident wave packet is a superposition of plane waves,

$$\psi_{\mathrm{inc}}(x, t) = \int_{-\infty}^{\infty} A(\nu) \exp \mathrm{i}\nu(x - ct)\mathrm{d}\nu, \tag{1.32}$$

and the corresponding scattered wave is

$$\psi_{\mathrm{sc}}(r, t) = \int_{-\infty}^{\infty} A(\nu)F(\nu)\frac{\exp \mathrm{i}\nu(r - ct)}{r}\mathrm{d}\nu, \tag{1.33}$$

where $F(\nu)$ is the forward scattering amplitude. Now, for a scattering process at time t, let $G(t)$ be the incident wave seen by the target at $x = 0$ and $H(t)$ be the scattered wave at some arbitrary but fixed observation point R,

$$G(t) \equiv \psi_{\text{inc}}(0,t) = \int\limits_{-\infty}^{\infty} A(\nu)\mathrm{e}^{-\mathrm{i}\nu t}\mathrm{d}\nu \tag{1.34}$$

$$H(t) \equiv \psi_{\text{sc}}(R,t) = \int\limits_{-\infty}^{\infty} A(\nu)F(\nu)\frac{\mathrm{e}^{\mathrm{i}\nu(R-t)}}{R}\mathrm{d}\nu \tag{1.35}$$

The output of the scattering process can be rewritten using

$$L(t,t') \equiv \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f(\nu')\frac{\mathrm{e}^{\mathrm{i}\nu R}}{R}\mathrm{e}^{\mathrm{i}\nu(t-t')} \tag{1.36}$$

so that

$$\begin{aligned} H(t) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}t'\int\limits_{-\infty}^{\infty}\mathrm{d}\nu' f(\nu')\frac{\mathrm{e}^{\mathrm{i}\nu R}}{R}\mathrm{e}^{\mathrm{i}\nu' t'}\underbrace{\int\limits_{-\infty}^{\infty}\mathrm{d}\nu A(\nu)\mathrm{e}^{\mathrm{i}\nu t'}}_{=G(t')} \\ &= \int\limits_{-\infty}^{\infty}\mathrm{d}t' L(\tau)G(t') \end{aligned} \tag{1.37}$$

where we have used that L is obviously linear, $L(t,t') = L(t-t') =: L(\tau)$. Imposing the causality condition which is a property of physical amplitudes gives the constraint

$$L(\tau) = 0 \quad \text{for} \quad \tau < 0. \tag{1.38}$$

By virtue of *Titchmarsh's theorem*, this implies for $L(\tau)$ that its Fourier transform $\widetilde{L}(\nu)$ is analytic in the upper half of the complex energy plane. By comparing the Fourier relation with the definition of L, eq. (1.36), we find

$$L(\tau) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\widetilde{L}(\nu)\mathrm{e}^{-\mathrm{i}\nu\tau}\mathrm{d}\nu \tag{1.39}$$

$$\Rightarrow \widetilde{L}(\nu) = F(\nu)\frac{\mathrm{e}^{-\mathrm{i}\nu R}}{R}. \tag{1.40}$$

It immediately follows that $F(\nu)$ is analytic. Due to the *Schwarz reflection principle*, $F(\nu^*) = F^*(\nu)$, the amplitude is also analytic in the lower half plane.

1.3.2 Dispersion Relations

For a massive particle, the amplitude $F(\nu)$ will have branch cuts along the real axis, see fig. 1.1. Due to analyticity we can apply Cauchy's theorem,

$$F(\nu) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{\mathrm{d}\nu'}{\nu' - \nu} F(\nu'). \tag{1.41}$$

The integration path is shown in fig. 1.1. Since we are usually interested in physical

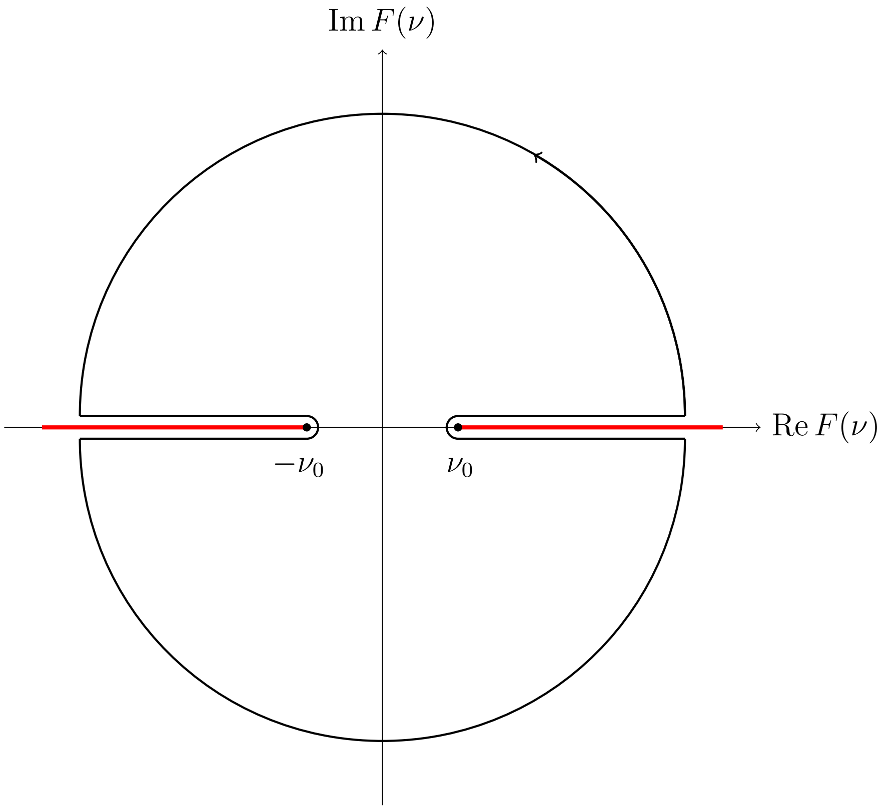


Figure 1.1: Contour of integration in the complex energy plane used to derive the dispersion relation. The half-circles are blown up to infinity.

energies, we choose to evaluate $F(\nu)$ at $\nu = x + \mathrm{i}\varepsilon$ for real x. We can split the integral into curve integrals along the semicircles of radius R and parts along the real axis. If we let $R \to \infty$, the parts along the contour vanish—assuming a sufficiently well-behaving

$F(\nu)$ (*no-subtraction hypothesis*)—and from the integrals along the branch cuts around the singularities we obtain an unsubtracted dispersion relation,

$$\operatorname{Re} F(\nu) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}\nu' \frac{\operatorname{Im} F(\nu')}{\nu' - \nu}. \tag{1.42}$$

The no-subtraction condition necessary for the validity of this dispersion relation is merely an assumption which is put into the theory. There is no guarantee that it holds. A more elaborate discussion on the subject of subtractions can be found, among others, in [Nus72], along with a detailed introduction into the subject of dispersion theory.

As we require symmetry under crossing for physical amplitudes, this dispersion relation can be rewritten as

$$\operatorname{Re} F(\nu) = \frac{2}{\pi}\mathcal{P}\int_{0}^{\infty} \mathrm{d}\nu' \frac{\operatorname{Im} F(\nu')}{\nu'^2 - \nu^2}. \tag{1.43}$$

Originally, this relation was derived in the context of dispersion of electromagnetic waves in sparse media by Kramers and Kroenig. In case of the Kramers-Kroenig relation, real and imaginary part of $F(\nu)$ are related to the real refraction index $n(\nu)$ and the extinction coefficient $\beta(\nu)$,

$$\operatorname{Re} F(\nu) = n(\nu) - 1 \quad \text{and} \quad \operatorname{Im} F(\nu) = \beta(\nu). \tag{1.44}$$

Apart from the pure mathematical derivation of the relation, the idea of the application to particle scattering processes is that of a dispersion relation in the limit of infinitely sparse media, where the medium effectively contains of a singular discrete scattering center, i.e., a single target particle, and the dispersion is effectively the asymptotic interaction with the target.

For a more elaborate treatise on the original Kramers-Kroenig relation and its application to Compton scattering, refer to the detailed review by D. Drechsel et al. [DPV02].

Chapter 2

Lagrangian Description of Charged Massive Vector Bosons

In this chapter we consider the field-theoretic description of charged massive spin-1 particles to study its electromagnetic interactions. The purpose of this study is to derive low-energy theorem for electromagnetic moments from Compton scattering. The "folk theorem" given by Weinberg and Leutwyler [Wei79, Leu94] states that:

> For a given set of asymptotic states, perturbation theory with the most general Lagrangian containing all terms allowed by the assumed symmetries will yield the most general S-matrix elements consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetries.

Therefore, if we derive a low-energy expansion from an effective Lagrangian of charged massive vector bosons consistent with electromagnetic gauge symmetry, we should be able to obtain the low-energy theorems. In this way we are led to consider the following general choice of the Lagrangian density:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}W^*_{\mu\nu}W^{\mu\nu} + M^2 W^*_\mu W^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ ie\ell_1 W^*_\mu W_\nu F^{\mu\nu} + ie\ell_2 W^*_{\mu\nu} W^\alpha \partial_\alpha F^{\mu\nu} - ie\ell_2 W^*_\alpha W_{\mu\nu}\partial^\alpha F^{\mu\nu},\end{aligned} \tag{2.1}$$

where ℓ_1 and ℓ_2 are the magnetic and quadrupole moment constants, respectively, $W_{\mu\nu} := D_\mu W_\nu - D_\nu W_\mu$ is the covariant vector boson field strength tensor, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$ is the photon field tensor.

The spin-1 constants ℓ_i are related to the electromagnetic moments by

$$\mu = (1+\ell_1)\frac{e}{2M} \quad \text{and} \quad Q = (\ell_2 - \ell_1)\frac{e}{M^2}. \tag{2.2}$$

In the following, we will motivate this choice of the effective Lagrangian.

2.1 The Proca Field

First, we will introduce the Proca[1] equation, which is the equation of motion for a free massive vector boson. The representation of particles with spin $j = 1$ requires at least a four-dimensional base space. This is due to the fact that $j = 1$ fields have three *d.o.f.*s, i.e. the representation needs at least be three-dimensional. In favor of a Lorentz covariant notation, we choose the Minkowski space M^4 as base space.

A vector boson of mass M is described by a real relativistic spinor field $\chi_\mu(x)$, also known as the *Proca field*. It obeys the Klein-Gordon equation in each of its four components, i.e.

$$\left(\Box + M^2\right) \chi_\mu = 0. \tag{2.3}$$

The field transforms covariantly under Lorentz transformations $a_{\mu\nu}$,

$$\chi_\mu \left(x\right) \to \chi'_\mu \left(x'\right) = a_{\mu\nu} \chi^\nu \left(x\right). \tag{2.4}$$

Using the free-field Lagrangian

$$\mathcal{L}_{\mathrm{P}} = -\frac{1}{4} \chi_{\mu\nu} \chi^{\mu\nu} + \frac{1}{2} M^2 \chi_\mu \chi^\mu, \tag{2.5}$$

where $\chi_{\mu\nu} = \partial_\mu \chi_\nu - \partial_\nu \chi_\mu$ is the corresponding field strength tensor, we recover from the Euler-Lagrange differential equations the free-field equation of motion (EOM)

$$\left[(\Box + M^2)\delta^\nu_\mu - \partial_\mu \partial^\nu\right] \chi_\nu(x) = 0, \tag{2.6}$$

the *Proca equation* [Tak69]. Note that this equation goes over into the homogeneous Maxwell equations if $M \to 0$. In contrast to the Maxwell field, however, the Proca field fulfills the Lorentz condition, $\partial_\mu \chi^\mu = 0$, as can be derived by applying the differential operator ∂ on the Proca current:

$$\begin{aligned} &\partial_\mu \partial_\nu \chi^{\mu\nu} = M^2 \partial_\mu \chi^\mu \stackrel{!}{=} 0 \\ &\Rightarrow \partial_\mu \chi^\mu = 0. \end{aligned} \tag{2.7}$$

[1] Alexandru Proca, 1897-1955, Romanian Theoretical Physicist [Poe05]

2.2 Charged Proca Fields

In order to describe a massive vector particle with electric charge, we need two real spinor fields $\chi^{(1)}$ and $\chi^{(2)}$ which are described by the Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{P}}\left(\chi_\mu^{(1)},\ \chi_\mu^{(2)}, \partial_\mu\chi_\nu^{(1)}, \partial_\mu\chi_\nu^{(2)}\right)& \\ &= \mathcal{L}_1 + \mathcal{L}_2 \\ &= \sum_{i=1}^{2}\left(\frac{1}{4}\chi_{\mu\nu}^{(i)}\chi^{(i)\mu\nu} + \frac{1}{2}M^2\chi_\mu^{(i)}\chi^{(i)\mu}\right)\end{aligned} \tag{2.8}$$

where $\chi_{\mu\nu}^{(i)} := \partial_\mu\chi_\nu^{(i)} - \partial_\nu\chi_\mu^{(i)}$.

This can be described equivalently by introducing the complex fields

$$W_\mu = \frac{1}{\sqrt{2}}\left(\chi_\mu^{(1)} + i\chi_\mu^{(2)}\right) \quad \text{and} \quad W_\mu^* = \frac{1}{\sqrt{2}}\left(\chi_\mu^{(1)} - i\chi_\mu^{(2)}\right) \tag{2.9}$$

which obviously fulfill the Proca equation. One can thus find a Lagrangian

$$\mathcal{L}' = \mathcal{L}'\left(W_\mu, W_\mu^*, \partial_\mu W_\nu, \partial_\mu W_\nu^*\right) \tag{2.10}$$

which is equivalent to the real field Lagrangian $\mathcal{L}\big(\chi_\mu^{(1)}, \chi_\mu^{(2)}, \partial_\mu\chi_\nu^{(1)}, \partial_\mu\chi_\nu^{(2)}\big)$.

Proof.

$$\begin{aligned}\mathcal{L}_P &= \mathcal{L}_1 + \mathcal{L}_2 \\ &= \sum_{i=1}^{2}\left(-\frac{1}{4}\chi_{\mu\nu}^{(i)}\chi^{(i)\mu\nu} + \frac{1}{2}M^2\chi_\mu^{(i)}\chi^{(i)\mu}\right) \\ &= -\frac{1}{2}\left(\partial_\mu W_\nu^*\right)\left(\partial^\mu W^\nu\right) + \frac{1}{2}\left(\partial_\mu W_\nu^*\right)\left(\partial^\nu W^\mu\right) \\ &\quad + \frac{1}{2}\left(\partial_\nu W_\mu^*\right)\left(\partial^\mu W^\nu\right) - \frac{1}{2}\left(\partial_\nu W_\mu^*\right)\left(\partial^\nu W^\mu\right) \\ &\quad + \frac{M^2}{4}\left(W\cdot W + 2W\cdot W^* + W^*\cdot W^* - W\cdot W + 2W\cdot W^* - W^*\cdot W^*\right) \\ &= -\frac{1}{2}\widetilde{W}_{\mu\nu}^*\widetilde{W}^{\mu\nu} + M^2 W_\mu^* W^\mu.\end{aligned} \tag{2.11}$$

□

Complex Proca Spinors

The spinor for a complex, i.e. charged, Proca field with polarization λ is defined as

$$W^{\mu}(p,\lambda) := \left(\frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}_\lambda}{M}, \boldsymbol{\zeta}_\lambda + \frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}_\lambda}{M(M+E)}\boldsymbol{p}\right), \tag{2.12}$$

where the $\boldsymbol{\zeta}_\lambda$ are the polarization vectors for the different polarizations λ,

$$(\zeta^{\mu}_{\pm}) = \frac{1}{\sqrt{2}}\,(0, \mp 1, -i, 0)^{\mathsf{T}} \quad \text{and} \quad (\zeta^{\mu}_{0}) = \hat{\boldsymbol{e}}_3. \tag{2.13}$$

The spinors fulfill the completeness relations

$$\sum_r \zeta_\mu(\boldsymbol{p}, r)\zeta_\nu(\boldsymbol{p}, r) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}, \tag{2.14}$$

as well as, in case of Compton scattering which is discussed in this work, the identities with regard to initial and final momenta p, p' for the bosons and q, q' for the photons,

$$\begin{aligned} p_\mu W^{\mu}(p) = 0, &\quad p'_\mu W^{*\mu}(p') = 0, \\ q_\mu \varepsilon^{\mu}(q) = 0, &\quad q'_\mu \varepsilon^{*\mu}(q') = 0. \end{aligned} \tag{2.15}$$

2.3 Construction of an Effective Lagrangian

In this section, we will construct a general Lagrange density for massive vector bosons fulfilling Lorentz invariance, gauge invariance and hermiticity. Following the usual approach of field theory, we start with the Lagrangian for a free particle with spin $j = 1$,

$$\mathcal{L}_0 = -\frac{1}{2}\widetilde{W}^{*}_{\mu\nu}\widetilde{W}^{\mu\nu} + M^2 W^{*}_{\mu} W^{\mu} \tag{2.16}$$

with the field tensor $\widetilde{W}_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$.

The free Lagrangian $\mathcal{L}_0$ is invariant under global $U(1)$ gauge transformations

$$W_\mu \mapsto W'_\mu = e^{i\alpha} W_\mu, \tag{2.17}$$

as will be discussed explicitly below.

To achieve invariance under *local* gauge transformations—meaning that the phase α is a function of the coordinate x—we introduce a *covariant derivative* (c.f. section 1.2) $\partial_\mu \mapsto D_\mu = \partial_\mu - \mathrm{i}eA_\mu$ which satisfies

$$D_\mu W^\nu \mapsto [D_\mu W^\nu]' = D'_\mu W'^\nu = e^{\mathrm{i}\alpha(x)} D_\mu W^\nu. \tag{2.18}$$

For this we need a gauge field $A_\mu(x)$, later to be identified with the electromagnetic field. This definition is also referred to as *Gell-Mann–Lévy trick*, which is analogous to the *minimal coupling* (or *minimal substitution*) in classical field theory. Hence, for derivatives of our spinor it follows that

$$\partial_\mu W^\nu \mapsto D_\mu W^\nu = (\partial_\mu - \mathrm{i}eA_\mu(x))\, W^\nu, \tag{2.19}$$

so that we can define the *covariant field strength tensor*,

$$D_\mu W_\nu - D_\nu W_\mu =: W_{\mu\nu}. \tag{2.20}$$

Let us only regard the field term of the Lagrangian for now:

$$\begin{aligned} W^*_{\mu\nu} W^{\mu\nu} &= \left(D^*_\mu W^*_\nu - D^*_\nu W^*_\mu\right)\left(D^\mu W^\nu - D^\nu W^\mu\right) \\ &= \widetilde{W}^*_{\mu\nu}\widetilde{W}^{\mu\nu} + \mathrm{i}e\widetilde{W}^*_{\mu\nu}T^{\mu\nu} - \mathrm{i}eT^*_{\mu\nu}\widetilde{W}^{\mu\nu} + e^2 T^*_{\mu\nu}T^{\mu\nu} \\ &= \widetilde{W}^*_{\mu\nu}\widetilde{W}^{\mu\nu} + 2\mathrm{i}e\widetilde{W}^*_{\mu\nu}A^\nu W^\mu - 2\mathrm{i}eA_\nu W^*_\mu \widetilde{W}^{\mu\nu} \\ &\quad + 2e^2 A^2 |W|^2 - 2e^2 A_\nu W^*_\mu A^\mu W^\nu, \end{aligned} \tag{2.21}$$

where we defined the coupling tensor $T_{\mu\nu} := W_\mu A_\nu - W_\nu A_\mu$.
Following from this, the interaction Lagrangian resulting from minimal substitution is given by

$$\mathcal{L}_{int} = -\mathrm{i}e\widetilde{W}^*_{\mu\nu}A^\nu W^\mu + \mathrm{i}eW^*_\mu A_\nu \widetilde{W}^{\mu\nu} - e^2 A^2 |W|^2 + e^2 W^*_\mu A_\nu W^\mu A^\nu$$

In contrast to these coupling terms coming from minimal substitution, higher order terms in W and A can be deduced by taking into account the symmetries of the expected field theory. According to Lorcé [Lor09], for a spin-1 particle, we will have contributions of electromagnetic moments up to $2S+1=3$, so there are additional terms proportional to the anomalous magnetic and quadrupole moments ℓ_1 and ℓ_2.

The magnetic dipole term is introduced in the coupling term

$$\mathcal{L}_2 = \mathrm{i}e\ell_1 W_\mu^* W_\nu F^{\mu\nu}, \tag{2.22}$$

which is the only contribution proportional to ℓ_1 in this model. For the quadrupole term, the boson field couples to a derivative of $F_{\mu\nu}$. Other forms of coupling, such as polarizabilities, are not included in the theory as we assume that these do not contribute to GDH and QSR. The most general term involving this coupling is

$$\mathcal{L}_3 = \mathrm{i}e\ell_2 W_{\mu\nu}^* W^\alpha \partial_\alpha F^{\mu\nu} - \mathrm{i}e\ell_2 W_\alpha^* W_{\mu\nu} \partial^\alpha F^{\mu\nu}. \tag{2.23}$$

We want to stress that one has to use the covariant field strength tensor $W_{\mu\nu}$ in all of the terms. Otherwise, one will miss, apart from the interaction terms derived above, additional quadrupole terms coming from $\mathcal{L}_3$:

$$\begin{aligned}\mathcal{L}_3 = &\ \mathrm{i}e\ell_2 \widetilde{W}_{\mu\nu}^* W^\alpha \partial_\alpha F^{\mu\nu} - \mathrm{i}e\ell_2 W_\alpha^* \widetilde{W}_{\mu\nu} \partial^\alpha F^{\mu\nu} \\ &- 2e^2\ell_2 A_\mu W_\nu^* W^\alpha \partial_\alpha F^{\mu\nu} - 2e^2\ell_2 W_\alpha^* A_\mu W_\nu \partial^\alpha F^{\mu\nu}.\end{aligned} \tag{2.24}$$

If these were neglected, gauge invariance of the amplitudes would be broken. In many of the calculations within this thesis, this might not even be noticed as we are using natural values, i.e. $l_2 = 0$. However, the derivation of the quadrupole moment sum rule in the physical field theory would yield an incorrect result. Thus it is very important to check the intermediate results after each step. Gauge invariance is a decisive property of the theory. It is also a good test to find out if there occur errors in the calculation, or if the theory might be incomplete at this order.

2.4 Gauge Invariance

The Lagrangian $\mathcal{L}_{\text{Eff}}$ is, by construction, a gauge theory, i.e. it has to be invariant under symmetry transformations of the gauge group U(1). To make sure we constructed it correctly, we explicitly check gauge invariance in this section. The Lagrangian should thus be invariant under the simultaneous translations

$$W_\mu \to e^{i\alpha(x)} W_\mu \qquad \text{and} \qquad A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha(x). \tag{2.25}$$

We look at the Lagrangian $\mathcal{L}_{\text{Eff}}$ term-by-term:

- $W_{\mu\nu}$ transforms covariantly (c.f. section 1.2), hence the quadratic term in the field strength tensor is invariant;

- $F_{\mu\nu}$ is invariant by construction;

- $\left(W_\mu^* W^\mu\right)' = e^{-i\alpha(x)} W_\mu^* e^{i\alpha(x)} W^\mu = W_\mu^* W^\mu$ since W and $e^{\pm i\alpha(x)}$ commute;

- $\left(W_{\mu\nu}^* W^\mu\right)' = e^{-i\alpha(x)} W_{\mu\nu}^* e^{i\alpha(x)} W^\mu = W_{\mu\nu}^* W^\mu$.

As all appearing terms are gauge invariant, so are linear combinations. It follows that $\mathcal{L}_{\text{Eff}}$ is invariant under U(1) gauge transformations.

2.5 Feynman Rules for $\mathcal{L}_{\text{Eff}}$

In order to describe interactions with the newly-constructed effective Lagrangian $\mathcal{L}_{\text{Eff}}$, we need to derive the corresponding Feynman rules. In this theory, two types of vertices appear: the 3-point vertex γWW and the 4-point contact vertex $\gamma\gamma WW$. In order to recover the correct Feynman rules, the states have to be contracted with the fields coming from the Lagrangian, or more precise, the action (c.f. section 1.3). For a quick overview over the resulting Feynman rules, see App. A.1.

A Taylor expansion of the S-Matrix yields the interaction part

$$\begin{aligned} S &= \langle f|\mathcal{T}\exp\left(i\int dx \mathcal{L}_{\text{Eff}}\right)|i\rangle \\ &= \cancelto{0}{\langle f|1|i\rangle} + i\int d^4x \langle f|\mathcal{L}_{\text{Eff}}|i\rangle \end{aligned} \tag{2.26}$$

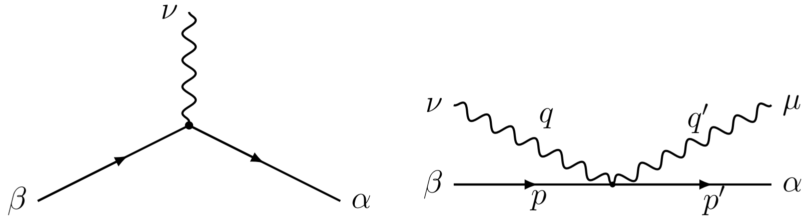
which corresponds to a Feynman rule describing the transition from the initial state $|i\rangle$ to the final state $|f\rangle \neq |i\rangle$. The contraction is evaluated using the plane-wave expansions which can be found in App. A.2.1. There, we have also derived the contraction relations. In the following we will neglect the integration $\int d^4x$; however, it is still implied in this symbolic notation.

Propagator The propagator for a massive vector boson is that of the Proca fields

$$\Delta_{\mathrm{P}}^{\alpha\beta}(p) = -\frac{g^{\alpha\beta} - {}^{p^\alpha p^\beta}/{}_{M^2}}{p^2 - M^2 + i0^+}. \tag{2.27}$$

By calculating the appropriate Feynman rule for the free field, one recovers the inverse propagator, which confirms that this is the correct definition:

$$\langle W^\alpha(p)|W^*_{\mu\nu}W^{\mu\nu}|W^\beta(p)\rangle = -g^{\alpha\beta}\left(p^2 - M^2\right) + p^\alpha p^\beta \tag{2.28}$$



(a) 3-point vertex γWW (b) contact vertex $\gamma\gamma WW$

Figure 2.1: Massive vector boson scattering vertices for derivation of Feynman rules

γWW–Vertex To provide a better overview, we will calculate the vertices term-by-term. Crossing symmetry implies here that the same Feynman rules will be obtained whether the photon is initial or final. The vertex is defined as

$$\Gamma^{\alpha\beta\mu} := \langle W^\alpha(p')|\mathcal{L}_{\mathrm{Eff}}|W^\beta(p)A^\mu(q)\rangle, \tag{2.29}$$

which contains the following four sub-terms:

$$\begin{aligned}
&\mathrm{i}e\frac{1}{2}\langle W^\alpha|(T^*_{\rho\tau}W^{\rho\tau}|W^\beta A^\mu\rangle - \mathrm{i}e\frac{1}{2}\langle W^\alpha|W^*_{\rho\tau}T^{\rho\tau})|W^\beta A^\mu\rangle \\
&\quad = \mathrm{i}\frac{e}{2}\langle 0|\left(\delta^\alpha_\rho\delta^\mu_\tau - \delta^\alpha_\tau\delta^\mu_\rho\right)\left(-\mathrm{i}p_\rho\, g^{\tau\beta} + \mathrm{i}p_\tau\, g^{\rho\beta}\right)|0\rangle \\
&\quad\quad - \mathrm{i}\frac{e}{2}\langle 0|\left(\mathrm{i}p'_\rho\,\delta^\alpha_\tau - \mathrm{i}p'_\tau\,\delta^\alpha_\rho\right)\left(g^{\rho\beta}g^{\tau\mu} - g^{\tau\beta}g^{\rho\mu}\right)|0\rangle \\
&\quad = e\left(p^\alpha g^{\mu\beta} + p'^\beta g^{\mu\alpha} - P^\mu g^{\alpha\beta}\right),
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
\mathrm{i}e\ell_1\langle W^\alpha|W^*_\rho W_\tau F^{\rho\tau}|W^\beta A^\mu\rangle &= \mathrm{i}e\langle 0|\delta^\alpha_\rho\delta^\beta_\tau\left(-iq_\rho\, g^{\tau\mu} + iq_\tau\, g^{\rho\mu}\right)|0\rangle \\
&= e(q^\alpha g^{\beta\mu} - q^\beta g^{\alpha\mu}),
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
&\mathrm{i}e\ell_2\langle W^\alpha|W^*_{\rho\tau}W_\sigma\partial^\sigma F^{\rho\tau}|W^\beta A^\mu\rangle \\
&\quad= \mathrm{i}e\ell_2\left(\left(\mathrm{i}p'_\rho\,\delta^\alpha_\tau - \mathrm{i}p'_\tau\,\delta^\alpha_\rho\right)\delta^\beta_\sigma(-\mathrm{i}q^\sigma)\left(-\mathrm{i}q^\rho g^{\tau\mu} + \mathrm{i}q^\tau g^{\rho\mu}\right)\right)\langle 0|0\rangle \\
&\quad= 2e\ell_2\left(p'\cdot q\, g^{\alpha\mu} - p'^\mu q^\alpha\right)q^\beta,
\end{aligned}\tag{2.32}$$

$$\begin{aligned}
&-\mathrm{i}e\ell_2\langle W^\alpha|W^*_\sigma W_{\rho\tau}\partial^\sigma F^{\rho\tau}|W^\beta A^\mu\rangle \\
&\quad= -\mathrm{i}e\ell_2\left(\delta^\alpha_\sigma\left(-\mathrm{i}p_\rho\delta^\beta_\tau + \mathrm{i}p_\tau\delta^\beta_\rho\right)(-\mathrm{i}q^\sigma)\left(-\mathrm{i}q^\rho g^{\tau\mu} + \mathrm{i}q^\tau g^{\rho\mu}\right)\right) \\
&\quad= 2e\ell_2\left(p\cdot q\, g^{\beta\mu} - p^\mu q^\beta\right)q^\alpha.
\end{aligned}\tag{2.33}$$

So, put together, this results in the 3-vertex Feynman rule

$$\begin{aligned}
\Rightarrow \Gamma^{\alpha\beta\mu}(p,p') = &-e\left(g^{\alpha\beta}P^\mu - p'^\beta g^{\alpha\mu} - p^\alpha g^{\beta\mu}\right) \\
&+\left(q^\beta g^{\alpha\mu} - q^\alpha g^{\beta\mu}\right)\ell_1 \\
&-2\left(q^\alpha q^\beta P^\mu - p\cdot q q^\alpha g^{\beta\mu} - p'\cdot q q^\beta g^{\alpha\mu}\right)\ell_2,
\end{aligned}\tag{2.34}$$

where $P^\mu = p^\mu + p'^\mu$.

$\gamma\gamma$WW–Vertex

$$\Gamma^{\alpha\beta\mu\nu}(p',p) := \langle W^\alpha(p'), A^\nu(k)|\mathcal{L}_{\text{Eff}}|W^\beta(p)A^\mu(q)\rangle \tag{2.35}$$

First, we focus on the direct interaction parts. Note that we have to consider crossing, which means here that the photons can couple to both the initial and the final state.

$$-e^2\langle W^\alpha(p')\,A^\mu|A^2 W^*_\tau W^\tau|W^\beta A^\nu\rangle = -2e^2\left(g^{\mu\nu}g^{\alpha\beta}\right), \tag{2.36}$$

$$\begin{aligned}
&e^2\langle W^\alpha(p')\,A_\mu|A_\rho W^*_\tau A^\tau W^\rho|W^\beta A^\nu\rangle + e^2\langle W^\alpha(p')\,A_\mu|A_\rho W^*_\tau A^\tau W^\rho|W^\beta A^\nu\rangle \\
&\quad= e^2\left(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}\right),
\end{aligned}\tag{2.37}$$

$$\begin{aligned}
&e^2\ell_2\frac{1}{2}\langle W^\alpha A^\mu|T^*_{\rho\tau}W_\sigma\partial^\sigma F^{\rho\tau}|W^\beta A^\nu\rangle + e^2\ell_2\frac{1}{2}\langle W^\alpha A^\mu|T^*_{\rho\tau}W_\sigma\partial^\sigma F^{\rho\tau}|W^\beta A^\nu\rangle \\
&\quad= -e^2\ell_2\left(2q^\alpha q^\beta g^{\mu\nu} - q^\beta q^\mu g^{\alpha\nu} - q^\beta q^\nu g^{\alpha\mu}\right),
\end{aligned}\tag{2.38}$$

$$
\begin{aligned}
&e^2\ell_2\frac{1}{2}\langle W^\alpha A^\mu|W^*_\sigma T_{\rho\tau}\partial^\sigma F^{\rho\tau}\big|W^\beta A^\nu\rangle + e^2\ell_2\frac{1}{2}\langle W^\alpha A^\mu|W^*_\sigma T_{\rho\tau}\partial^\sigma F^{\rho\tau}\big|W^\beta A^\nu\rangle \\
&\qquad = -e^2\ell_2\left(2q^\alpha q^\beta g^{\mu\nu} - q^\alpha q^\mu g^{\beta\nu} - q^\alpha q^\nu g^{\beta\mu}\right),
\end{aligned} \tag{2.39}
$$

$$
\begin{aligned}
\Rightarrow \Gamma^{\alpha\beta\mu\nu}(p,p') = &-\left(2g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right)e^2 \\
&-\left(4q^\alpha q^\beta g^{\mu\nu} - q^\alpha q^\mu g^{\beta\nu} - q^\alpha q^\nu g^{\beta\mu}\right. \\
&\left. -q^\beta q^\mu g^{\alpha\nu} - q^\beta q^\nu g^{\alpha\mu}\right)e^2\ell_2.
\end{aligned} \tag{2.40}
$$

For the photon, we simply use the familiar photon propagator and the appropriate contractions (see App. A.2.2 for reference).

2.6 Electromagnetic Moments and Natural Values

The ℓ_i are a way to parametrize the electromagnetic moments. Following closely the paper by Lorcé [Lor09], we give a short introduction on anomalous electromagnetic moments for particles of any spin, followed by a presentation of the concept of natural values.

In general, a particle of spin j, has $2j+1$ electromagnetic moments corresponding to the total number of independent covariant vertex structures if Lorentz, parity and time-reversal symmetries are respected. Due to parity, the particle has only *even* electric and *odd* magnetic multipoles. The electromagnetic moments can be obtained via multipole decomposition of the electromagnetic current

$$
J^\mu(\boldsymbol{q}) = \int \mathrm{d}^3 r \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} J^\mu(x) \equiv \frac{e}{2M}\langle p', j|J^\mu(0)|p, j\rangle. \tag{2.41}
$$

For the electric moments, the multipole decomposition in the Breit frame is

$$
\rho(\boldsymbol{q}) \equiv J^0(\boldsymbol{q}) = e\sum_{\substack{l=0 \\ l\ \text{even}}}^{2j} (-\tau)^{l/2}\sqrt{\frac{4\pi}{2l+1}}\frac{l!}{(2l-1)!!}G_{\mathrm{E}l}(Q^2)Y_{l0}(0) \tag{2.42}
$$

$$
= e\left[G_{\mathrm{E}0}(Q^2) - \frac{2}{3}\tau G_{\mathrm{E}2}(Q^2) + \dots\right], \tag{2.43}
$$

where we used $\tau \equiv Q^2/4M^2$ and $Q^2 \equiv -q^2$ is the momentum transfer squared. Similarly, the magnetic moments are obtained from a decomposition of the magnetic density which is related to the charge current $\boldsymbol{J}(\boldsymbol{q})$ by

$$\rho_{\mathrm{M}}(\boldsymbol{q}) \equiv \boldsymbol{J}(\boldsymbol{q}) = \mathrm{i}e\sqrt{\tau} \sum_{\substack{l=1 \\ l \text{ odd}}}^{2j} (l+1)(-\tau)^{(l-1)/2} \sqrt{\frac{4\pi}{2l+1}} \frac{l!}{(2l-1)!!} G_{\mathrm{M}l}(Q^2) Y_{l0}(0) \tag{2.44}$$

$$= 2\mathrm{i}e\sqrt{\tau} \left[G_{\mathrm{M1}}(Q^2) - \frac{4}{5}\tau G_{\mathrm{M3}}(Q^2) + \dots \right] \tag{2.45}$$

The electromagnetic moments are defined as the low-energy constants of the *multipole*, or *Sachs form factors* at zero momentum transfer, i.e. $Q^2 = 0$. The l^{th} electric moment Q_l is thus given by

$$Q_l = \frac{e}{M^l} \frac{(l!)^2}{2^l} G_{\mathrm{E}l}(0), \tag{2.46}$$

while the l^{th} magnetic moment is defined as

$$\mu_l = \frac{e}{M^l} \frac{(l!)^2}{2^{l-1}} G_{\mathrm{M}l}(0). \tag{2.47}$$

For $j = 1$, the most general electromagnetic interaction current is

$$\begin{aligned} J^{\mu}_{(1)} = -W^*_{\alpha}(p', \lambda') \Big[& g^{\alpha\beta} P^{\mu} F_1(Q^2) + (g^{\mu\beta} q^{\alpha} - g^{\mu\alpha} q^{\beta}) F_2(Q^2) \\ & - \frac{q^{\alpha} q^{\beta}}{2M^2} P^{\mu} F_3(Q^2) \Big] W_{\beta}(p, \lambda). \end{aligned} \tag{2.48}$$

The interaction is thus described in terms of the independent covariant vertex structures

$$\begin{aligned} & - g^{\alpha\beta} P^{\mu}, \\ & g^{\mu\beta} q^{\alpha} - g^{\mu\alpha} q^{\beta}, \quad \text{and} \\ & \frac{q^{\alpha} q^{\beta}}{2M^2} P^{\mu}. \end{aligned} \tag{2.49}$$

Fixing $\lambda = \lambda' = +1$, we can obtain the relation between the Sachs form factors and the form factors F_i corresponding to these structures. For the electric moment, the charge density evaluates to

$$J^0_{(1)} = 2p_0 \left(F_1(Q^2) + \tau(F_1(Q^2) - F_2(Q^2) + (1-\tau)F_3(Q^2)) \sin^2\theta \right) \tag{2.50}$$

with θ the scattering angle, while for the magnetic part, we get

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{J}_{(1)} \times \boldsymbol{q} \right) = \mathrm{i}\sqrt{\tau} 2p_0 F_2(Q^2) 2\sqrt{4\pi} 3 Y_{10}(0). \tag{2.51}$$

Hence, we can make the identifications

$$G_{E0}(Q^2) = \sqrt{1+\tau}\left(F_1(Q^2) + \frac{2}{3}\tau G_{E2}(Q^2)\right), \tag{2.52a}$$

$$G_{E2}(Q^2) = \sqrt{1+\tau}\left(F_1(Q^2) - F_2(Q^2) + (1+\tau)F_3(Q^2)\right), \quad \text{and} \tag{2.52b}$$

$$G_{M1}(Q^2) = \sqrt{1+\tau}F_2(Q^2). \tag{2.52c}$$

The natural value of an electromagnetic moment of any particle is the value obtained for an elementary particle. It can be obtained under the assumption that at tree-level and $Q^2 = 0$, the light-cone helicity is conserved [Lor09]. The deviation from the natural value is an indicator for the internal structure of non-elementary, i.e. composite, particles which is measured by *anomalous* moments. The natural values for an elementary particle with unit electric charge $Z = +1$ up to the quadrupole moment are given by Lorcé [Lor09] as

$$G_{\mathrm{E0}}(0) = 1, \tag{2.53a}$$

$$G_{\mathrm{M1}}(0) = 2j, \quad \text{and} \tag{2.53b}$$

$$G_{\mathrm{E2}}(0) = -j(2j-1). \tag{2.53c}$$

For a spin-1 particle, the natural electromagnetic moments are

$$\mu = \frac{e}{2M} \quad \text{and} \quad Q = \frac{e}{M^2} \tag{2.54}$$

which implies in terms of our electromagnetic moment constants ℓ_i

$$\ell_1 = 1 \quad \text{and} \quad \ell_2 = 0. \tag{2.55}$$

Chapter 3

Compton Scattering and Sum Rules

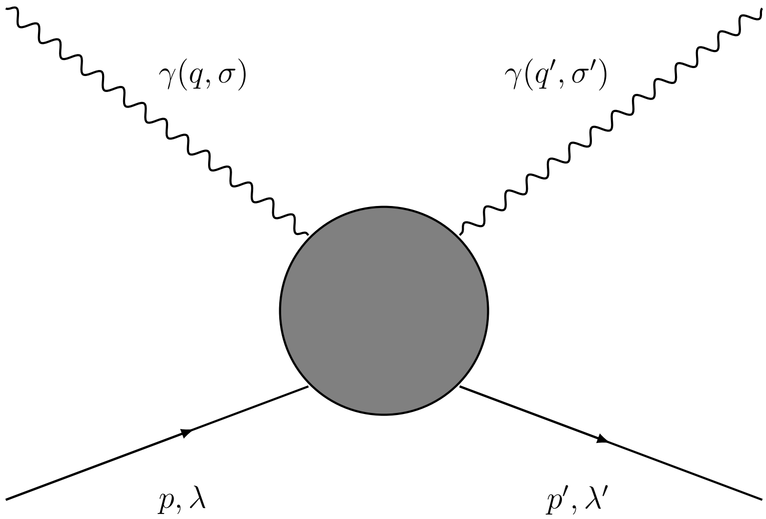


Figure 3.1: Complete real forward Compton scattering diagram for polarized scattering off particles with arbitrary spin j. The wavy lines are the photons of momentum q and polarization σ probing the polarized target which is depicted by the bold lines. The grey circle denotes all possible intermediate states.

The purpose of this study is to derive low-energy theorems for Compton scattering (i.e. elastic photon scattering) from a target with spin 1. The real forward Compton scattering (RFCS) amplitude of a polarized photon $\gamma(q, \varepsilon_\sigma)$ off a (massive) particle with arbitrary spin j is depicted in fig. 3.1, where the grey circle in the center denotes a sum over all possible intermediate states. The CS amplitude can be written as a function of only the external photon lines:

$$\begin{aligned} T &= e^2 \boldsymbol{\varepsilon}_\nu \boldsymbol{\varepsilon}^*_\mu T^{\mu\nu} \\ &= e^2 \boldsymbol{\varepsilon}_\nu \boldsymbol{\varepsilon}^*_\mu \,\mathrm{i} \int d^4x e^{iq\cdot x} \langle p, \lambda | \mathcal{T} J^\mu(x) J^\nu(0) | p, \lambda \rangle, \end{aligned}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^*$ are the polarization vectors of the photon. Here, we use circularly polarized photons exclusively. Compton scattering (CS) is a useful means to gain an insight into the electromagnetic properties of a particle. Real scattering means that

$$Q^2 \equiv -q^2 = 0. \tag{3.1}$$

3.1 Decomposition of the Polarized Amplitude

The polarization of the amplitude depends solely on the spin states of the photon, denoted by polarization vectors $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^*)$, and the target, denoted by the spin vector $\boldsymbol{S}$. For the spin vectors, we can use the properties of the underlying spin algebra, which we will discuss briefly in the following.

3.1.1 Spin Algebra for Vector Particles

The *spin group* for particles of arbitrary spin j is an $n = 2j + 1$-dimensional representation of the spin symmetry algebra $\mathfrak{su}(2)$. For a given spin j, $2j + 1$ relevant electromagnetic moment structures appear, as Lorcé [Lor09] has pointed out (c.f. section 2.6).

The structures which appear can be decomposed into structures proportional to

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^*, \quad [\boldsymbol{S} \cdot \boldsymbol{\varepsilon}^*, \boldsymbol{S} \cdot \boldsymbol{\varepsilon}] \,(\boldsymbol{S} \cdot \boldsymbol{q})^{2n-1}\,, \qquad \text{and} \qquad \{\boldsymbol{S} \cdot \boldsymbol{\varepsilon}^*, \boldsymbol{S} \cdot \boldsymbol{\varepsilon}\} \,(\boldsymbol{S} \cdot \boldsymbol{q})^{2n-2}\,, \tag{3.2}$$

where $0 \leq n \leq 2j - 1$. Here, $\boldsymbol{S}$ is the spin operator.

For example, in the case $j = 1/2$ only the first two structures appear, where $n = 0$:

$$T(\nu) = W^\dagger \left[\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^* f_0(\nu) + \nu f_1(\nu) \mathrm{i} \boldsymbol{S} \cdot (\boldsymbol{\varepsilon}^* \times \boldsymbol{\varepsilon}) \right]. \tag{3.3}$$

The decomposition for arbitrary j, where $j > 1/2$, is

$$\begin{aligned} T(\nu) = W^\dagger \Big[& \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^* f_0(\nu) \\ & + \nu \sum_{n \in \{\mathbb{N}+\frac{1}{2}\}}^{j} f_{2n}(\nu) \left[\boldsymbol{S} \cdot \boldsymbol{\varepsilon}^*, \boldsymbol{S} \cdot \boldsymbol{\varepsilon} \right] (\boldsymbol{S} \cdot \boldsymbol{q})^{2n-1} \\ & + \nu^2 \sum_{n \in \mathbb{N}}^{j} f_{2n}(\nu) \left\{ \boldsymbol{S} \cdot \boldsymbol{\varepsilon}^*, \boldsymbol{S} \cdot \boldsymbol{\varepsilon} \right\} (\boldsymbol{S} \cdot \boldsymbol{q})^{2n-2} \Big] W, \end{aligned} \tag{3.4}$$

where the f_i are the e.m. structure functions. The spin vector $\boldsymbol{S}$ can be constructed via its relation to the Clebsch-Gordan coefficients [Sch07]. To accomplish this, it is helpful to construct the $(2j+1) \times (2j+1)$ polarization matrices

$$\left(\mathcal{C}_\sigma^{(S)}\right)_{\lambda'+j+1, \lambda+j+1} = \sqrt{j(j+1)} C(1\sigma, j\lambda; j\lambda'), \tag{3.5}$$

where

$$C(j_1 m_1, j_2 m_2; jm) \equiv \langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle \tag{3.6}$$

are the Clebsch-Gordan coefficients. The indices run as $\sigma = (-1, 1)$ and $\lambda = (-s, s)$. The components of the spin vector are given by

$$S_1 = \frac{1}{\sqrt{2}} \left(\mathcal{C}_{+1} - \mathcal{C}_{-1} \right), \tag{3.7a}$$

$$S_2 = \frac{\mathrm{i}}{\sqrt{2}} \left(\mathcal{C}_{+1} + \mathcal{C}_{-1} \right), \tag{3.7b}$$

$$S_3 = \mathcal{C}_0. \tag{3.7c}$$

It can be easily confirmed that these matrices satisfy the spin algebra $\mathfrak{su}(2)$, $[S_k, S_l] = \mathrm{i}\varepsilon_{klm} S_m$. They also fulfill the additional properties of a spin operator,

$$\boldsymbol{S}^2 = j(j+1) \qquad \text{and} \qquad (S_3)_{\lambda'\lambda} = \lambda\, \delta_{\lambda'\lambda}. \tag{3.8}$$

We choose the 3-axis as the direction of propagation for the photons. The photon momentum is $\boldsymbol{q} = \nu \hat{\boldsymbol{e_3}}$. Since we are using circular polarized photons with respect to the 3-direction, the polarization vectors are defined as

$$\boldsymbol{\varepsilon} = -\frac{1}{\sqrt{2}} \left(\hat{\boldsymbol{e_1}} + \mathrm{i} \hat{\boldsymbol{e_2}} \right), \tag{3.9}$$

and the commutator and anticommutator can be rewritten as

$$[\boldsymbol{S}\cdot\boldsymbol{\varepsilon}^*, \boldsymbol{S}\cdot\boldsymbol{\varepsilon}] = \mathrm{i}\boldsymbol{S}\cdot\boldsymbol{\varepsilon}^*\times\boldsymbol{\varepsilon}, \tag{3.10}$$

$$\{\boldsymbol{S}\cdot\boldsymbol{\varepsilon}^*, \boldsymbol{S}\cdot\boldsymbol{\varepsilon}\} = \boldsymbol{S}^2 - S_3^2, \tag{3.11}$$

respectively. The first relation is familiar, so we will prove only the latter:

Proof. We can define operators

$$S_+ := \boldsymbol{S}\cdot\boldsymbol{\varepsilon} = -\frac{1}{\sqrt{2}}(S_1 + \mathrm{i}S_2) \tag{3.12a}$$

$$S_- := \boldsymbol{S}\cdot\boldsymbol{\varepsilon}^* = -\frac{1}{\sqrt{2}}(S_1 - \mathrm{i}S_2). \tag{3.12b}$$

Thus it follows

$$\begin{aligned}\{\boldsymbol{S}\cdot\boldsymbol{\varepsilon}^*, \boldsymbol{S}\cdot\boldsymbol{\varepsilon}\} &= \{S_-, S_+\} \\ &= \frac{1}{2}(S_1 - \mathrm{i}S_2)(S_1 + \mathrm{i}S_2) + \frac{1}{2}(S_1 + \mathrm{i}S_2)(S_1 - \mathrm{i}S_2) \\ &= S_1^2 + S_2^2 + \cancelto{0}{\frac{\mathrm{i}}{2}\left([S_1, S_2] + [S_2, S_1]\right)} \\ &\equiv \boldsymbol{S}^2 - S_3^2.\end{aligned}$$

□

3.1.2 Decomposition

In terms of the target polarization λ, the decomposition can be rewritten as

$$T_\lambda^{(S)}(\nu) = \frac{e^2}{M}\sum_{n=0}^{2j} f_n'(\nu)\left(\frac{\lambda\nu}{jM}\right)^n. \tag{3.13}$$

We consider three cases explicitly: First, the decomposition for the historical case, $j = 1/2$, is performed. Next, the spin-1 amplitude will be decomposed which is the focus of this work. In addition to this, the case $j = 3/2$ is considered to make a point on further applications. As we will see, due to the decomposition, an additional term will appear for every half-spin step, each corresponding to a higher-order electromagnetic moment (c.f. e.g. Lorcé [Lor09]).

Nucleons: $j = 1/2$

For this case the original GDH was derived. The amplitude (here, the χ are the Dirac spinors for fermions) has the decomposition in terms of the spin matrices

$$T(\nu) = \chi^\dagger \left[\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^* f_0(\nu) + \nu f_1(\nu) \mathrm{i} \boldsymbol{S} \cdot (\boldsymbol{\varepsilon}^* \times \boldsymbol{\varepsilon})\right] \chi. \tag{3.14}$$

In terms of Eq. (3.13), it follows that (note that the prefactors $\frac{e^2}{M}$ and $\left(\frac{1}{M}\right)^n$ are from now on embedded into the f_i')

$$T^{(1/2)}_{\pm 1/2}(\nu) = f_0' \pm \nu f_1'. \tag{3.15}$$

Massive vector bosons: $j = 1$

The focus of this work is the description of massive vector bosons W. In terms of the spins, the amplitude is written as

$$T(\nu) = W^* \left[\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^* f_0(\nu) + \nu f_1(\nu) \mathrm{i} \boldsymbol{S} \cdot (\boldsymbol{\varepsilon}^* \times \boldsymbol{\varepsilon}) + \nu^2 f_2(\nu) \left(\boldsymbol{S}^2 - S_3^2\right)\right] W, \tag{3.16}$$

leading to a decomposition in terms of the helicity,

$$\begin{aligned} T^{(1)}_{\pm 1} &= f_0' \pm \nu f_1' + \nu^2 f_2', && (3.17\text{a}) \\ T^{(1)}_0 &= f_0'. \end{aligned}$$

Note how an additional parameter f_2' appeared, which is the structure function leading to the electric quadrupole moment.

Rarita-Schwinger particles: $j = 3/2$

Rarita and Schwinger have developed a representation of $j = 3/2$ particles as a tensor product of $j = 1/2$ spinors and spin-1 polarization vectors. Proceeding as before, we obtain the following equations:

$$T^{(3/2)}_{\pm 1/2}(\nu) = f_0' \pm \frac{\nu}{3} f_1' + \frac{\nu^2}{9} f_2' \pm \frac{\nu^3}{27} f_3', \tag{3.18a}$$

$$T^{(3/2)}_{\pm 3/2}(\nu) = f_0' \pm \nu f_1' + \nu^2 f_2' \pm \nu^3 f_3'. \tag{3.18b}$$

Using these decompositions we will be able to derive the structure functions from the Compton scattering diagrams. In order to accomplish this, we need to choose an appropriate frame of reference. Therefore, we will first briefly discuss the kinematics of real Compton scattering, or more generally, scattering processes with two particles in the initial and final states ($2 \to 2$) in the following section.

3.2 Scattering Kinematics

In this section we will discuss the kinematics of Compton scattering as a special case of a $2 \to 2$ scattering process in different frames of reference. We will start from an invariant description and discuss how to transfer to and between the *laboratory system* (LS), also known as *lab frame*, and the *center-of-momentum system* (CMS). While all frames of reference are equivalent, calculations can be of different magnitudes of complexity depending on the choice of the frame. A good introduction to this subject is the work by Byckling and Kajantie [BK73].

The general $2 \to 2$ scattering process is depicted in fig. 3.2. In case of Compton scattering, we identify $p_a = p$, $p_b = q$, $p_1 = p'$, and $p_2 = q'$, where $p^{(\prime)}$ and $q^{(\prime)}$ are the initial (final) target particle and photon momenta, respectively.

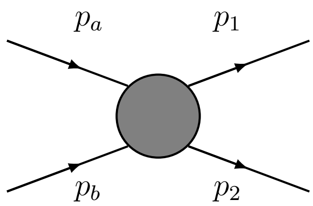


Figure 3.2: Kinematics for $2 \to 2$ scattering processes.

It is possible to describe scattering kinematics invariantly using the invariant variables introduced by Mandelstam [Man58]. For Compton scattering, these are defined as

$$\begin{aligned} s &= (p+q)^2 = (p'+q')^2, \\ t &= (p-p')^2 = (q-q')^2, \\ \text{and} \qquad u &= (p-q')^2 = (q-p')^2. \end{aligned} \tag{3.19}$$

s, u, and t are related to three different reaction channels: s is the invariant mass, of the incoming and outgoing particles, t is the momentum transfer, and u is the crossed momentum transfer.

A generalized parametrization of the 4-momenta is

$$\begin{aligned} p &= (E, \boldsymbol{p}), \qquad & p' &= (E', \boldsymbol{p'}), \\ q &= (\omega, \boldsymbol{q}), \qquad & q' &= (\omega', \boldsymbol{q'}), \end{aligned} \tag{3.20}$$

where $E^{(\prime)}$ and $\omega^{(\prime)}$ are the initial-(final-)state target particle and photon energies in a given frame, respectively. In all frames, the energy-momentum conservation holds:

$$p^\mu + q^\mu = p'^\mu + q'^\mu. \tag{3.21}$$

Due to the on-shell condition for external particles, the Mandelstam variables are constrained by the relation

$$s + t + u = \sum_i m_i^2 = 2M^2 \tag{3.22}$$

In general, the differential cross section for unpolarized $2 \to 2$ scattering is defined as

$$\mathrm{d}\sigma(s) = \frac{1}{8\pi^2 \lambda^{\frac{1}{2}}\left(s, m_a^2, m_b^2\right)} \int \frac{\mathrm{d}^3 p_1}{2E_1} \frac{\mathrm{d}^3 p_2}{2E_2} \delta^{(4)}\left(p_a + p_b - p_1 - p_2\right) \overline{\left|\mathcal{M}_{\text{fi}}\right|}^2. \tag{3.23}$$

Here, $\overline{\left|\mathcal{M}_{\text{fi}}\right|}^2$ is the averaged sum over all spin states of the matrix element,

$$\overline{\left|\mathcal{M}_{\text{fi}}\right|}^2 := \frac{1}{4j} \sum_{s_i, r_i} \left|\mathcal{M}_{\text{fi}}\right|^2. \tag{3.24}$$

The cross section contains the kinematic triangle function, which is defined as

$$\lambda\left(s, m_a^2, m_b^2\right) := \left(\left(\sqrt{m_a^2} + \sqrt{m_b^2}\right)^2\right)\left(s - \left(\sqrt{m_a^2} - \sqrt{m_b^2}\right)^2\right), \tag{3.25}$$

The relativistically covariant integral over the final state momenta is the N-body phase space integral which has the form

$$R_2(s; m_1^2, m_2^2) := \int \mathrm{d}^4 p_1 \mathrm{d}^4 p_2 \delta(p_1^2 - m_1^2) \delta(p_2^2 - m_2^2) \delta^{(4)}(p_a + p_b - p_1 - p_2). \tag{3.26}$$

After integrating over p_2 and identifying $p_a \equiv p, p_b \equiv q, p_1 \equiv p'$ as above, we obtain

$$R_2(s) = \int \frac{\mathrm{d}^3 p}{2E} \delta(s + p'^2 - 2(p+q) \cdot p') \tag{3.27}$$

Laboratory system (LS)

In the LS, the target particle is initially at rest. Thus we have the following initial and final momenta, where ν is the photon energy:

$$\begin{aligned}
&\textbf{initial} && \textbf{final} \\
&p = (m, \mathbf{0})\,, && p' = (E, k\sin\phi, 0, k\cos\phi)\,, && (3.28a)\\
&q = (\nu, 0, 0, \nu)\,, && q' = (\nu', \nu'\sin\theta, 0, \nu'\cos\theta)\,. && (3.28b)
\end{aligned}$$

We consider only real Compton scattering, which implies the on-shell conditions for initial and final state,

$$p^2 = M^2 = p'^2 \quad \text{and} \quad q^2 = 0 = q'^2. \tag{3.29}$$

The Mandelstam variables are related to the LS by

$$\begin{aligned}
s &= M^2 + 2\nu M = M^2 + 2\nu'\left(E - k\cos\left(\theta + \phi\right)\right),\\
t &= 2M^2 - 2EM = -2\nu\nu'\left(1 - \cos\theta\right),\\
u &= M^2 - 2\nu' M = M^2 - 2\nu\left(E - k\cos\phi\right).
\end{aligned} \tag{3.30}$$

From these relations we can deduce Compton's formula for the shift in the photon wavelength:

$$\nu' = \frac{\nu}{1 + \frac{\nu}{M}\left(1 - \cos\theta\right)}. \tag{3.31}$$

Note that in this frame, the photon energy ν is related to s by $\nu(s) = \frac{s-M^2}{2M}$. Hence, the LS prefactor is given by

$$\varphi\left(\nu\right) = \frac{1}{64\pi\nu^2 M^2}. \tag{3.32}$$

The angles θ and ϕ in the LS are related in a complicated manner. Deriving the amplitude in this frame requires involved calculations. It is much more convenient if we choose the CMS for our calculations.

Center-of-Momentum System (CMS)

In the CMS, both particles are initially travelling towards each other with equal absolute 3-momenta. After the collision, both will deviate from their former course by a scattering angle ϑ, while maintaining their velocity.

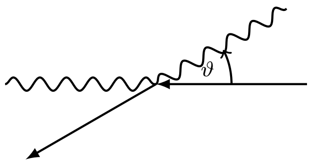


Figure 3.3: Compton scattering in the center-of-momentum frame.

Expressed as initial and final 4-momenta, this is written as

initial	**final**	
$p = (E_{\text{cm}}, 0, 0, -\omega)\,,$	$p' = (E'_{\text{cm}}, -\omega \sin\vartheta, 0, -\omega\cos\vartheta)\,,$	(3.33a)
$q = (\omega, 0, 0, \omega)\,,$	$q' = (\omega, \omega\sin\vartheta, 0, \omega\cos\vartheta)\,.$	(3.33b)

Note that in the CMS, the photon energy is $\omega \equiv \omega_{\text{cm}}(s) = \frac{s-M^2}{2\sqrt{s}}$. The CMS prefactor $\varphi(s)$ in terms of ω is

$$\varphi(\omega) = \frac{1}{64\pi\left(2E\omega^3 + 2\omega^4 + \omega^2 M^2\right)}. \tag{3.34}$$

With regard to the CMS, we can write the differential cross section as

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1^{\text{cm}}} &= \frac{1}{4\pi}\frac{1}{8\pi^2\lambda^{\frac{1}{2}}(s, M^2, 0)}\int \frac{\mathrm{d}^3 p_{\text{cm}}}{2E_{\text{cm}}}\frac{\mathrm{d}^3 q'_{\text{cm}}}{2\omega}\delta^{(4)}(p+q-p'-q')\overline{|\mathcal{M}_{\text{fi}}|}^2 \\ &= \frac{1}{64\pi^2 s}\frac{|\boldsymbol{p}'_{\text{cm}}|}{|\boldsymbol{p}_{\text{cm}}|}\overline{|\mathcal{M}_{\text{fi}}|}^2.\end{aligned} \tag{3.35}$$

However, we want to express the differential cross section in an invariant manner. Therefore, we change the integration variable to the Mandelstam variable t. Its differential is related to the CMS differential solid angle via the substitution

$$\begin{aligned}\mathrm{d}t &= 2\,|\boldsymbol{p}_b^{\text{cm}}|\,|\boldsymbol{p}_2^{\text{cm}}|\,\mathrm{d}\cos\vartheta = \frac{1}{\pi}\,|\boldsymbol{p}_b^{\text{cm}}|\,|\boldsymbol{p}_2^{\text{cm}}|\,\mathrm{d}\Omega_2^{\text{cm}} \\ &= 2\omega^2 \mathrm{d}\cos\vartheta,\end{aligned} \tag{3.36}$$

so that we finally obtain the invariant cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \underbrace{\frac{1}{16\pi\lambda(s, M^2, 0)}}_{=\varphi(s)}\overline{|\mathcal{M}_{\text{fi}}(s,t)|}^2. \tag{3.37}$$

3.3 Low-Energy Theorems

3.3.1 Original Theorem for Spin 1/2

In 1954, Low [Low54], Gell-Mann and Goldberger [GMG54] derived a low-energy theorem (LET) for nuclei, i.e. particles with spin $S = 1/2$. The LET states that to first order in the perturbation expansion in the photon energy ν, the amplitude $T(\nu)$ is equal to the Born contribution,

$$T^{(LO)}(\nu) = T_{\text{Born}}(\nu). \tag{3.38}$$

In the limit $\nu \to 0$, the Born term is the exact solution to the amplitude functions f_i (c.f. section 3.1).
In the nucleon vertex, two underlying structures γ^μ and $\sigma^{\mu\nu}$ appear which correspond to the electromagnetic moments, i.e. to the aforementioned structure functions f_i. The structure γ^μ corresponds to the electric dipole, the structure $\sigma^{\mu\nu}$ leads to the anomalous magnetic moment.

For the nuclei, the Born terms are therefore

$$f^{Born}(\nu) \equiv f_0(0) = -\frac{Z^2\alpha}{M} \qquad \text{and} \tag{3.39}$$

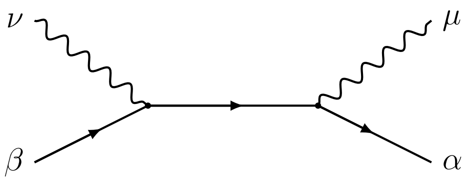
$$g^{Born}(\nu) \equiv f_1(0) = -\frac{\mu_a^2}{2\pi} = -\frac{\alpha\kappa^2}{2M^2}, \tag{3.40}$$

where the right-hand side of $f_0(0)$, found by Thirring [Thi50], is the classical Thomson limit, and $f_1(0)$ is the anomalous magnetic moment LET [Low54, GMG54].

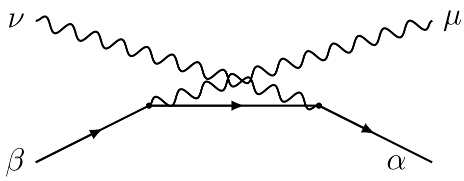
Validity of the LETs As Pantfoerder [Pan98] points out, it is worth mentioning that this LET is only a fictitious LO-in-α contribution to the anomalous magnetic moment. This is in contrast to the Thirring LET which is exact in all orders of α. This is due to the fact that there are no radiative corrections to the physical electric charge. The anomalous magnetic moment LET, on the other hand, is not guaranteed to hold in all orders of the expansion; in analogy to the smallness of the Schwinger correction to the lepton anomalous magnetic moment, we assume that higher order corrections are also small, so that in principle the LET holds. Cheng [Che68, Che69] and Roy and Singh [SR70] showed that this is the case up to NLO; still, higher order contributions to κ might necessitate modifications to eq. (3.40).

3.3.2 LET for Massive Vector Bosons

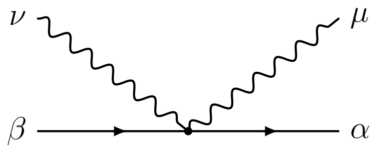
In order to derive the LET values for the structure functions, we first need to calculate the tree-level forward scattering amplitude. As there are, for arbitrary spin j, $2j+1$ elecromagnetic structure functions, we will have three structure functions f_0, f_1, and f_2 for massive vector bosons, i.e. $j = 1$.



(a) s-channel contribution



(b) crossed (u-channel) contribution



(c) seagull vertex

Figure 3.4: Born diagrams for polarized Compton scattering off massive vector bosons

At tree level, the three scattering diagrams found in (fig. 3.4) contribute to the electromagnetic moments (the formulas to these diagrams are given in App. C.1). In order to identify which structure from the calculated amplitude corresponds to which electromagnetic moment from the decomposition, it is useful to know that the massive vector boson spinors are related to the spin operators by

$$W_i^* W_j - W_j^* W_i = [S_i, S_j] \quad \text{and} \tag{3.41a}$$

$$W_i^* W_j + W_j^* W_i = 2\delta_{ij} - \{S_i, S_j\} \, . \tag{3.41b}$$

In the rest frame of the target we obtain for fixed photon polarization the correspondence

$$W^* \cdot W \, \varepsilon^* \cdot \varepsilon = 1, \tag{3.42a}$$

$$W^* \cdot \varepsilon^* \, W \cdot \varepsilon - W^* \cdot \varepsilon \, W \cdot \varepsilon^* = [S_i, S_j] \, \varepsilon_i^* \varepsilon_j, \tag{3.42b}$$

$$W^* \cdot \varepsilon^* \, W \cdot \varepsilon + W^* \cdot \varepsilon \, W \cdot \varepsilon^* = \{S_i, S_j\} \, \varepsilon_i^* \varepsilon_j, \tag{3.42c}$$

$$W \cdot q \, W^* \cdot q \, \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon} = -\nu^2 \left(1 - (S_3)^2\right), \tag{3.42d}$$

which can be directly related to the target helicity using

$$[S_i, S_j]\,\varepsilon_i^*\varepsilon_j = -S_3 = -\lambda, \tag{3.43a}$$

$$\{S_i, S_j\}\,\varepsilon_i^*\varepsilon_j = (S_3)^2 = \lambda^2, \tag{3.43b}$$

$$-\nu^2\left(1-(S_3)^2\right) = -\nu^2(1-\lambda^2). \tag{3.43c}$$

Without further ado, the result for the tree-level forward Compton scattering amplitude using our effective Lagrangian is given by:

$$\begin{aligned} T_{fi} = &-\frac{e^2}{M}W^*\cdot W\,\varepsilon^*\cdot\varepsilon \\ &-\frac{e^2\nu}{4M^2}(\ell_1-1)^2(W^*\cdot\varepsilon^*\,W\cdot\varepsilon - W^*\cdot\varepsilon\,W\cdot\varepsilon^*) \\ &-\frac{e^2}{M^3}\ell_2 W\cdot q\,W^*\cdot q\,\boldsymbol{\varepsilon^*}\cdot\boldsymbol{\varepsilon} \\ &-\frac{e^2}{M}\frac{3\nu^2}{4M^2}(\ell_1-1)\ell_2(W^*\cdot\varepsilon^*\,W\cdot\varepsilon + W^*\cdot\varepsilon\,W\cdot\varepsilon^*) \\ &-\frac{e^2}{M}\frac{\nu}{16M^3}\ell_2^2\nu^2(W^*\cdot\varepsilon^*\,W\cdot\varepsilon - W^*\cdot\varepsilon\,W\cdot\varepsilon^*), \end{aligned} \tag{3.44}$$

from which we can deduce the specific expressions for the functions f_n,

$$f_0(\nu) = -1 - \frac{\nu^2}{M^2}\ell_2, \tag{3.45}$$

$$f_1(\nu) = -\frac{1}{4M}(1-\ell_1)^2 + \frac{\nu^3}{16M^3}\ell_2^2, \tag{3.46}$$

$$f_2(\nu) = \frac{3\nu^2}{4M^2}(1-\ell_1)\ell_2. \tag{3.47}$$

In the low-energy limit $\nu \to 0$, we finally obtain the LET values of the structure functions:

$$f_0(0) = -1, \tag{3.48}$$

$$f_1(0) = -\frac{1}{4M}(\ell_1-1)^2 = -\frac{1}{4M}\kappa^2, \tag{3.49}$$

$$f_2(0) = \frac{3\nu^2}{4M^2}(\ell_1-1)\ell_2 = \frac{3\nu^2}{4M^2}\kappa(\kappa+Q_{\mathrm{a}}). \tag{3.50}$$

3.4 Optical Theorem

The optical theorem relates the absorptive part of the amplitude, $\mathrm{Abs}\, T$, to the total photoabsorption cross section,

$$\sigma(\nu) = \frac{4\pi}{\nu} \mathrm{Abs}\, T(\nu) = \frac{e^2}{2M\nu} \varepsilon^*_\mu \varepsilon_\nu \mathrm{Abs}\, T^{\mu\nu}. \tag{3.51}$$

A derivation of the optical theorem can be found in [Pan98]. Using the decompositions in sect. 3.1, we obtain for the polarized forward Compton scattering amplitude, via eq. (3.15) and (3.18a), the optical theorems for $j = 1/2$ and $j = 1$, respectively:

Optical Theorem for Spin 1/2

$$\mathrm{Im} f_0(\nu) = \frac{\nu}{8\pi} \left(\sigma_{\frac{1}{2}}(\nu) + \sigma_{\frac{3}{2}}(\nu) \right), \tag{3.52a}$$

$$\mathrm{Im} f_1(\nu) = \frac{1}{8\pi} \left(\sigma_{\frac{1}{2}}(\nu) - \sigma_{\frac{3}{2}}(\nu) \right). \tag{3.52b}$$

Optical Theorem for Massive Vector Bosons

$$\mathrm{Im} f_0(\nu) = \frac{\nu}{8\pi} \underbrace{(\sigma_{+1}(\nu) - \sigma_0(\nu) + \sigma_{-1}(\nu))}_{=:\sigma_{\mathrm{T}}(\nu)}, \tag{3.53a}$$

$$\mathrm{Im} f_1(\nu) = \frac{1}{8\pi} \underbrace{(\sigma_{-1}(\nu) - \sigma_{+1}(\nu))}_{=:\Delta\sigma(\nu)}, \tag{3.53b}$$

$$\mathrm{Im} f_2(\nu) = \frac{1}{16\pi\nu} \underbrace{(2\sigma_0(\nu) - (\sigma_{-1}(\nu) + \sigma_{+1}(\nu)))}_{=:\sigma_{\mathrm{Q}}(\nu)}. \tag{3.53c}$$

3.5 Derivation of Forward Dispersion Relations

We have now laid the foundation to be able to derive sum rules for the electromagnetic moments from our theory: We have expanded our general Compton scattering amplitude in terms of the target helicity λ (c.f. section 3.1, eqs. (3.4) and (3.18a)). We have then constructed a gauge invariant phenomenological effective Lagrangian $\mathcal{L}_{\mathrm{Eff}}$ to describe our interactions (section 2.3) and have derived Feynman rules from it (section 2.5). Based on this, we have derived the low-energy theorems (LETs) for the structure functions from $\mathcal{L}_{\mathrm{Eff}}$, including a new quadrupole LET.

Now we will use the properties of physical scattering amplitudes to derive the GDH and quadrupole sum rules. We base our argument on the decomposition of the $j = 1$ forward scattering amplitude,

$$T(\nu) = W^* \left[\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^* f_0(\nu) + \nu f_1(\nu) \mathrm{i} \boldsymbol{S} \cdot (\boldsymbol{\varepsilon}^* \times \boldsymbol{\varepsilon}) + \nu^2 f_2(\nu) \left(\boldsymbol{S}^2 - S_3^2 \right) \right] W. \tag{3.54}$$

Physical scattering amplitudes possess the properties of analyticity and unitarity. Since $T(\nu)$ is causal, so are the decomposition components $f_i(\nu)$, which means they are holomorphic in the upper half of the complex energy plane, by virtue of *Titchmarsh's theorem*. From unitarity, i.e. *Parseval's theorem*, we can deduce the validity of the *Schwarz reflection principle* $f_i(\nu^*) = f_i^*(\nu)$, so that the integration can be continued into the lower half plane. Thus, we can derive for the f_i *dispersion relations* between absorptive and reflective part of the f_i,

$$\mathrm{Re}\, f_i(\nu) = \frac{1}{\pi} \mathcal{P} \int\limits_{-\infty}^{\infty} \mathrm{d}\nu' \, \frac{\mathrm{Im}\, f_i(\nu')\nu'}{\nu'^2 - \nu^2}, \tag{3.55}$$

analogous to the derivation in sect. 1.3, where we also have discussed the matter of integration paths. $\mathcal{P}$ denotes the *Cauchy principal value.*Due to the crossing symmetry of the amplitude, the f_i have the following properties under crossing:

$$f_0(-\nu) = f_0(\nu), \tag{3.56a}$$

$$f_1(-\nu) = -f_1(\nu), \tag{3.56b}$$

$$f_2(-\nu) = f_2(\nu), \tag{3.56c}$$

so we can rewrite the dispersion relations as

$$\mathrm{Re}\, f_i(\nu) = \frac{2}{\pi} \mathcal{P} \int\limits_{0}^{\infty} \mathrm{d}\nu' \frac{\nu' \mathrm{Im}\, f_i(\nu')}{\nu'^2 - \nu^2}, \tag{3.57}$$

which is the *Kramers-Kroenig relation.* Note that the physical amplitude is proportional to the real part of the functions f_i, so in the following, we will suppress the Re , so that $f_i(\nu)$ implicitly means $\mathrm{Re}\, f_i(\nu)$.

Applying the optical theorem on all of the dispersion relations, we obtain

$$f_0(\nu) = \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}\nu' \frac{\nu'^2 \sigma_{\mathrm{T}}(\nu')}{\nu'^2 - \nu^2}, \quad \text{and} \tag{3.58a}$$

$$f_1(\nu) = \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}\nu' \frac{\nu' \Delta\sigma(\nu')}{\nu'^2 - \nu^2}, \tag{3.58b}$$

which are valid in this form for both the $j = 1/2$ and $j = 1$ case; additionally, for massive vector bosons we obtain the dispersion relation

$$f_2(\nu) = \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}\nu' \frac{\sigma_{\mathrm{Q}}(\nu')}{\nu'^2 - \nu^2}. \tag{3.58c}$$

The Born terms are equal to the amplitude in the low-energy limit in the lowest order, $\nu \to 0$, and replacing the $f_i(0)$ with their respective LETs, we finally obtain the electromagnetic moment sum rules:

$$-\frac{e^2}{M} = \frac{1}{\pi} \int_0^\infty \mathrm{d}\nu' \sigma_{\mathrm{T}}(\nu') + \int_C \mathrm{d}\nu' \sigma_{\mathrm{T}}(\nu') \tag{3.59}$$

for the Thomson limit; this is not a real sum rule. As the integral over the contour C at infinity does not vanish, the no-subtraction hypothesis is not valid here. To recover the finite Thomson limit one has to include the full path. Next, we obtain the GDH sum rule,

$$-\frac{e^2}{M}\kappa^2 = \frac{1}{\pi} \int_0^\infty \mathrm{d}\nu' \frac{\Delta\sigma(\nu')}{\nu'}; \tag{3.60}$$

and finally, a novel sum rule for spin-1 particles, the *quadrupole sum rule* (QSR):

$$\boxed{\frac{3e^2}{4M^2}\kappa(\kappa + Q_{\mathrm{a}}) = \int_0^\infty \mathrm{d}\nu' \frac{\sigma_{\mathrm{Q}}(\nu')}{\nu'^2}.} \tag{3.61}$$

At the order of the QSR integrand, polarizabilities might contribute due to quadratic terms in the energy to the QSR at tree level. If this is indeed the case, the Lagrangian would have to be expanded by appropriate polarizability terms. Through this, the polarizability might even contribute to the tree-level value of any electromagnetic moment. Consequently, we need to check the sum rules at tree-level to ascertain if this is the case. As the GDH has been confirmed for $j = 1$, we should recover this result. For the quadrupole, this has not been done yet, so we will check if the theory is sufficient to derive a complete quadrupole sum rule. In the following part, we will evaluate the sum rules at LO and NLO to test our theory.

Chapter 4

Testing the Sum Rules in QFT

We have now successfully derived the GDH sum rule from our effective Lagrangian, as well as a new quadrupole sum rule (c.f sect. 3.5) which we will now put to test in a QFT. We use a perturbative approach. Due to the properties of the dispersion relations, this corresponds directly to a perturbative expansion of the sum rules. Hence, at every order of the expansion both sides of the sum rules should agree.

To recapitulate, the sum rules are

$$-\frac{2\pi\alpha}{M^2}\kappa^2 = \frac{1}{\pi}\int_0^\infty \frac{d\nu}{\nu}\Delta\sigma(\nu), \tag{4.1a}$$

$$\frac{\pi\alpha}{M^2}\kappa(\kappa+Q_{\mathrm{a}}) = \frac{1}{2\pi}\int_0^\infty \frac{d\nu}{\nu^2}\sigma_{\mathrm{Q}}, \tag{4.1b}$$

where κ is the anomalous magnetic moment and Q_{a} is the anomalous quadrupole moment, and

$$\Delta\sigma(\nu) = \sigma_{-1} - \sigma_{+1} \qquad \text{and} \tag{4.2}$$

$$\sigma_{\mathrm{Q}} = 2\sigma_0(\nu) - \sigma_-(\nu) - \sigma_+(\nu) \tag{4.3}$$

are the linear combinations of the doubly polarized cross section. We assume no contributions from polarizabilities to the sum rules.

The most convenient way to test the sum rules is to assume that the target particles are point-like, i.e. elementary particles. This implies that we will perform the calculations in the limit of natural values for the electromagnetic moments (c.f. sect. 2.6). Then, the anomalous electromagnetic moments κ and Q_a should vanish at tree-level,

$$\kappa = Q_\mathrm{a} = 0. \tag{4.4}$$

For the verification, this means that we should recover that the right-hand side, i.e. the spectral integrals in eq. (4.1b), evaluate to zero if there are indeed no contributions from polarizabilities.

Applying the natural limit to the effective Lagrangian, the Feynman rules reduce to

$$\begin{aligned} \Rightarrow \Gamma^{\alpha\beta,\mu}(p,p') = &-e\left(g^{\alpha\beta}P^\mu - p'^\beta g^{\alpha\mu} - p^\alpha g^{\beta\mu}\right) \\ &+ e\left(q^\beta g^{\alpha\mu} - q^\alpha g^{\beta\mu}\right) \end{aligned} \tag{4.5}$$

for the 3–point vertex, and

$$\Rightarrow \Gamma^{\alpha\beta,\mu\nu}(p,p') = -\left(2g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right)e^2 \tag{4.6}$$

for the seagull. In this limit, we actually obtain a truncated massive Yang-Mills theory. In the following section we will describe an SU(2) Yang-Mills theory for the massive vector bosons inspired by the electroweak unification.

4.1 QFT: Yang-Mills Theory

4.1.1 Motivation

For a moment, let us interpret the previously discussed theory of massive vector bosons in the framework of a well-known example: the theory of electroweak interaction. The original theory, set aside fermion interactions, is an SU(2)×U(1) theory with four gauge boson fields, namely the three W bosons and the B^0 boson [BRS95]. Of those, $W^\pm$ are real bosons which appear in *charged current* reactions, measured e.g. at HERA [Kuz99, Kuz08]. The W^0 and the B^0 are unphysical fields. The massless photon and the massive Z^0, the latter being responsible for *neutral current* reactions, are superpositions of the W^0 and B^0 with a mixing angle θ_W, also called *Weinberg* *angle* [Wei72]. The Weinberg angle is a free parameter of the electroweak theory. In consequence, it is valid to choose

$$\theta_{\mathrm{W}} = 90^\circ, \qquad \text{or} \qquad \sin(\theta_{\mathrm{W}}) = 1, \tag{4.7}$$

so that the two neutral bosons coincide with the fields,

$$|\gamma\rangle = |W^0\rangle \qquad \text{and} \qquad |Z^0\rangle = |B^0\rangle. \tag{4.8}$$

The mass of the Z^0 is related to the $W^\pm$ mass via $M_{\mathrm{Z}} = {}^{M_{\mathrm{W}}}/_{\cos\theta_{\mathrm{W}}}$, hence in this case it diverges, $M_{\mathrm{Z}} \to \infty$. Thus, the Z^0 decouples. Furthermore, $g' \to \infty$, and from the definition of the electric charge we find

$$e \equiv \frac{gg'}{\sqrt{g^2 + g'^2}} = g. \tag{4.9}$$

We therefore obtain a Yang-Mills theory containing three bosons, i.e. two (massive) bosons W and a (massless) photon. Note that the non-zero mass breaks the SU(2) gauge symmetry down to U(1). One could introduce the mass without breaking the symmetry explicitly, i.e. through the Higgs mechanism, see e.g. [BD65, PS95]. However, this point is not relevant to our forthcoming discussion. In what follows we consider the electroweak theory for $\theta_{\mathrm{W}} = 90^\circ$ and examine to which extent it coincides with our previously constructed Lagrangian. It will be seen that the latter is a truncated YM, as it lacks the boson self-interaction term.

4.1.2 Yang-Mills Theory

In sect. 1.2 we introduced SU(N) Yang-Mills theories. We will now concentrate on the case $N = 2$. The generators of this group, in matrix notation, are

$$T^a_{ij} = -\frac{\mathrm{i}}{2}\left(\tau_a\right)_{ij} \tag{4.10}$$

and fulfill the algebra

$$\left[T^a, T^b\right] = \frac{1}{4}\left[\tau_a, \tau_b\right] = \mathrm{i} f_{abc} T_c. \tag{4.11}$$

The coupling constant of the theory g is obtained from the commutator of the covariant derivative. In case of the $\sin\theta_{\mathrm{W}} = 1$ electroweak theory, g is the electric charge, $g = e$. The structure constants of the theory are

$$f_{klm} = \varepsilon_{klm}. \tag{4.12}$$

Since the YM Lagrangian contains, besides quadratic, also terms cubic and quartic in the gauge field, the YM field $\mathcal{A}_\mu$ can be self-interacting. This will play an important role in the following.

In the SU(2) case of Yang-Mills theory, the Lagrangian is of the form

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\sum_{a=1}^{3} \mathcal{F}^a_{\mu\nu}\mathcal{F}^{a,\mu\nu} \tag{4.13}$$

where $\mathcal{F}^a_{\mu\nu}$ is the field density (note that $\mathcal{A}^a_\mu$ are real fields, and hence $\mathcal{F}^a$ are real tensors):

$$\mathcal{F}^a_{\mu\nu} = \underbrace{\partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu}_{=:\mathcal{G}^a_{\mu\nu}} + e\varepsilon_{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu. \tag{4.14}$$

4.1.3 Derivation of the Yang-Mills Lagrangian

In analogy to the complex Proca fields (see sect. 2.2), we express our charged vector boson fields in terms of the real YM fields $\mathcal{A}^a$, as follows:

$$W_\mu = \frac{\mathcal{A}^1_\mu + i\mathcal{A}^1_\mu}{\sqrt{2}}, \qquad W^*_\mu = \frac{\mathcal{A}^2_\mu - i\mathcal{A}^2_\mu}{\sqrt{2}}, \qquad A_\mu = \mathcal{A}^3_\mu. \tag{4.15}$$

With the definition (4.14) we obtain the field strength tensors

$$\begin{aligned}
\mathcal{F}^1_{\mu\nu} &= \mathcal{G}^1_{\mu\nu} + g\varepsilon_{1bc}\mathcal{A}^b_\mu\mathcal{A}^c_\nu && \text{(4.16a)}\\
&\overset{(4.17)}{=} \frac{1}{\sqrt{2}}\left(\widetilde{W}_{\mu\nu} + \widetilde{W}^*_{\mu\nu} - ig\left(T_{\mu\nu} - T^*_{\mu\nu}\right)\right)\\
&= \frac{1}{\sqrt{2}}\left(W_{\mu\nu} + W^*_{\mu\nu}\right),\\
\mathcal{F}^2_{\mu\nu} &= \mathcal{G}^2_{\mu\nu} + g\varepsilon_{2bc}\mathcal{A}^b_\mu\mathcal{A}^c_\nu && \text{(4.16b)}\\
&= \frac{-i}{\sqrt{2}}\left(\widetilde{W}_{\mu\nu} - \widetilde{W}^*_{\mu\nu} - ig\left(T_{\mu\nu} - T^*_{\mu\nu}\right)\right),
\end{aligned}$$

and

$$\mathcal{F}^3_{\mu\nu} = F_{\mu\nu}, \tag{4.16c}$$

where $\widetilde{W}_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$ and we introduced the tensors

$$T_{\mu\nu} := W_\mu A_\nu - W_\nu A_\mu, \tag{4.17}$$

The $\mathcal{G}^a_{\mu\nu}$ are the free-field tensors,

$$\begin{aligned}
\mathcal{G}^1_{\mu\nu} &= \frac{1}{\sqrt{2}}\left(\widetilde{W}_{\mu\nu} + \widetilde{W}^*_{\mu\nu}\right), \\
\mathcal{G}^2_{\mu\nu} &= \frac{-i}{\sqrt{2}}\left(\widetilde{W}_{\mu\nu} - \widetilde{W}^*_{\mu\nu}\right), \\
\mathcal{G}^3_{\mu\nu} &= F_{\mu\nu},
\end{aligned} \tag{4.18}$$

so we can derive the free-field contribution to the Yang-Mills Lagrangian:

$$\sum_a \mathcal{G}^a_{\mu\nu}\mathcal{G}^{a,\mu\nu} = \widetilde{W}^*_{\mu\nu}\widetilde{W}^{\mu\nu} + F_{\mu\nu}F^{\mu\nu}. \tag{4.19}$$

With $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$ the full YM Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{YM} = &-\frac{1}{2}W^*_{\mu\nu}W^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&+\frac{1}{2}ieW^*_\mu W_\nu F^{\mu\nu} + \frac{1}{2}e^2\left(|W\cdot W|^2 - |W|^4\right)
\end{aligned} \tag{4.20}$$

4.1.4 Vertex for W – W Interaction

The propagator for the W-fields is the familiar Proca propagator. The Feynman rules for the γWW and $\gamma\gamma WW$ vertices are listed in the following (c.f. the derivation of the Feynman rules for the effective Lagrangian in sect. 2.5):

$$\begin{aligned}
\Gamma^{\alpha\beta\mu}(p',p) = &-\left(g^{\alpha\beta}P^\mu - p'^\alpha g^{\mu\beta} - p^\beta g^{\mu\alpha}\right) \\
&-2\left(q^\alpha g^{\mu\beta} - q^\beta g^{\mu\alpha}\right)
\end{aligned} \tag{4.21}$$

and

$$\Gamma^{\alpha\beta\mu\nu} = -e^2 \left(2g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right). \tag{4.22}$$

The two vertices are identical to those for the effective Lagrangian in the natural limit, c.f. eqs. (4.5) and (4.6). In addition to those, a new vertex describing the W self-interaction appears. The Feynman rule for it is:

$$\begin{aligned}\langle W^\rho W^\tau | \int d^4x e^2 \left(W \cdot W W^* \cdot W^* - |W|^4\right) |W^\alpha W^\beta\rangle \\ = e^2 \left(2g^{\alpha\beta}g^{\rho\tau} - g^{\alpha\tau}g^{\rho\beta} - g^{\alpha\rho}g^{\tau\beta}\right) =: \Gamma^{\alpha\beta\rho\tau}_{\text{YM}}.\end{aligned} \tag{4.23}$$

The additional bosonic self-interaction term does not contribute to the tree-level Compton scattering amplitude. Consequently, an ambiguity arises between the effective and YM Lagrangians with regard to the tree level. However, the self-interaction does contribute to Compton scattering at loop-level through the tadpole graph, as will be discussed in the following chapter. Before we come to that we will compute the sum rules at tree level in the following sections.

4.2 Gauge Invariance: Ward-Takahashi Identities

Ward-Takahashi identities (WTIs) [Tak57] are a translation of Noether's theorem into the framework of QFT. They are a generalization of the *Ward identity* [War50] which ensures the on-shell condition of the external particles. For example, for a physical QED process with one external photon, the photon has to be transversal. Translated into a Ward identity, the condition

$$k_\mu \mathcal{M}^\mu_{\text{fi}}(k) = 0 \tag{4.24}$$

has to be fulfilled, which implies that longitudinal photons are unphysical and should not contribute to the scattering matrix. This is an equivalent formulation of the principle of *gauge invariance*.

In these calculations, we check the U(1) Ward identities to ensure that we correctly derived the Feynman rules gauge invariantly. As the test of the sum rules is done in the natural limit of the theory (see sect. 2.6), we calculate the WTIs with natural values $\ell_1 = 1$ and $\ell_2 = 0$. The Feynman rules can be found in app. A.1. The WTI for the tree level 3-point vertex (fig. 2.1(a)) in the off-shell case is

$$q \cdot \Gamma^{\alpha\beta} = \left(\Delta^{-1}(p') - \Delta^{-1}(p)\right)^{\alpha\beta}, \tag{4.25}$$

where Δ^{-1} is the inverse propagator,

$$\Delta^{-1}_{\alpha\beta}(p) := -g_{\alpha\beta}\left(p^2 - M^2\right) + p_\alpha p_\beta \tag{4.26}$$

which yields identity if contracted with the propagator,

$$\Delta^{-1}_{\alpha\beta}(p)\Delta^{\beta\tau}(p) = \delta^{\tau}_{\alpha}. \tag{4.27}$$

One can easily see that this WTI reduces to zero if initial and final state are assumed to be on shell, i.e. $p^2 = p'^2 = M^2$, and it is contracted with the initial and final state polarization vectors:

$$\begin{aligned}
&W^*_\alpha(p')\left(q \cdot \Gamma^{\alpha\beta}\right) W_\beta(p) \\
&\quad = W^*_\alpha(p')\left(\Delta^{-1}(p') - \Delta^{-1}(p)\right)^{\alpha\beta} W_\beta(p) \\
&\quad = -W^*_\alpha(p')\Big(g^{\alpha\beta}\cancel{\left(p'^2 - M^2\right)} - p'^{\alpha\prime}p^{\beta}\Big)W_\beta(p) \\
&\qquad + W^*_\alpha(p')\Big(g^{\alpha\beta}\cancel{\left(p^2 - M^2\right)} - p^{\alpha}p^{\beta}\Big)W_\beta(p) \\
&\quad = \cancelto{0}{W^*_\alpha(p')p'^\alpha}\, p'^\beta W_\beta(p) - W^*_\alpha(p')p^\alpha\, \cancelto{0}{p^\beta W_\beta(p)},
\end{aligned} \tag{4.28}$$

where in the last line, we used the transversality property of the spinors, c.f. sect. 2.2. It then follows that on shell,

$$q \cdot \Gamma = 0. \tag{4.29}$$

4.3 Tree-Level Verification

In order to obtain the tree level contribution to the GDH, we have to calculate the polarized scattering cross sections $\sigma_\lambda(s)$. In what follows we fix the photon helicity at $\sigma = \sigma' = 1$, and directly use the target helicity amplitude

$$T_{\lambda\lambda'} = W^*_\alpha\left(p', \lambda'\right)\varepsilon^*_\mu\left(q'\right)\mathcal{M}^{\alpha\beta,\mu\nu}\left(p, p', q, q'\right)W_\beta\left(p, \lambda\right)\varepsilon_\nu\left(q\right). \tag{4.30}$$

The polarized cross section for the helicity λ is defined in the CMS as

$$\begin{aligned}\frac{\mathrm{d}\sigma\,(\omega)}{\mathrm{d}\Omega} &= \frac{\mathrm{d}t}{\mathrm{d}\Omega}\frac{1}{16\pi\lambda(s,M,0)}\left|T\right|^2 = \frac{\omega^2}{\pi}\frac{1}{16\pi\left(s-M^2\right)^2}\left|T\right|^2 \\ &= \frac{1}{16\pi^2}\frac{1}{4s}\frac{1}{2(2j+1)}\sum_{\lambda\lambda'}\left|T_{\lambda\lambda'}\right|^2,\end{aligned} \tag{4.31}$$

where $T_{\Lambda\Lambda'}$ is the helicity amplitude. On tree level, the amplitude consists of the three diagrams shown in fig. (3.4). The Feynman amplitude $\mathcal{M}$ is thus defined as:

$$\begin{aligned}\mathcal{M}^{\alpha\beta,\mu\nu} &= \Gamma^{\alpha\sigma,\mu}\left(p',p+q\right)\Delta_{\sigma\rho}(p+q)\Gamma^{\rho\beta,\nu}\left(p+q,p\right) \\ &+ \Gamma^{\alpha\sigma,\nu}\left(p',p-q\right)\Delta_{\sigma\rho}(p-q)\Gamma^{\rho\beta,\mu}\left(p-q',p\right) \\ &+ \Gamma^{\alpha\beta,\mu\nu}\left(p,p'\right),\end{aligned} \tag{4.32}$$

which is the sum of s-channel, u-channel and seagull term, see section 2.5 and App. A.1.

As already noted in section 2.2, we use the following definitions for the boson polarization vectors:

$$\boldsymbol{\zeta}^{\mu}_{\pm} = \frac{1}{\sqrt{2}}\left(\pm 1,-i,0\right) \equiv \boldsymbol{\varepsilon}_{\pm}, \qquad \boldsymbol{\zeta}_0 z = \hat{\boldsymbol{e}}_3, \quad \text{and} \tag{4.33a}$$

$$W^{\mu}_{\lambda}(p) := \left(\frac{\boldsymbol{p}.\boldsymbol{\zeta}_{\lambda}}{M}, \boldsymbol{\zeta}_{\lambda} + \frac{\boldsymbol{p}.\boldsymbol{\zeta}_{\lambda}}{M(M+E)}\,\boldsymbol{p}\right). \tag{4.33b}$$

The cross section calculation is performed in the CMS (c.f. section 3.2). Hence, we define the initial and final momenta

$$\begin{aligned} p &= \left(E_n,0,0,-\omega\right), \qquad & p' &= \left(E_n,-\omega\sin\theta,0,-\omega\cos\theta\right), \\ q &= \left(\omega,0,0,\omega\right), & q' &= \left(\omega,\omega\sin\theta,0,\omega\cos\theta\right).\end{aligned} \tag{4.34}$$

The respective energies of photon and massive vector boson are

$$\omega = \frac{s-M^2}{2\sqrt{s}} \qquad \text{and} \qquad E_n = \frac{s+M^2}{2\sqrt{s}}. \tag{4.35}$$

After evaluating eq. (4.30) and forming its absolute square, we can integrate the polarized cross sections. The sum rules are integrated in the lab frame. The analytical expression for the polarized cross section difference $\Delta\sigma$ is

$$\Delta\sigma(\nu) = \frac{\left(5+2\frac{\nu}{M}\right)}{\nu^2}\log\left(2\frac{\nu}{M}+1\right) - \frac{2(M+\nu)\left(15M+66\nu+76\frac{\nu^2}{M}\right)}{3\nu\left(M+2\nu\right)^3}. \tag{4.36}$$

Plotted over the lab photon energy ν (fig. 4.1), one can suspect from the form of the integrand that the integral will vanish, as the integration over the complete spectrum shows. For the GDH integral, we obtain

$$\int_0^\infty \frac{\Delta\sigma(\nu)}{\nu}\,\mathrm{d}\nu = -\left(\frac{\left(5\frac{M}{\nu}+4\right)\log\left(\frac{2\nu}{M}+1\right)-10}{2\nu M}+\frac{1+\frac{8}{3}\frac{\nu}{M}}{(M+2\nu)^2}\right)\Bigg|_{\nu=0}^{\infty} = 0 \tag{4.37}$$

which corresponds to the expected tree-level value for the anomalous magnetic moment.

For the quadrupole sum rule, the case is different. The quadrupole LET is linear in the quadrupole moment, but it also contains a contribution linear in κ. The right-hand side integrand has a so-called tensorial structure (due to the structure of the contributing cross section polarizations). The result for this tensorial cross section is

$$\begin{aligned}\sigma_{\mathrm{Q}}(\nu) = &-\frac{(18M^5+99M^4\nu+183M^3\nu^2+132M^2\nu^3+52M\nu^4+24\nu^5)}{3M^2\nu^2\,(M+2\nu)^3}\\ &+\frac{(6M^2+3\nu M+2\nu^2)}{2\nu^3M^3}\log\left(2\frac{\nu}{M}+1\right).\end{aligned} \tag{4.38}$$

Calculation of the integral in Mathematica yields

$$\int_0^\infty \frac{\sigma_{\mathrm{Q}}(x)}{x^2}\,\mathrm{d}x = -\frac{1}{M^4}. \tag{4.39}$$

Seemingly, the QSR contains polarizability contributions, as it yields a non-zero result. It is an interesting point for further discussion to find how exactly this relation is defined and how the Lagrangian would have to be modified in order to obtain a pure quadrupole sum rule, i.e. one which vanishes at tree level. However, it is vital to note that the QSR without any polarizability contributions does *not* hold.

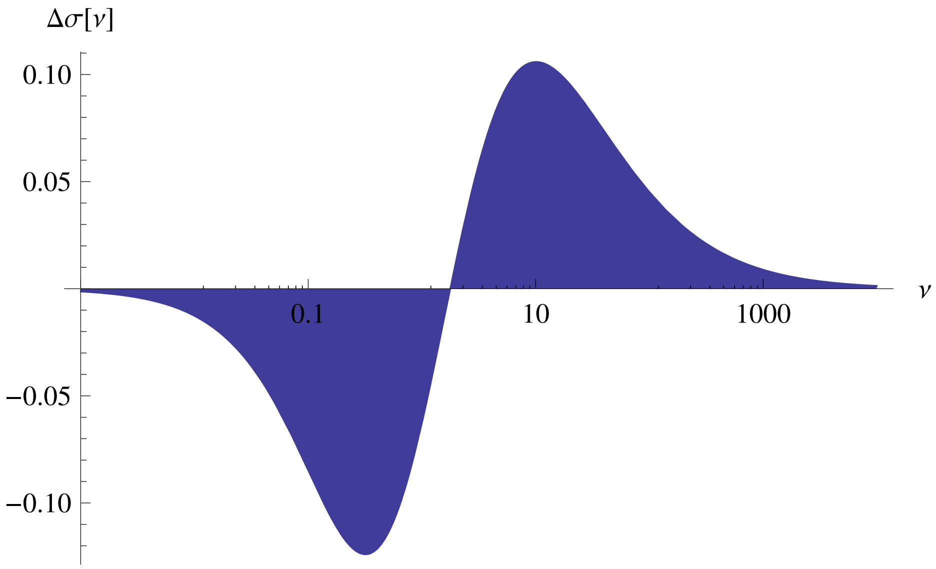


Figure 4.1: Logarithmic plot of the GDH integrand $\Delta\sigma(\nu)$ over the lab photon energy spectrum up to high energies. The mass is chosen as unit mass, $M \equiv 1$. One can see that the integrand converges in both high and low energy regimes.

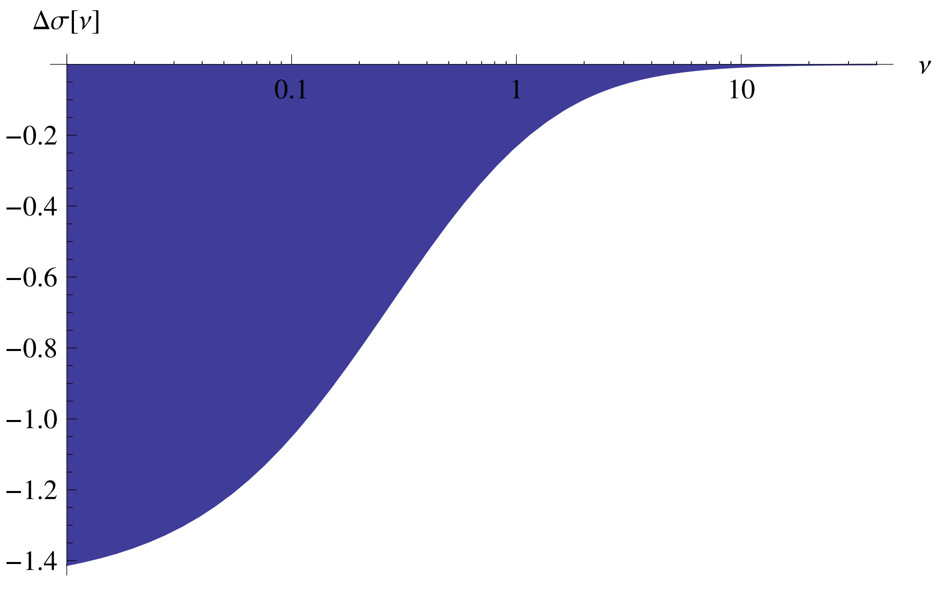


Figure 4.2: Logarithmic plot of the quadrupole sum rule integrand over the photon energy spectrum up to high energies using a unit mass $M \equiv 1$. It is easily concluded that despite the convergence in the high-energy limit, the integral will yield a non-zero result.

Chapter 5

Quantum One-Loop Corrections: Symmetry of the Theory

For the one-loop calculation, we use derivatives of the sum rules to simplify calculations, an approach which has been proposed by Pascalutsa et al. [PHV04]. For reasons explained below, we will test the sum rules in the framework of two quantum field theories:

(i) charged massive point-like vector bosons (described by the effective Lagrangian of Chapter 2 in the natural limit),

(ii) SU(2) Yang-Mills theory, partially massive (electroweak theory with $\theta_{\mathrm{W}} = 90°$ considered in the previous chapter).

The ambiguity of these theories is investigated, and the result prefering the YM case is obtained. In order to verify the GDH at loop level, we have to calculate the anomalous magnetic moment κ from both vertex correction and the GDH integral. The calculation for the vertex correction (for spin $1/2$ known as *Schwinger correction*) is quite straightforward, only the three graphs shown in fig. 5.3 contribute. As we will see in sect. 5.4, this is not sufficient to obey the GDH sum rule; a tadpole diagram from boson self-interaction has to be considered.

The right-hand side of the GDH, i.e. the integral, requires more involved calculations. The reason for this is that the left-hand side is proportional to κ^2, and hence of order $\mathcal{O}(\alpha^3)$. Therefore, higher order diagrams have to be considered, such as one-loop correction to the Compton scattering cross-sections, as well as pair production processes. For electrons, this has been done by Dicus and Vega [DV01]. Alternatively one can use an approach called the *derivative GDH*, in short δGDH. We will discuss this in the following section.

5.1 Derivative GDH

In 2004, Pascalutsa, Vanderhaeghen and Holstein derived a sum rule with a linear relation between anomalous magnetic moment and a derivative of the photoabsorption cross section [PHV04].

Starting from the GDH for arbitrary spin j,

$$-\frac{\pi\alpha}{jM^2}\kappa^2 = \frac{1}{\pi}\int_0^\infty \frac{d\nu}{\nu}\Delta\sigma(\nu), \tag{5.1}$$

a "classical" value κ_0 of the anomalous magnetic moment is introduced such that

$$\kappa = \kappa_0 + \delta\kappa. \tag{5.2}$$

At level of the field theory, this implies that the Lagrangian acquires an explicit term containing the classical anomalous magnetic moment, in analogy to the Pauli term from the classical description of the nucleon. Although we use a tree-level coupling term only similar to Pauli's idea, we call it *Pauli coupling* to emphasize the analogy. In the spin-1 case, the additional term is

$$\mathcal{L}_{\text{class}} = i\frac{e\kappa_0}{4M}W_\mu W_\nu^* F^{\mu\nu}. \tag{5.3}$$

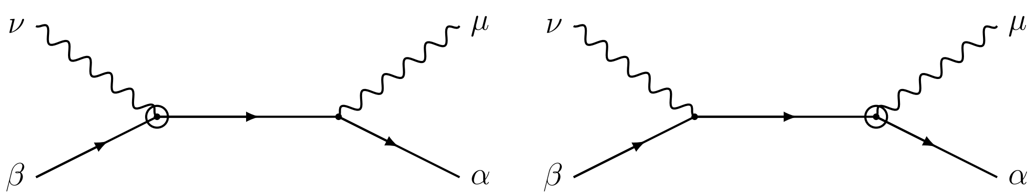


Figure 5.1: Pauli-like anomalous magnetic moment coupling in the left and right scattering 3-point vertices at tree level. Note that we also have to consider the crossed terms. The seagull, in contrast, is not affected by the coupling.

The Pauli coupling is added to one of the 3-point vertices in any of the respective scattering processes, c.f. fig. 5.1. It is important to note that this tree-level value is not physical and cannot be observed in experiment; however it provides a means to calculate the anomalous magnetic moment equivalently to NLO calculations.

Substituting eq. (5.2) into (5.1) for the $j = 1$ case, we obtain:

$$-\frac{\pi\alpha}{M^2}\left[\kappa_0^2 + \delta\kappa^2 + 2\kappa_0\delta\kappa\right] = \frac{1}{\pi}\int_0^\infty \frac{d\nu}{\nu}\Delta\sigma(\nu;\kappa_0). \tag{5.4}$$

Note the explicit dependence on κ_0 of $\Delta\sigma$. Now, if we differentiate the sum rule by κ_0 at $\kappa_0 = 0$, we obtain

$$-\frac{2\pi\alpha}{M^2}\delta\kappa\left(1 + \left[\frac{\partial}{\partial\kappa_0}\delta\kappa\right]_{\kappa_0=0}\right) = \frac{1}{\pi}\int_0^\infty \frac{d\nu}{\nu}\underbrace{\left(\frac{\partial}{\partial\kappa_0}\Delta\sigma(\nu;\kappa_0)\bigg|_{\kappa_0=0}\right)}_{=:\Delta\sigma_1(\nu)}, \tag{5.5}$$

which, as it has been derived non-perturbatively, must be valid at any order. Hence, at leading order, with $\delta\kappa \to \kappa$ we obtain a new relation linear in κ,

$$-\frac{2\pi\alpha}{M^2}\kappa = \frac{1}{\pi}\int_0^\infty \frac{d\nu}{\nu}\Delta\sigma_1(\nu). \tag{5.6}$$

The calculation of the integral in eq. (5.6) is analogous to the calculation in sect. 4.3. We obtain:

$$\Delta\sigma_1(\nu) = \frac{3(4\frac{\nu}{M}+15)\log(2\frac{\nu}{M}+1)}{6\nu^2} - \frac{2\frac{\nu}{M}\left(188\frac{\nu^3}{M^3} + 396\frac{\nu^2}{M^2} + 237\frac{\nu}{M} + 45\right)}{6\nu^2(2\frac{\nu}{M}+1)^3}, \tag{5.7}$$

and hence

$$\kappa = \frac{5\alpha}{3\pi}. \tag{5.8}$$

One might argue that this is not an explicit proof as the results have not been obtained by the calculation of the loop-level GDH integral. On the other hand, the Pauli coupling has historically been an ansatz to explain the experimentally confirmed deviation of the anomalous magnetic moment from its tree-level expectation, i.e. $g = 2$. Additionally, the δGDH has been validated in QED by Pascalutsa et al. [PHV04], as they could recover the results obtained from the loop calculations performed by Dicus and Vega [DV01]. In the same publication, Pascalutsa et al. also have used the δGDH in chiral effective theories for the nucleon to further verify its validity.

This result now has to be compared to the one-loop contribution to κ which is obtained through the vertex correction. First, we will test the gauge invariance of the loop-level graphs via WTIs. After gauge invariance has been confirmed we will perform the loop calculations.

5.2 Gauge Invariance

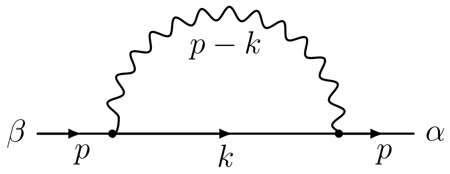


Figure 5.2: Self energy diagram for massive vector bosons

First, let us introduce the massive vector boson self energy, defined as

$$\Sigma(p)^{\alpha\beta} = \int \mathrm{d}\tilde{k}\ \Gamma^{\alpha\rho\xi}(p,k)\Delta_{\rho\tau}(k)\Gamma^{\tau\beta\zeta}(k,p)D_{\xi\zeta}(p-k) \tag{5.9}$$

as depicted in fig. 5.2. The vertex function (see below, fig. 5.3) in tensor notation is

$$\Lambda^{\mu\alpha\beta} = \int \mathrm{d}\tilde{k}\ \Gamma^{\alpha\rho\zeta}\left(p',k'\right)\Delta_{\rho\tau}\left(k'\right)\Gamma^{\tau\sigma\mu}\left(k',k\right)\Delta_{\sigma\eta}\left(k\right)\Gamma^{\eta\beta\xi}\left(k,p\right)D_{\xi\zeta}\left(p-k\right). \tag{5.10}$$

where we have used the loop integration measure

$$\mathrm{d}\tilde{k} := \frac{\mathrm{d}^D k}{(2\pi)^D}. \tag{5.11}$$

When we apply the WTI, i.e. contract Λ with the incoming photon vector q^μ, this expression can be reduced to the tree-level vertex WTI $q\cdot\Gamma^{\alpha\beta}$:

$$q\cdot\Lambda^{\alpha\beta} = \int \mathrm{d}\tilde{k}\quad \Gamma^{\alpha\rho\zeta}\left(p',k'\right)\Delta_{\rho\tau}\left(k'\right)\left(q\cdot\Gamma^{\tau\sigma}\right)\Delta_{\sigma\eta}\left(k\right)\Gamma^{\eta\beta\xi}\left(k,p\right)D_{\xi\zeta}\left(p-k\right). \tag{5.12}$$

As we know from sect. 4.2, the WTI for the tree level vertex is

$$q\cdot\Gamma = \Delta^{-1}(p') - \Delta^{-1}(p), \tag{5.13}$$

which reduces to zero for on-shell particles. In the vertex function, however, $q\cdot\Gamma$ is off-shell, yielding a non-zero contribution. Thus, we can finally write the WTI for the vertex function (using the abbreviations $\Gamma^{\alpha\rho\zeta} \equiv \Gamma^{\alpha\rho\zeta}(p,k)$ and $\Gamma'^{\alpha\rho\zeta} \equiv \Gamma^{\alpha\rho\zeta}(p',k')$) as

$$\begin{aligned} q\cdot\Lambda &= \int \mathrm{d}\tilde{k}\, \Gamma'^{\alpha\rho\zeta}\Delta_{\rho\tau}(k+q)\left(\Delta^{-1}(k+q)-\Delta^{-1}(k)\right)^{\tau\sigma}\Delta_{\sigma\eta}(k)\,\Gamma^{\eta\beta\xi}D_{\xi\zeta}(p-k) \\ &= \int \mathrm{d}\tilde{k}\, \Gamma^{\alpha\sigma\zeta}\Delta_{\sigma\eta}(k)\,\Gamma'^{\eta\beta\xi}D_{\xi\zeta}(p-k) - \int \mathrm{d}\tilde{k}\, \Gamma'^{\alpha\rho\zeta}\Delta_{\rho\eta}(k+q)\,\Gamma^{\eta\beta\xi}D_{\xi\zeta}(p-k)\,, \end{aligned} \tag{5.14}$$

which is of the expected form. On-shell, the WTI reads $q\cdot\Lambda = 0$. We verified this using FORM by explicit contraction with the boson polarization vectors. However, it is useful to further investigate to find a more general proof. To compute $q\cdot\Lambda$, it is vital to know how the propagator and the 3-point vertex transform under shifts $p \to p' = p+q$ and $k \to k' = k+q$:

$$\Gamma^{\alpha\beta\mu}(p',k') = \Gamma^{\alpha\beta\mu}(p,k) - 2g^{\alpha\rho}q^{\mu} + g^{\mu\rho}q^{\alpha} + g^{\mu\alpha}q^{\rho}, \tag{5.15}$$

$$\Delta_{\rho\tau}(k') = \Delta_{\rho\tau}(k)\left(1 - \frac{2k\cdot q + q^2}{(k+q)^2 - M^2}\right) + \frac{q_\rho k_\tau + q_\tau k_\rho + q_\rho q_\tau}{M^2\left((k+q)^2 - M^2\right)}. \tag{5.16}$$

Using these relations, it is possible to rewrite the 3-point vertex function WTI as the difference of the self energies Σ (eq. (5.9)) at different momenta, plus a remainder:

$$q\cdot\Lambda = \Sigma^{\alpha\beta}(p') - \Sigma^{\alpha\beta}(p) + R^{\alpha\beta}(q)\,, \tag{5.17}$$

where

$$\begin{aligned} R^{\alpha\beta}(q) := &\int \mathrm{d}\tilde{k}\ \Gamma^{\mu,\alpha\beta}(p',k')\,\Delta_{\rho\tau}(k')\left(-2g^{\tau\rho}q_\mu + \delta^\rho_\mu q^\tau + \delta^\tau_\mu q^\rho\right)\frac{1}{(p'-k')^2} \\ &+ \int \mathrm{d}\tilde{k}\ \left(-2g^{\alpha\rho}q^\mu + g^{\mu\rho}q^\alpha + g^{\mu\alpha}q^\rho\right)\Delta_{\rho\tau}(k)\,\Gamma^{\tau\beta}_{\mu}(k,p)\,\frac{1}{(p-k)^2}. \end{aligned}$$

The sum of left- and right-hand WTI yield exactly $-R^{\alpha\beta}$, cancelling the remaining tensor in eq. (5.17). The complete vertex correction for massive vector bosons is the sum of vertex function and left- and right-hand seagull correction. Owing to that, the complete off-shell vertex correction WTI reduces to

$$q\cdot\Lambda = \Sigma(p') - \Sigma(p)\,. \tag{5.18}$$

The on-shell self energy is independent of the momentum, as can be quickly shown, so the difference of both self energy contributions vanish on-shell:

$$\Sigma^{\alpha\beta}(p') - \Sigma^{\alpha\beta}(p) = 0. \tag{5.19}$$

Following this, the remainder $R^{\alpha\beta}$ itself has to vanish on-shell. Under contraction with the polarization vectors, this is indeed the case:

$$W^*_\alpha R^{\alpha\beta} W_\beta = 0. \tag{5.20}$$

Hence, we have confirmed that we have derived all diagrams correctly. We will now continue by testing the sum rules at loop-level. Additionally, the ambiguity between effective Lagrangian and YM theory is investigated.

5.3 Vertex Correction

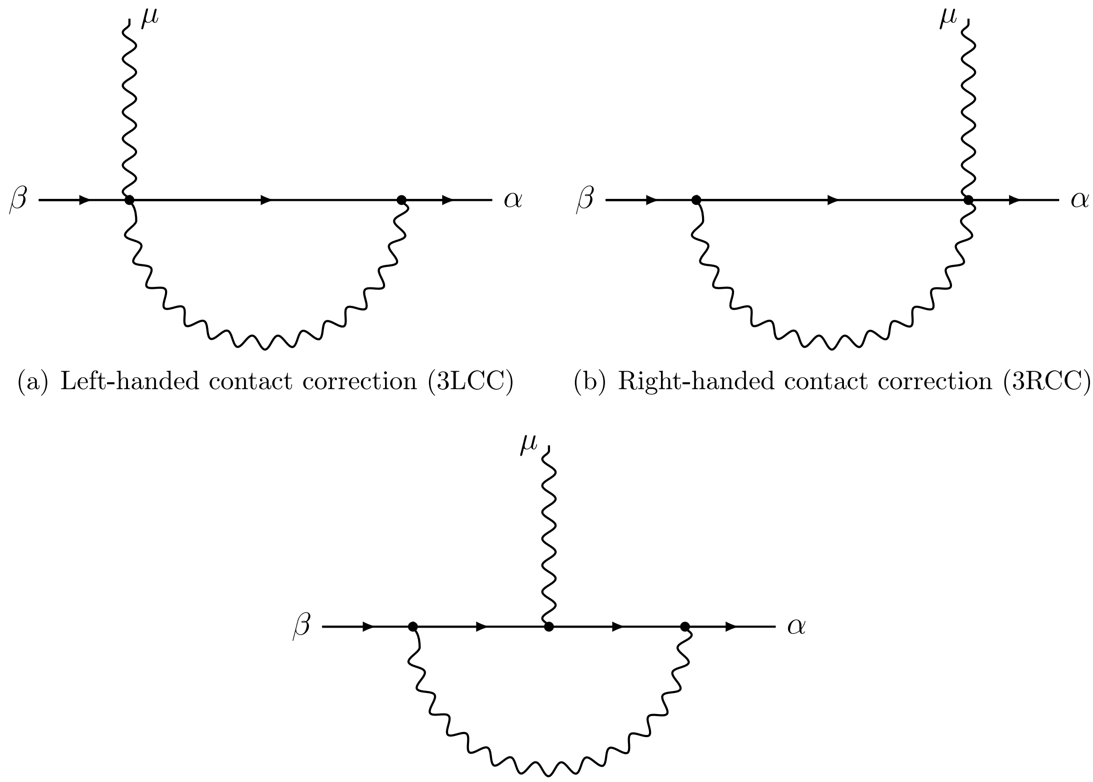
In order to calculate the full vector boson Compton scattering amplitude perturbatively, higher order (*loop*) contributions have to be taken into account. These corrections determine the anomalous electromagnetic moment. At one-loop, the 3- and 4-point interaction vertices $\Gamma^{\alpha\beta\mu}$ and $\Gamma^{\alpha\beta,\mu\nu}$ receive vertex corrections as depicted in fig. 5.3. From these corrections, the anomalous magnetic moment can be determined. The corrections are calculated in the forward limit, i.e. $t = 0$. The left- and right-hand 3-point vertex corrections (3LCC, 3RCC, see fig. 5.3) are easy to calculate and can be done by hand. However, we have also done the calculations in FORM [Ver00], together with the third diagram, the 3-point vertex term (3VC) which is familiar from QED. The latter could be calculated by hand as well. However this would be quite tedious due to a large number of terms in the numerator.

The 3-point vertex term is given by

$$T_{3VC} = W^*_\alpha(p')\Gamma^{\alpha\rho\zeta}(p', l')\Delta_{\rho\tau}(l')\Gamma^{\tau\eta\mu}(l', l)\Delta_{\eta\sigma}(l)\Gamma^{\sigma\beta\xi}(l, p)\varepsilon_\mu D_{\xi\zeta}(p - l)W_\beta(p). \tag{5.21}$$

We combine the left- and right-handed corrections to the contact correction (CC):

$$\begin{aligned} T_{CC} &= W^*_\alpha(p')\Gamma^{\alpha\rho\zeta}(p', k')\Delta_{\rho\tau}(k')\Gamma^{\tau\beta\mu\xi}(k', p)\varepsilon_\mu D_{\xi\zeta}(p' - k)W_\beta(p) \\ &\quad + W^*_\alpha(p')\Gamma^{\alpha\rho\mu\zeta}(p', k)\Delta_{\rho\tau}(k)\Gamma^{\tau\beta\xi}(k, p)\varepsilon_\mu D_{\xi\zeta}(p - k)W_\beta(p). \end{aligned} \tag{5.22}$$



(a) Left-handed contact correction (3LCC)

(b) Right-handed contact correction (3RCC)

(c) 3-point vertex term (3VC)

Figure 5.3: Vertex corrections to vector boson-photon interactions

The calculation is performed in the limit of natural values, i.e. $\ell_1 = 1$ and $\ell_2 = 0$. All three diagrams contain one loop-momenta which have to be integrated over. These diagrams are UV- and IR-divergent. To handle the UV divergence, we will use the method of dimensional regularization. For an overview of the concept refer to App. B.1. First, however, we have to concern ourselves with the denominators of the integrals. We use the parametrization method of FEYNMAN and SCHWINGER to bring all denominators into this form (c.f. App. B.2). This is then followed by shifting the loop momenta such that all denominators become of the required form

$$\ell^2 - \mathcal{M}^2, \tag{5.23}$$

where ℓ is the loop momentum (not to be confused with the electromagnetic moment constants ℓ_i !) and $\mathcal{M}$ is the *shifted mass*. Here, we have

$$\mathcal{M} = xM. \tag{5.24}$$

The corresponding momentum shifts are given in table C.2 in App. C.2. After the integrals are handled, we simplify the terms using the transversality conditions

$$W \cdot p = 0, \tag{5.25a}$$
$$p' \cdot W^* = 0, \tag{5.25b}$$
$$q \cdot \epsilon = 0, \tag{5.25c}$$

and the resulting additional identities

$$\epsilon \cdot p' = \epsilon \cdot p, \tag{5.26a}$$
$$W \cdot p' = W \cdot q, \tag{5.26b}$$
$$W^* \cdot p = -W^* \cdot q. \tag{5.26c}$$

Since we are considering only real scattering, initial and final state particle satisfy the on-shell conditions, i.e.

$$\begin{aligned} &q^2 = 0, && p \cdot q = 0 = p' \cdot q, \\ &p^2 = M^2 = p'^2, && p' \cdot p = M^2. \end{aligned} \tag{5.27}$$

Once this has been done, using L as defined in eq. (B.14) we obtain the following expressions:

$$T_{3VC} = e^3 \left(\frac{1085}{72} - \frac{41}{12} L + 8 \ln \omega \right) G_{\mathrm{E0}} + e^3 \left(\frac{229}{48} + \frac{3}{8} L \right) T_2 + e^3 \frac{4}{9} T_3, \tag{5.28}$$

$$T_{CC} = e^3 \left(-\frac{1085}{72} + \frac{89}{12} L \right) G_{\mathrm{E0}} + e^3 \left(-\frac{159}{16} + \frac{33}{8} L \right) T_2, \tag{5.29}$$

where we have used the definitions

$$G_{\mathrm{E0}} = T_1 - T_2, \tag{5.30a}$$

$$T_1 = W^* \cdot W \varepsilon \cdot p, \tag{5.30b}$$

$$T_2 = W^* \cdot \varepsilon W \cdot q - W^* \cdot q W \cdot \varepsilon, \tag{5.30c}$$

$$T_3 = \frac{1}{M^2} W^* \cdot q W \cdot q \varepsilon \cdot p. \tag{5.30d}$$

G_{E0}, T_2, and T_3 represent the charge, magnetic moment, and quadrupole moment terms, respectively. The vertex correction then is the sum of the two contributions above:

$$T = e^3 \left(\left[4L + 8 \ln \omega \right] G_{E0} + \left[-\frac{31}{6} + \frac{9}{2} L \right] T_2 + \frac{4}{9} T_3 \right). \tag{5.31}$$

As can be seen, the charge has UV-divergent (L) and IR-divergent ($\ln \omega$) contributions. They are expected to be cancelled by the charge renormalization and soft bremsstrahlung contributions, respectively. There is also an infinity L contributing to the anomalous magnetic moment. This is an unfortunate result, since we have no counterterm to renormalize this infinity away. In a sensible theory, the anomalous magnetic moment contribution should be finite, as indicated by the δGDH sum rule result (c.f. 5.1).

To recapitulate, our first theory—the truncated YM—while reproducing the GDH sum rule at tree-level, fails to so at NLO. Since we confirmed that there is no anomaly by checking the WTIs for all terms in detail (see sect. 5.2), all there is left to do about this remarkable finding is examining the full YM case.

5.4 Yang-Mills Tadpole Contribution

So far we used the truncated Yang-Mills theory of the effective Lagrangian by virtue of the natural limit. However, the complete YM contains an additional self-interaction vertex which contributes at NLO through the so-called tadpole graph (see fig. 5.4) which is effectively a boson loop (i.e. a particle/antiparticle intermediate state). We use

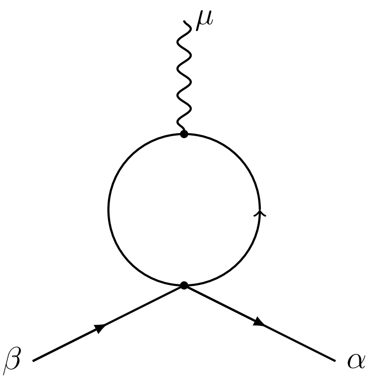


Figure 5.4: YM tadpole loop graph from $WWWW$ self-interaction vertex as resulting from complete Yang-Mills theory of γW interaction

the Feynman rules as derived in sect. 4.1.4 to calculate the tadpole. The corresponding matrix element reads

$$T_{YM} = W_\alpha^*(p')\Gamma_{YM}^{\alpha\beta\tau\rho} D_{\tau\zeta}(h')\Delta_{\xi\rho}(h)\Gamma^{\zeta\xi\mu}(h',h)\varepsilon_\mu W_\beta. \tag{5.32}$$

Before we do the actual calculations, we routinely check the gauge invariance of the tadpole. Of course, since the Yang-Mills theory is invariant, so is this graph and it fulfills a WTI of its own:

$$q \cdot \Lambda_{Tadpole} \stackrel{!}{=} 0 \tag{5.33}$$

$$\begin{aligned}
\Leftarrow q \cdot \Lambda_{Tadpole} &= \int \mathrm{d}^4\tilde{k}\Delta\left(k'\right) q \cdot \Gamma\left(k',k\right) \Delta\left(k\right) \\
&= \int \mathrm{d}^4\tilde{k}\Delta\left(k'\right) \left[\Delta^{-1}\left(k\right) - \Delta^{-1}\left(k\right)\right] \Delta\left(k\right) \\
&= \left[\int \mathrm{d}^4\tilde{k}\Delta\left(k'\right) - \int \mathrm{d}^4\tilde{k}\Delta\left(k\right)\right] \\
&= 0
\end{aligned} \tag{5.34}$$

where in the last line, we substituted $k \to k - q$ in the left-hand integral.

The tadpole graph calculation is performed analogous to sect. 5.3. However, the momentum shift in this case is

$$h \to \ell - xq \qquad\qquad h' = h + q \tag{5.35}$$

which leads to the shifted mass simply being the original mass, $\mathcal{M} = M$. After handling the loop integrals, the tadpole contributes

$$T_{YM} = T_2 e^3 \left(-\frac{9}{2} - \frac{9}{2} L \right). \tag{5.36}$$

Adding the tadpole value to the prior result, we obtain

$$T = e^3 \left(4L + 8 \ln(\omega) \right) G_{E0} - e^3 \frac{20}{3} T_2 + e^3 \frac{4}{9} T_3. \tag{5.37}$$

As we can see, the divergence of the magnetic moment has vanished. This is a very intriguing result. The ambiguity which appears at tree-level is resolved at one-loop: only if the full SU(2) symmetry is considered and consequently the bosonic self-interaction term is included does the theory obey the GDH sum rule at one-loop level..

This implies an intricate relation between the gauge symmetry and fundamental principles of space-time continuum, such as causality and unitarity, which go into the derivation of sum rules.

5.5 Discussion

To validate the GDH at one-loop level, we have to make sure that both sides of the derivative GDH (eq. (5.6)) are identical. In sect. 5.1, we have seen that a sum rule linear in κ can be derived by introducing a classical anomalous magnetic moment κ_0,

$$-\frac{2\pi\alpha}{M^2} \kappa = \frac{1}{\pi} \int \frac{d\omega}{\omega} \Delta\sigma_1 (\omega) \,, \tag{5.38}$$

where (c.f. sect. 5.1)

$$\Delta\sigma_1(\omega) = \frac{\partial}{\partial \kappa_0} \Delta\sigma(\omega) \bigg|_{\kappa_0 = 0} . \tag{5.39}$$

The vertex correction calculation yields the contribution

$$\delta\kappa^{(loop)} = -\frac{e^2}{(4\pi)^2} \left(-\frac{20}{3} \right) = \frac{5\alpha}{3\pi} \approx 0.004, \tag{5.40}$$

which is more than three times larger in magnitude than the value obtained by SCHWINGER for the spin-$1/2$ case, $\kappa = \alpha/2\pi$. For the deuteron which is practically the only spin-1 system with well-measured anomalous magnetic moment, the value is [A+08a, MT00]

$$\kappa_d = -0.143. \tag{5.41}$$

However, since we obtain for κ the value mentioned above, our one-loop correction evaluates to the value in eq. (5.40) which is significantly smaller than the measured quantity.

This is not surprising as most of the anomalous magnetic moment contribution will be due to the internal structure of the deuteron. Pion production and other contributions are not even considered here. Our result should be seen as a theoretic expectation value for a pure electromagnetic theory of massive vector bosons.

The theory could also be used to give an approximation for a charged spin-1 particle without internal structure, exemplified by W gauge bosons. In the standard electroweak theory it is required that at tree level $G_M(0) = 2$ and $G_Q(0) = -1$. For elementary particles, any deviations from these values would indicate new, beyond standard model, physics, and will show up in the presence of anomalous $WW\gamma$ couplings, usually parameterized in terms of two new couplings κ_γ and λ_γ, appearing in an effective Lagrangian. In terms of those parameters, the W magnetic dipole moment and quadrupole moment take on the values [HPZH87]:

$$\mu_W = \frac{e}{2M_W}\left\{2 + (\kappa_\gamma - 1) + \lambda_\gamma\right\}, \tag{5.42}$$

$$Q_W = -\frac{e}{M_W 2}\left\{1 + (\kappa_\gamma - 1) - \lambda_\gamma\right\}, \tag{5.43}$$

with M_W the W-boson mass. In the Standard Model, $G_M(0) = 2$ and $G_Q(0) = -1$ equivalently correspond with $\kappa_\gamma = 1$, $\lambda_\gamma = 0$ at tree level. The measurement of the gauge boson couplings and the search for possible anomalous contributions due to the effects of new physics beyond the Standard Model have been among the principal physics aims at LEP-II. The anomalous $WW\gamma$ couplings have been prominently studied at LEP-II in the $e^+e^- \to W^+W^-$ process through an s-channel virtual photon exchange mechanism. The most recent PDG fit for the anomalous $WW\gamma$ couplings based on an analysis of all LEP data is given by [A+08b] :

$$\kappa_\gamma = 0.973^{+0.044}_{-0.045} \qquad \lambda_\gamma = -0.028^{+0.020}_{-0.021}. \tag{5.44}$$

The result for κ_γ is well in conformance with our tree-level result. At one-loop, our estimate of the deviation $\Delta\kappa_\gamma = \delta\kappa$ from the tree-level value is below one percent,

$\kappa_\gamma + \delta\kappa \approx 1.004$. In fig. 5.5 which was taken from [A+06], we have marked our result for κ_γ which is well within the prediction region. However, the value of κ_γ is non-trivially related to the value for the electric quadrupole, λ_γ. So, in order to make a coherent prediction for the anomalous magnetic moment, a theory consistent up to NLO with regard to the QSR has to be found first.

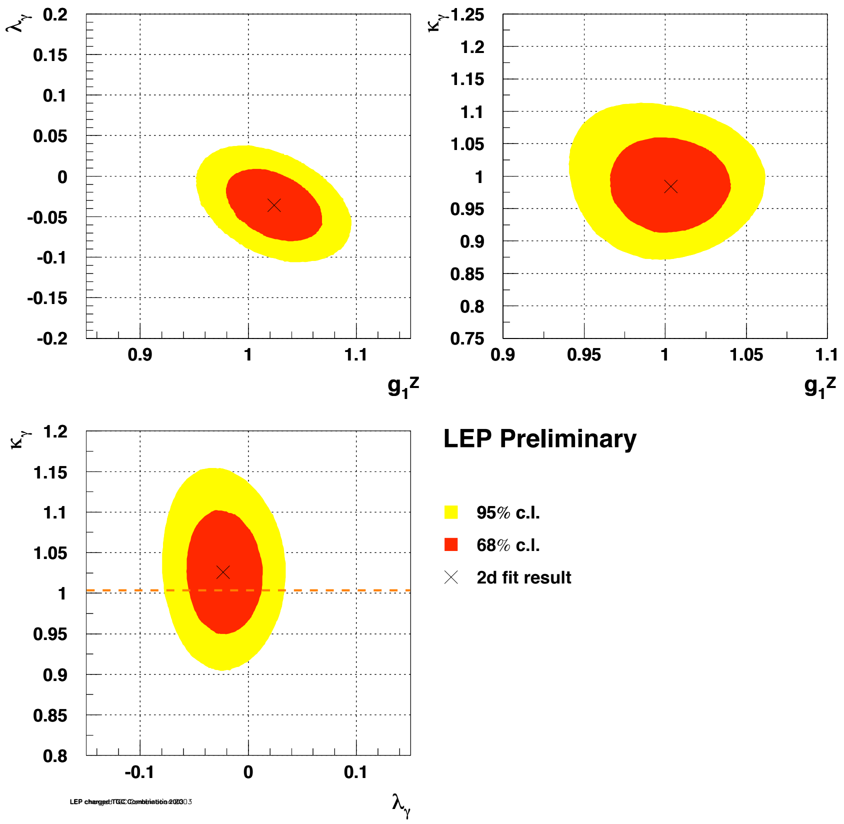


Figure 5.5: The ellipse is the 95% C.L. (confidence level) limit contour in $\kappa_\gamma - \lambda_\gamma$ space. The red dashed line marks our determined value in terms of κ_γ of $\kappa_\gamma \approx 1.004$. Plot taken from Review of Particle Physics *[A+06].*

Chapter 6

Conclusion and Outlook

We have constructed an effective Lagrangian for massive vector bosons from which we derived the well-established GDH sum rule. Additionally, we derived a novel quadrupole sum rule (QSR). We then have tested the sum rules in two different quantum field theories. One is obtained by minimal coupling of the electromagnetic field to a massive spin-1 field (Proca field), and then setting the magnetic moment of that field to the *natural value*, $\mu = {}^{e}/_{M}$. The latter step ensures the absence of the anomalous magnetic moment at tree-level, which is crucial for testing the sum rule. The second theory is obtained by considering the gauge boson sector of the electroweak theory with the Weinberg angle taken at 90°, such that Z^0 decouples. The resulting theory describes massive $W^{\pm}$ bosons interacting with the electromagnetic field, and includes self-interaction.

The tree-level Compton scattering for the two theories is equivalent, and therefore yields identical results for the sum rules test. Namely, the left-hand side is trivially zero, while the right-hand side has been computed with the usual techniques of calculating the cross sections. The GDH could be confirmed at tree-level. The QSR, on the other hand, yields a non-zero result due to non-trivial polarizability contributions. As we can see, even at tree-level polarizabilities can contribute due to the order of the QSR integrand being quadratic in energy.

At one-loop level, the two theories yield dramatically different results. The derivation of the theory used for the derivation of the sum rules was constructed based on fundamental principles like causality, crossing symmetry and unitarity. Nevertheless, this has proven to be unsufficient to describe the GDH coherently at NLO. The test was performed in QFT in the natural limit. QFTs, by definition, satisfy the aforementioned principles. The higher symmetry of the YM theory, motivated from electroweak theory, is a plausible formulation for the W bosons which avoids the UV divergence in the anomalous magnetic moment. The breakdown if the SU(2) symmetry is not considered might imply that one of the fundamental principles is violated, for example microcausality. The case might be different for other massive bosons, e.g. if we wanted to apply the sum rules to the deuteron. The QFT without boson self-coupling seems to have worse UV behaviour. A possible explanation is that the no-subtraction hypothesis is violated. A method recently developed to handle this kind of UV divergences is *UV completion*. It might be possible to derive a theory based on this which does not need additional symmetries. This is a subject which could be investigated.

In a follow-up study, the polarizability contribution to the quadrupole sum rule at tree-level needs to be examined further. It is our conjecture that the left-hand side of the QSR is incomplete as further polarizability terms need to be considered for the Lagrangian. These would make the Lagrangian complete with respect to the QSR and would lead to a modification of the left-hand side of the QSR. We presume that the additional contributions are the origin of the non-zero tree-level result.

Another interesting point of study would be the derivation of higher-order sum rules for higher-spin particles. The next plausible ansatz would be the $S = 3/2$, or Rarita-Schwinger particles. A first goal would be to recover the GDH; then, quadrupole and octupole sum rules could be derived. However, the situation for spin $3/2$ is even more complicated. Not only do we expect to have polarizabilities contribute to the tree-level value of quadrupole and octupole, but the additional constraints posed on the fields due to the superfluous *d.o.f.*s from the compositeness (Rarita-Schwinger particle wave functions are composed from spin-1 polarization vectors and spin-$1/2$ spinors) make for a non-trivial description of the interactions (see e.g. [Lor09]).

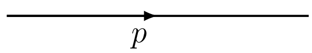
Appendix A

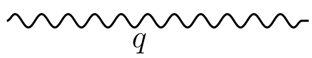
Feynman Rules

A.1 Feynman Graphs

In this appendix, the Lagrangian of the two QFTs and the corresponding Feynman rules and diagrams for the massive vector bosons are presented for convenient reference.

Propagators

Boson β  α $\qquad \Delta_{\mathrm{P}}^{\alpha\beta}(p) = -i\frac{\left(g_{\alpha\beta} - p_\alpha p_\beta / M^2\right)}{p^2 - M^2 + \mathrm{i}0^+}$.

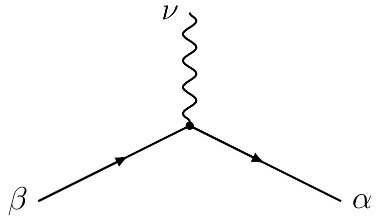
Photon μ  ν $\qquad D_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2 + \mathrm{i}0^+}$.

Yang-Mills Theory

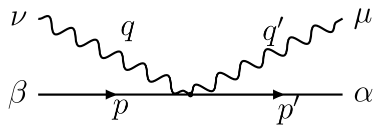
The Lagrangian for the full Yang-Mills theory reads:

$$\begin{aligned}\mathcal{L}_{YM} = &-\frac{1}{2}W^*_{\mu\nu}W^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{1}{2}ieW^*_\mu W_\nu F^{\mu\nu} + \frac{1}{2}e^2\big(\,|W \cdot W|^2 - |W|^4\,\big)\end{aligned} \tag{A.1}$$

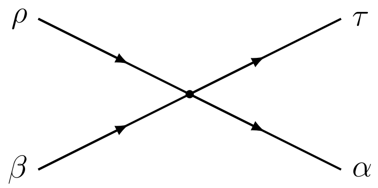
where $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$ is the covariant field tensor. The corresponding Feynman rules are given in Table A.1.



$$\Gamma^{\alpha\beta\mu}(p',p) = -e^2\left(g^{\alpha\beta}P^\mu - p'^\alpha g^{\mu\beta} - p^\beta g^{\mu\alpha}\right) -2e^2\left(q^\alpha g^{\mu\beta} - q^\beta g^{\mu\alpha}\right)$$



$$\Gamma^{\alpha\beta\mu\nu} = -e^2\left(2g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right)$$



$$\Gamma^{\alpha\beta\mu\nu}_{YM} = e^2\left(2g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right)$$

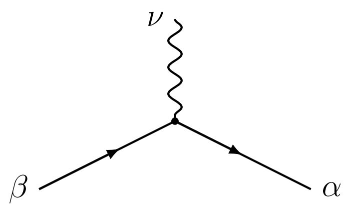
Table A.1: Feynman graphs and corresponding rules for the Yang-Mills theory

Effective Lagrangian

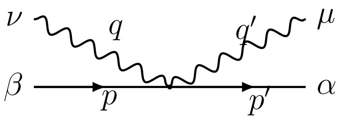
The effective Lagrangian for massive vector bosons reads:

$$\begin{aligned}\mathcal{L}_{\mathrm{Eff}} = &-\frac{1}{2}W^*_{\mu\nu}W^{\mu\nu} + M^2W^*_\mu W^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+\frac{1}{2}ie\ell_1 W^*_\mu W_\nu F^{\mu\nu} \\ &+ie\ell_2 W^*_{\mu\nu}W^\alpha\partial_\alpha F^{\mu\nu} - ie\ell_2 W^*_\alpha W_{\mu\nu}\partial^\alpha F^{\mu\nu}.\end{aligned} \tag{A.2}$$

The corresponding Feynman rules are given in Table A.2



$$\begin{aligned}\Gamma^{\alpha\beta\mu}(p,p') = &-e\left(g^{\alpha\beta}P^\mu - p'^\beta g^{\mu\alpha} - p^\alpha g^{\mu\beta}\right) \\ &+\left(q^\beta g^{\mu\alpha} - q^\alpha g^{\mu\beta}\right)\ell_1 \\ &-2\left(q^\alpha q^\beta P^\mu - p\cdot q q^\alpha g^{\mu\beta} - p'\cdot q q^\beta g^{\mu\alpha}\right)\ell_2\end{aligned}$$



$$\begin{aligned}\Gamma^{\alpha\beta\mu\nu}(q) = &-\left(2g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right)e^2 \\ &-\left(4q^\alpha q^\beta g^{\mu\nu} - q^\alpha q^\mu g^{\beta\nu} - q^\alpha q^\nu g^{\beta\mu}\right. \\ &\left.-q^\beta q^\mu g^{\alpha\nu} - q^\beta q^\nu g^{\alpha\mu}\right)e^2\ell_2\end{aligned}$$

Table A.2: Feynman graphs and corresponding rules for the effective Lagrangian

A.2 Plane Wave Expansions

In the following we will define the plane wave expansions of the particle fields, along with useful relations the expansions fulfill.

A.2.1 Massive Spin-1 Particles

First, we define the plane wave expansion of the massive vector boson fields and its derivative. For this we introduce the momentum-space creation operators $a(p,r)$ and the polarization vectors $\zeta(p,r)$, where p is the momentum and r is the polarization.

$$W^{\mu}(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{\sqrt{2p^0}}\sum_{r=0}^{3}\left(a(p,r)\zeta^{\mu}(p,r)e^{-ip\cdot x}\right) \tag{A.3a}$$

$$W^{\dagger\mu(x)} = \frac{1}{(2\pi)^3}\int \frac{d^3p}{\sqrt{2p^0}}\sum_{r=0}^{3}\left(a^{\dagger}(p,r)\zeta^{*\mu}(p,r)e^{ip\cdot x}\right) \tag{A.3b}$$

$$\partial^{\nu}W^{\mu}(x) = \frac{-ip^{\nu}}{(2\pi)^3}\int \frac{d^3p}{\sqrt{2p^0}}\sum_{r=0}^{3}\left(a(p,r)\zeta^{\mu}(p,r)e^{-ip\cdot x}\right) \tag{A.3c}$$

$$\partial^{\nu}W^{\dagger\mu(x)} = \frac{ip^{\nu}}{(2\pi)^3}\int \frac{d^3p}{\sqrt{2p^0}}\sum_{r=0}^{3}\left(a^{\dagger}(p,r)\zeta^{*\mu}(p,r)e^{ip\cdot x}\right) \tag{A.3d}$$

Commutation and Completeness Relations The momentum operators of the fields fulfill the following commutation relations:

$$[a(\boldsymbol{p},r),a^{\dagger}(\boldsymbol{p'},s)] = (2\pi)^3\delta_{rs}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p'}), \tag{A.4a}$$

$$[a,a] = [a^{\dagger},a^{\dagger}] = 0 \tag{A.4b}$$

while the polarization vectors fulfill the completeness relation

$$\sum_{r}\zeta_{\mu}(\boldsymbol{p},r)\zeta_{\nu}(\boldsymbol{p},r) = -g_{\mu\nu} + \frac{p_{\mu}p_{\nu}}{M^2} \tag{A.5}$$

Contractions with States In order to derive the Feynman rules, we contract the plane-wave expansions with the states.

$$\overline{W_{\mu}(x)|W}(p)\rangle = \zeta_{\mu}(p,r)e^{-ip\cdot x}|0\rangle \tag{A.6a}$$

$$\partial_{\nu}\overline{W_{\mu}(x)|W}(p)\rangle = -ip_{\nu}\zeta_{\mu}(p,r)e^{-ip\cdot x}|0\rangle \tag{A.6b}$$

$$\langle W(p')|W^\dagger(x) = \zeta^*_\mu(p,r)e^{ip\cdot x}\langle 0| \tag{A.6c}$$

$$\langle W(p')|\partial_\nu W^\dagger(x) = ip_\nu \zeta^*_\mu(p,r)e^{ip\cdot x}\langle 0| \tag{A.6d}$$

We introduce the following tensors for the field strength and the photon-boson coupling:

$$\begin{aligned} &\text{The covariant field strength tensor reads} \qquad && W_{\mu\nu} := D_\mu W_\nu - D_\nu W_\mu \\ &\text{and the coupling tensor is given by} && T_{\mu\nu} := W_\mu A_\nu - A_\mu W_\nu. \end{aligned} \tag{A.7}$$

For convenience, the contraction of these tensors with the fields is given in the following:

$$W_{\mu\nu}(x)|Wp\rangle = -i\left(p_\mu\zeta_\nu(p,r)e^{-ip\cdot x} - p_\nu\zeta_\mu(p,r)e^{-ip\cdot x}\right)|0\rangle, \tag{A.8a}$$

$$\langle W(p')|W^\dagger_{\mu\nu}(x) = i\left(p_\mu\zeta^*_\nu(p,r)e^{ip\cdot x} - p_\mu\langle 0|\zeta^*_\nu(p,r)e^{ip\cdot x}\right)\langle 0|, \tag{A.8b}$$

$$T_{\mu\nu}(x)|Wp\gamma(k,\sigma)\rangle = \left(\zeta^\beta(p,r)\varepsilon^\rho(k,\sigma)e^{-ip\cdot x} - \zeta^\rho(p,r)\varepsilon^\beta(k,\sigma)e^{-ip\cdot x}\right)|0\rangle. \tag{A.8c}$$

A.2.2 Photon Fields

The photon field can be written as a plane wave as well. Note that due to the photon mass being zero, the photon is its own antiparticle. The plane-wave expansion is given by

$$A^\mu(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2p^0}}\sum_{\sigma=0}^{3}\left(b(p,\sigma)\varepsilon^\mu(p,\sigma)e^{-ik\cdot x} + b^\dagger(p,\sigma)\varepsilon^{*\mu}(p,\sigma)e^{ik\cdot x}\right). \tag{A.9}$$

Commutation and Completeness Relations The photon fields also fulfill commutation and completeness relations as given by

$$[b(\boldsymbol{k},r), b^\dagger(\boldsymbol{k'},s)] = \delta_{rs}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'}), \tag{A.10a}$$

$$[b,b] = [b^\dagger,b^\dagger] = 0, \tag{A.10b}$$

and

$$\sum_\sigma \varepsilon_\mu(\boldsymbol{k},r)\varepsilon^*_\nu(\boldsymbol{k},r) = -g_{\mu\nu}. \tag{A.11}$$

Contractions with States The contractions of the photon fields with the states are given by

$$\overbracket{A_\mu(x)|\gamma(k}, \sigma)\rangle = \varepsilon_\mu(p,\sigma) e^{-ik\cdot x}|0\rangle, \tag{A.12a}$$

$$\partial_\nu \overbracket{A_\mu(x)|\gamma(k}, \sigma)\rangle = -ik_\nu \varepsilon_\mu(p,r) e^{-ik\cdot x}|0\rangle, \tag{A.12b}$$

$$\langle\gamma(\overbracket{k,\sigma)|A}_\mu(x) = \varepsilon_\mu(p,\sigma) e^{ik\cdot x}|0\rangle, \tag{A.12c}$$

and
$$\langle\gamma(k,\overbracket{\sigma)|\partial_\nu A_\mu(x}) = -ik_\nu \varepsilon_\mu(p,r) e^{ik\cdot x}|0\rangle. \tag{A.12d}$$

Appendix B

Loop Integration

B.1 Tensor Loop Integrals

Integrals of the type

$$\mathcal{J}_n\left(\mathcal{M}^2\right) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{1}{\left(\ell^2 - \mathcal{M}^2 + \mathrm{i}\epsilon\right)^n} \tag{B.1}$$

are divergent for $n \leq 2$ in the UV regime due to the upper limit. In order to evaluate these integrals a method of regularization has to be used. Here, we resort to dimensional regularization: The integral is evaluated in a fractional dimension $D = 4 - 2\epsilon$ in which it is still convergent. Afterwards, a Taylor expansion around $\epsilon = 0$ yields the solution of the integral.

In euclidian space, integrals of the type in eq. (B.1) can be reduced to the Euler Gamma function. However, the denominator is defined with respect to the Minkowski metric, i.e.

$$\ell^2 \equiv \ell_0^2 - \boldsymbol{\ell}^2. \tag{B.2}$$

The denominator has poles at

$$\ell^0 = \pm\sqrt{\boldsymbol{\ell}^2 + \mathcal{M}^2 - \mathrm{i}\varepsilon}. \tag{B.3}$$

Hence, the solution to the scalar integral can be derived using the equivalence of Minkowski and Euclidian metric if we allow ℓ^0 to take complex values (analytic continuation). The transition $\ell_0 \to \mathrm{i}\ell_0$ is called *Wick rotation*, see fig. B.1.

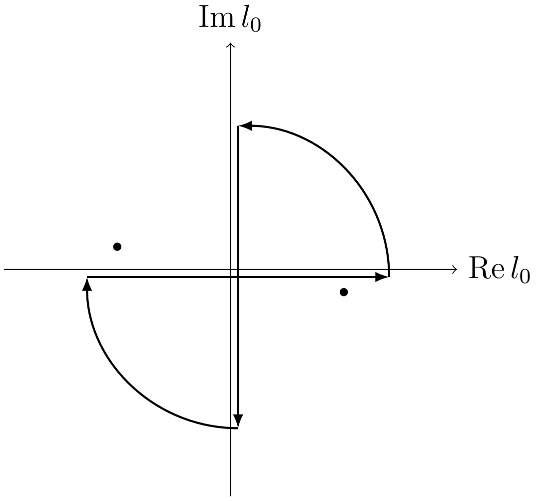


Figure B.1: Integration path of the Wick rotation.

In D-dimensional euclidic space we can rewrite the differential $\mathrm{d}^D\ell$ in polar coordinates,

$$\int_{-\infty}^{\infty} \mathrm{d}^D\ell = \int_{-\infty}^{\infty} \mathrm{d}l\, l^{D-1} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\vartheta_1 \sin\vartheta_1 \cdots \mathrm{d}\vartheta_{D-2} \sin\vartheta_{D-2} \tag{B.4}$$

with $l = |\ell|$, so that we obtain

$$\mathcal{J}_n(\mathcal{M}^2) = 2\pi^{\frac{D}{2}} \mathrm{i} \frac{(-1)^n}{\Gamma(\frac{D}{2})} \int_0^{\infty} \frac{\mathrm{d}l}{(2\pi)^D} \frac{l^{D-1}}{(l^2 + \mathcal{M}^2 - \mathrm{i}\varepsilon)}. \tag{B.5}$$

There are no angular dependencies so that a direct evaluation is possible. Using the relations to the Euler Gamma function,

$$\int_0^{\pi} \mathrm{d}\vartheta \sin^k\vartheta = \sqrt{\pi} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(1+\frac{k}{2})} \quad \text{and} \quad \int_0^{\infty} \mathrm{d}t^{2x-1} \frac{1}{(1+t^2)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}, \tag{B.6}$$

we find in the limit $D \to 4$

$$\mathcal{J}_n\left(\mathcal{M}^2\right) = \mathrm{i} \frac{(-1)^n}{(4\pi)^2} \frac{\Gamma(n-2)}{\Gamma(n)} \mathcal{M}^{-2(n-2)} \tag{B.7}$$

From the scalar integral, we can iteratively derive the formulas for higher order tensor integrals, that is for integrals with loop momenta *ell* in the numerator. In this work, integrals up to order $\mathcal{O}(\ell^4)$ appear. Odd integrals vanish due to the symmetry under

$\ell \to -\ell$. The scalar and 2– and 4–tensor loop integrals are given by

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\left(\ell^2 - \mathcal{M}^2\right)^n} = \mathcal{J}_n\left(\mathcal{M}^2\right), \tag{B.8a}$$

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{\left(\ell^2 - \mathcal{M}^2\right)^n} = \frac{1}{2(n-1)} g^{\mu\nu} \mathcal{J}_{n-1}\left(\mathcal{M}^2\right), \tag{B.8b}$$

$$\begin{aligned} &\int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu \ell^\rho \ell^\tau}{\left(\ell^2 - \mathcal{M}^2\right)^n} \\ &\qquad = \frac{1}{4(n-1)(n-2)} \mathcal{J}_{n-1}\left(\mathcal{M}^2\right) \left(g^{\mu\nu} g^{\rho\tau} + g^{\mu\rho} g^{\nu\tau} + g^{\mu\tau} g^{\rho\nu}\right). \end{aligned} \tag{B.8c}$$

The derivation is straightforward: The 2–tensor integral is of the form

$$\int \mathrm{d}\tilde{\ell} \frac{\ell^\alpha \ell^\beta}{(\ell^2 - \mathcal{M}^2)^n} = g^{\alpha\beta} A, \tag{B.9}$$

where A is a constant. To determine A, we contract both sides with $g^{\alpha\beta}$, using $g^2 = D$,

$$\begin{aligned} DA &= \int \mathrm{d}\tilde{\ell} \frac{\ell^2}{(\ell^2 - \mathcal{M}^2)^n} \\ &= \int \mathrm{d}\tilde{\ell} \left(\frac{1}{(\ell^2 - \mathcal{M}^2)^{n-1}} + \frac{\mathcal{M}^2}{(\ell^2 - \mathcal{M}^2)^n} \right) \\ &= \mathcal{J}_{n-1}(\mathcal{M}^2) + \mathcal{M}^2 \mathcal{J}_n(\mathcal{M}^2) = \frac{D}{2(n-1)} \mathcal{J}_{n-1}(\mathcal{M}^2), \end{aligned} \tag{B.10}$$

where in the last line, we have used

$$\mathcal{J}_n(\mathcal{M}^2) = \mathcal{J}_{n-1}(\mathcal{M}^2)(-1) \frac{n - 1 - \frac{D}{2}}{n-1} \mathcal{M}^{-1}. \tag{B.11}$$

Thus, we obtain eq. (B.8b). The 4–tensor integral is obtained in a similar manner. Here, we have to consider that the tensor is completely symmetric,

$$\int \mathrm{d}\tilde{\ell} \frac{\ell^\mu \ell^\nu \ell^\rho \ell^\tau}{\left(\ell^2 - \mathcal{M}^2\right)^n} = B\left(g^{\mu\nu} g^{\rho\tau} + g^{\mu\rho} g^{\nu\tau} + g^{\mu\tau} g^{\rho\nu}\right). \tag{B.12}$$

$$\begin{aligned} \overset{\text{contraction}}{\Rightarrow} \quad B(D^2 + 2D) &= \int \mathrm{d}\tilde{\ell} \frac{\ell^2 \ell^2}{(\ell^2 - \mathcal{M}^2)^n} \\ &= \frac{1}{4(n-1)(n-2)} \mathcal{J}_{n-2}(\mathcal{M}^2)\left(D^2 + 2D\right), \end{aligned} \tag{B.13}$$

so that we finally obtain eq. (B.8c).

It is useful to rewrite $\mathcal{J}_n$ for $n = 1, 2$ evaluating the Euler-Gammas such that

$$\mathcal{J}_1\left(\mathcal{M}^2\right) = \frac{-\mathrm{i}\mathcal{M}^2}{(4\pi)^2}\left(\frac{2}{D-4} + \gamma_E - 1 + \ln\frac{\mathcal{M}^2}{4\pi}\right) \tag{B.14a}$$

$$\mathcal{J}_2\left(\mathcal{M}^2\right) = \frac{-\mathrm{i}}{(4\pi)^2}\underbrace{\left(\frac{2}{D-4} + \gamma_E + \ln\frac{\mathcal{M}^2}{4\pi}\right)}_{=:L}. \tag{B.14b}$$

where $D = 4 \to D = 4^- = 4 - 2\epsilon$. Here, L is a symbolic notation for renormalizable divergences.

B.2 Feynman-Schwinger Parametrization

Often, the denominator of an interaction integral is inconvenient to integrate. However, Feynman invented a method to rewrite the denominator such that the calculation is greatly simplified, based on a parametrization by Schwinger. In principle, this corresponds to a substitution applying the parametric differentiation rule.
For a discussion of this concept, refer to [PS95], p.189 ff.

The general Feynman parametrization for n denominators is given by

$$\frac{1}{A_1 A_2 \dots A_n} = \int\limits_0^1 dx_1 dx_2 \dots dx_n \delta\left(\sum x_i - 1\right)\frac{(n-1)!}{\left[x_1 A_1 + x_2 A_2 + \dots + x_n A_n\right]^n}. \tag{B.15}$$

The formulas we need in our work are:

- For two propagation terms

$$\frac{1}{AB} = \int\limits_0^1 dx \frac{1}{\left[Ax + B(1-x)\right]^2} \tag{B.16}$$

- For three propagators

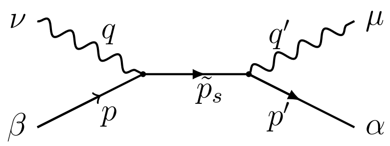
$$\frac{1}{ABC} = \int\limits_0^1 dx \int\limits_0^x dy \frac{2}{\left[Ay + B(x-y) + C(1-x)\right]^3} \tag{B.17}$$

Appendix C

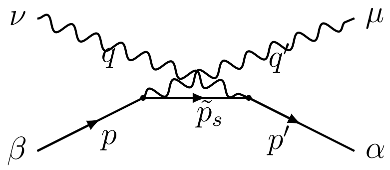
Diagrams

C.1 Tree-level Diagrams

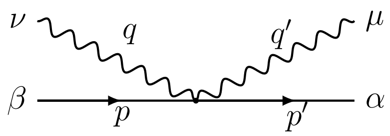
The tree level diagrams are given by the following diagrams:



$$\equiv W_\alpha^*(p')\varepsilon^*(q')\Gamma^{\alpha\rho\mu}(p',\tilde{p}_s)\Delta_{\rho\tau}(\tilde{p}_s)\Gamma^{\alpha\tau\nu}(\tilde{p}_s,p)\varepsilon_\nu(q)W_\beta(p)$$

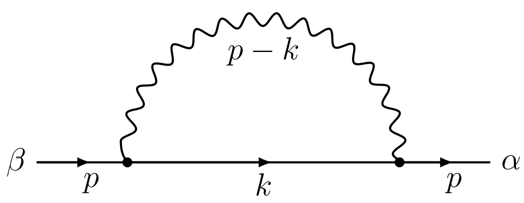


$$\equiv W_\alpha^*(p')\varepsilon^*(q)\Gamma^{\alpha\rho\nu}(p',\tilde{p}_u)\Delta_{\rho\tau}(\tilde{p}_u)\Gamma^{\alpha\tau\mu}(\tilde{p}_u,p)\varepsilon_\nu(q')W_\beta(p)$$

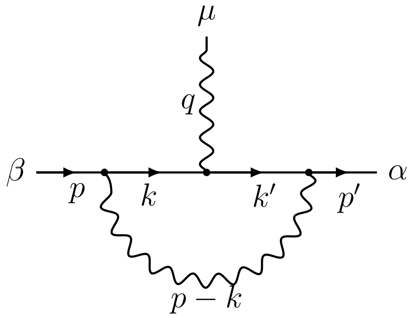


$$\equiv W_\alpha^*(p')\varepsilon^*(q')\Gamma^{\alpha\beta\mu\nu}(p',p)\varepsilon_\nu(q)W_\beta(p)$$

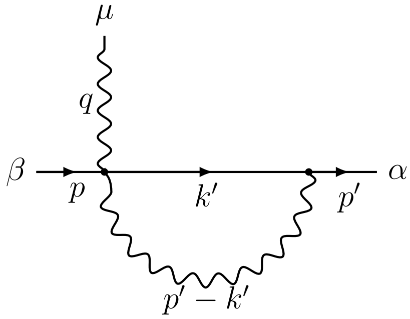
C.2 Vertex Correction Diagrams



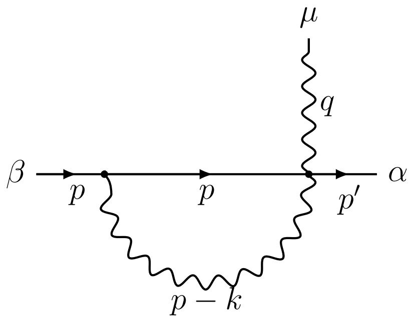
$$W_\alpha^*(p')\varepsilon^*(q')\Gamma^{\alpha\rho\mu}(p',\tilde{p}_s)\Delta_{\rho\tau}(\tilde{p}_s)\Gamma^{\alpha\tau\nu}(\tilde{p}_s,p)\varepsilon_\nu(q)W_\beta(p)$$



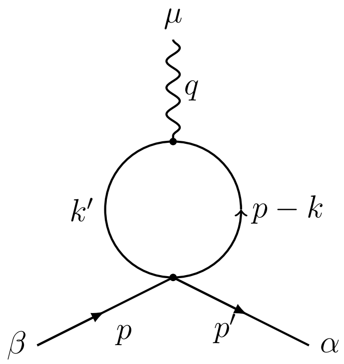
$$W^*_\alpha(p')\varepsilon^*_\mu(q)\Gamma^{\alpha\rho\xi}(p',p-k)\Delta_{\rho\tau}(k')\Gamma^{\tau\eta\mu}(k',k)\Delta_{\eta\sigma}(k)D_{\sigma\beta}(p-k)\Gamma^{\tau\eta\mu}(k,p)W_\beta(p)$$



$$W^*_\alpha(p')\Gamma^{\alpha\rho\xi}(p',p'-k')\Delta_{\rho\tau}(p'-k')D_{\xi\zeta}(k')\Gamma^{\beta\tau\zeta\mu}(p'-k',p)\varepsilon_\mu(q)W_\beta(p)$$



$$W^*_\alpha(p')\varepsilon_\mu(q)\Gamma^{\alpha\rho\xi}(p',p-k)\Delta_{\rho\tau}(p-k)D_{\xi\zeta}(k)\Gamma^{\beta\tau\zeta\mu}(p-k,p)W_\beta(p)$$



$$W^*_\alpha(p')\Gamma^{\alpha\beta\sigma\eta}_{\mathrm{YM}}(p',p)\Delta_{\sigma\tau}(k')\Gamma^{\tau\eta\mu}(k',k)\Delta_{\eta\tau}(p-k)\varepsilon^*_\mu(q)W_\beta(p)$$

C.3 Momentum Shifts

The denominators are all reduced to the same type, $\frac{1}{(\ell^2-\mathcal{M}^2)^n}$, with $\mathcal{M}^2 = x^2M^2$, by using the Feynman trick and appropriate shifts of the loop momenta.

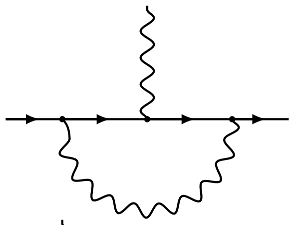
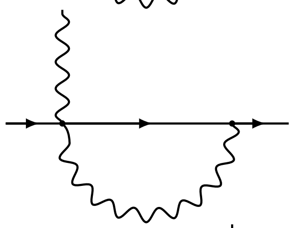
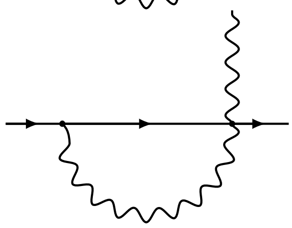
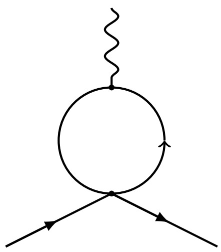
Diagram	Denominator	Shift
	$(k^2 - M^2)(k'^2 - M^2)(p-k)^2$	$k \to \ell + (1-x)p - yq$
	$(l'^2 - M^2)(p-l)^2$	$l \to \ell + (1-x)p'$
	$(l^2 - M^2)(p-l)^2$	$l \to \ell + (1-x)p$
	$(h'^2 - M^2)(p-h)^2$	$h' \to \ell + (1-x)q$

Table C.2: Vertex correction contributions and corresponding momentum shifts

Bibliography

[A$^+$06] J. Alcaraz et al. A Combination of preliminary electroweak measurements and constraints on the standard model. 2006.

[A$^+$08a] M. W. Ahmed et al. Near-threshold deuteron photodisintegration: An indirect determination of the Gerasimov-Drell-Hearn sum rule and forward spin polarizability (gamma0) for the deuteron at low energies. *Phys. Rev.*, C77:044005, 2008.

[A$^+$08b] C. Amsler et al. Review of particle physics. *Phys. Lett.*, B667:1, 2008.

[AAA$^+$08] V. M. Abazov, et al. First Study of the Radiation-Amplitude Zero in W gamma Production and Limits on Anomalous WW gamma Couplings at sqrt(s) = 1.96 TeV. *Physical Review Letters*, 100(24):241805, 2008.

[ACM72] G. Altarelli, N. Cabibbo, and L. Maiani. The Drell-Hearn sum rule and the lepton magnetic moment in the Weinberg model of weak and electromagnetic interactions. *Physics Letters B*, 40(3):415–419, 1972. ISSN 0370-2693.

[AFS04a] H. Arenhövel, A. Fix, and M. Schwamb. Spin asymmetry and GDH sum rule for real and virtual photons for the deuteron. In *Gerasimov-Drell-Hearn sum rule and its extensions*, pp. 294–303. Norfolk, 2004.

[AFS04b] H. Arenhövel, A. Fix, and M. Schwamb. Spin Asymmetry and Gerasimov-Drell-Hearn Sum Rule for the Deuteron. *Phys. Rev. Lett.*, 93:202301, 2004.

[AG68] H. D. I. Abarbanel and M. L. Goldberger. Low-Energy Theorems, Dispersion Relations, and Superconvergence Sum Rules for Compton Scattering. *Phys. Rev.*, 165(5):1594–1609, 1968.

[Ait80] I. J. R. Aitchison. *An Informal Introduction to Gauge Theories.* Oxford Publishing, 1980. OXFORD-TP-17-80.

[BCKT08] D. Binosi, et al. JaxoDraw: A graphical user interface for drawing Feynman diagrams. Version 2.0 release notes. 2008.

[BD65] J. D. Bjorken and S. D. Drell. *Relativistic Quantum Fields*. McGraw-Hill, 1965.

[Ber08] T. Beranek. *Strangeness Production and Exotic Baryons*. Master's thesis, Ruhr-Universität Bochum, 2008.

[BK73] E. Byckling and K. Kajantie. *Particle kinematics*. Wiley London, New York,, 1973. ISBN 0471128856.

[Bra05] A. Braghieri. Results from the GDH experiment at Mainz and Bonn. In *Tallahassee 2005, Physics of excited nucleons*, pp. 90–97. 2005. Prepared for International Workshop on the Physics of Excited Baryons (NSTAR 05), Tallahassee, Florida, 10-15 Oct 2005.

[BRS95] V. I. Borodulin, R. N. Rogalev, and S. R. Slabospitsky. CORE: COmpendium of RElations: Version 2.1, 1995.

[BT04] D. Binosi and L. Theussl. JaxoDraw: A graphical user interface for drawing Feynman diagrams. *Comput. Phys. Commun.*, 161:76–86, 2004.

[Che68] T. P. Cheng. Low-energy theorems for e-to-the-fourth Compton scattering amplitudes. *Phys. Rev.*, 176:1674–1679, 1968.

[Che69] T. P. Cheng. Low-energy theorems on radiative corrections. *Phys. Rev.*, 184:1805–1814, 1969.

[CJL04] J.-W. Chen, X. Ji, and Y. Li. Drell-Hearn-Gerasimov sum-rule for the deuteron in nuclear effective field theory. *Physics Letters B*, 603(1-2):6–12, 2004. ISSN 0370-2693.

[CN84] M. Chaichian and N. F. Nelipa. *Introduction to Gauge Field Theories*. Texts and Monographs In Physics. Springer Verlag, 1984.

[CPT92] A. P. Contogouris, S. Papadopoulos, and F. V. Tkachov. One-loop corrections for polarized deep-inelastic Compton scattering. *Phys. Rev. D*, 46(7):2846–2853, 1992.

[DH66] S. D. Drell and A. C. Hearn. Exact Sum Rule for Nucleon Magnetic Moments. *Phys. Rev. Lett.*, 16:908–911, 1966.

[DHK+04] H. Dutz, et al. Experimental Check of the Gerasimov-Drell-Hearn Sum Rule for $H1$. *Phys. Rev. Lett.*, 93(3):032003, 2004.

[DKT01] D. Drechsel, S. S. Kamalov, and L. Tiator. The GDH sum rule and related integrals. *Phys. Rev.*, D63:114010, 2001.

[DPV02] D. Drechsel, B. Pasquini, and M. Vanderhaeghen. Dispersion Relations in Real and Virtual Compton Scattering. *Physics Reports*, 2002.

[Dre00] D. Drechsel. The Gerasimov-Drell-Hearn sum rule. 2000. Prepared for 3rd Workshop on Chiral Dynamics - Chiral Dynamics 2000: Theory and Experiment, Newport News, Virginia, 17-22 Jul 2000.

[DT04] D. Drechsel and L. Tiator. The Gerasimov-Drell-Hearn sum rule and the spin structure of the nucleon. *Ann. Rev. Nucl. Part. Sci.*, 54:69–114, 2004.

[DV01] D. A. Dicus and R. Vega. The Drell-Hearn sum rule at order alpha**3. *Phys. Lett.*, B501:44–47, 2001.

[ELOP66] R. Eden, et al. *The Analytic S-Matrix.* Cambridge University Press, 1966.

[Ger66] S. B. Gerasimov. A Sum rule for magnetic moments and the damping of the nucleon magnetic moment in nuclei. *Sov. J. Nucl. Phys.*, 2:430–433, 1966.

[Ger06] S. B. Gerasimov. Dispersion total photoproduction sum rules for nucleons and few-body nuclei revisited. *Czech. J. Phys.*, 56:F195–F201, 2006.

[GMd99] M. Gomes, L. C. Malacarne, and A. J. da Silva. Spin-1 massive particles coupled to a Chern-Simons field. *Phys. Rev. D*, 60(12):125016, 1999.

[GMG54] M. Gell-Mann and M. L. Goldberger. Scattering of low-energy photons by particles of spin 1/2. *Phys. Rev.*, 96:1433–1438, 1954.

[GMGT54] M. Gell-Mann, M. L. Goldberger, and W. E. Thirring. Use of causality conditions in quantum theory. *Phys. Rev.*, 95:1612–1627, 1954.

[Hel06] K. Helbing. The Gerasimov-Drell-Hearn sum rule. *Prog. Part. Nucl. Phys.*, 57:405–469, 2006.

[HPZH87] K. Hagiwara, et al. Probing the Weak Boson Sector in e+ e- → W+ W-. *Nucl. Phys.*, B282:253, 1987.

[HY66] M. Hosoda and K. Yamamoto. Sum Rule for the Magnetic Moment of the Dirac Particle. *Progr. Theor. Phys.*, 36(2), 1966.

[JL04] X. Ji and Y. Li. Sum rules and spin-dependent polarizabilities of the deuteron in effective field theory. *Physics Letters B*, 591(1-2):76–82, 2004. ISSN 0370-2693.

[Jun02] F. Jung. PhysTeX: TeX für Physiker, 2002.

[Kni96] B. A. Kniehl. Dispersion relations in loop calculations. *Acta Phys. Polon.*, B27:3631–3644, 1996.

[Kuz99] M. Kuze. Search for physics beyond standard model at HERA. 1999.

[Kuz01] M. Kuze. Searches for new physics at HERA. 2001.

[Kuz08] M. Kuze. Search for Physics beyond Standard Model at HERA. 2008.

[LC75] K. Y. Lin and J. C. Chen. Forward Dispersion Relations and Low-Energy Theorems for Compton Scattering on Spin 1 Targets. *J. Phys.*, G1(4):394, 1975.

[Leu94] H. Leutwyler. On the foundations of chiral perturbation theory. *Ann. Phys.*, 235:165–203, 1994.

[Lin71] K. Y. Lin. Low-energy theorems for compton scattering on targets of arbitrary spin. *Nuovo Cim.*, A2:695–706, 1971.

[Lor09] C. Lorce. Electromagnetic properties for arbitrary spin particles: Natural electromagnetic moments from light-cone arguments. *Physical Review D (Particles and Fields)*, 79(11):113011, 2009.

[Low54] F. Low. Low-Energy Theorem. *Phys. Rev.*, 96:1428, 1954.

[Man58] S. Mandelstam. Determination of the Pion-Nucleon Scattering Amplitude from Dispersion Relations and Unitarity. General Theory. *Phys. Rev.*, 112(4):1344–1360, 1958.

[MT00] P. J. Mohr and B. N. Taylor. CODATA recommended values of the fundamental physical constants: 1998. *Rev. Mod. Phys.*, 72:351–495, 2000.

[Nus72] H. Nussenzveig. *Causality and Dispersion Relations.* Mathematics in Science and Engineering. Acad. Press, 1972.

[Pai67] A. Pais. Low-Energy Theorems for Spin $S \geq 1$. *Phys. Rev. Letters*, 19(9):544 – 546, 1967.

[Pai68] A. Pais. Compton Scattering on Stable Targets of Arbitrary Spin. *Nuovo Cim.*, 53 A(2):433 – 454, 1968. Contains a derivation of the Multipole Expansion Terms.

[Pan98] R. Pantfoerder. *Investigations on the foundation and possible modifications of the GDH sum rule.* Ph.D. thesis, Bonn University, hep-ph/9805434, 1998.

[Pas05] V. Pascalutsa. GDH sum rule in QED and ChPT. *Nucl. Phys.*, A755:657–660, 2005.

[PHV04] V. Pascalutsa, B. R. Holstein, and M. Vanderhaeghen. A derivative of the Gerasimov-Drell-Hearn sum rule. *Phys. Lett.*, B600:239–247, 2004.

[Pic98] A. Pich. Effective field theory. 1998.

[Poe05] D. N. Poenaru. Alexandru Proca (1897–1955) — The Great Physicist. ArXiv, 2005.

[PS95] M. Peskin and D. Schroeder. *Introduction to Quantum Field Theory.* Westview, 1995.

[Qui83] C. Quigg. *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions.* The Benjamin/Cummings Publishing Company, Inc., 1983.

[QV74] N. Queen and G. Violini. *Dispersion theory in high-energy physics.* McMillan Press Ltd., 1974.

[Sch07] S. Scherer. Gruppentheorie in der Physik I + II. 2007.

[SR70] V. Singh and S. M. Roy. Unitarity upper bound on the absorptive parts of elastic- scattering amplitudes. *Phys. Rev. Lett.*, 24:28–33, 1970.

[SS05] S. Scherer and M. R. Schindler. A chiral perturbation theory primer. 2005.

[Ste71] H. Stern. Low-energy theorems for compton scattering on a target of arbitrary spin. *Nuovo Cim.*, A6:393–429, 1971.

['t 71] G. 'T HOOFT. Renormalization of Massless Yang-Mills Fields. *Nucl. Phys.*, B33:173–199, 1971.

[Tak57] Y. TAKAHASHI. On the generalized Ward identity. *Nuovo Cim.*, 6:371, 1957.

[Tak69] Y. TAKAHASHI. *An Introduction to Field Quantization.* Pergamon Press Oxford, New York, [1st ed.] edition, 1969. ISBN 0080128246.

[Tan06] T. TANTAU. *Graphics with PGF and TikZ*, 2006.

[Thi50] W. E. THIRRING. Radiative corrections in the non-relativistic limit. *Phil. Mag.*, 41:1193–1194, 1950.

[Tho06] A. THOMAS. The Gerasimov-Drell-Hearn sum rule at MAMI. *Eur. Phys. J.*, A28S1:161–171, 2006.

[Tia00] L. TIATOR. Helicity amplitudes and sum rules for real and virtual photons. *ArXiv.org/nucl-th/0012045*, 2000.

[Tia02] L. TIATOR. MAID and the GDH sum rule in the resonance region. 2002. Prepared for 2nd International Symposium on the Gerasimov- Drell-Hearn Sum Rule and the Spin Structure of the Nucleon (GDH 2002), Genoa, Italy, 3-6 Jul 2002.

[TK06] L. TIATOR and S. KAMALOV. MAID analysis techniques. *ArXiv.org/nucl-th/0603012*, 2006.

[Ver00] J. VERMASEREN. New Features of FORM. *ArXiv.org*, math-ph/0010025, 2000.

[War50] J. C. WARD. An Identity in Quantum Electrodynamics. *Phys. Rev.*, 78(2):182, 1950.

[Wei72] S. WEINBERG. Mixing angle in renormalizable theories of weak and electromagnetic interactions. *Phys. Rev.*, D5:1962–1967, 1972.

[Wei79] S. WEINBERG. Phenomenological Lagrangians. *Physica*, A96:327, 1979.

[Whe37] J. A. WHEELER. On the Mathematical Description of Light Nuclei by the Method of Resonating Group Structure. *Phys. Rev.*, 52(11):1107–1122, 1937.

[YM54] C. N. YANG and R. L. MILLS. Conservation of Isotopic Spin and Isotopic Gauge Invariance. *Phys. Rev.*, 96(1):191–195, 1954.

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