Mesonic Lagrangian and Anomalous Processes

Diploma thesis

of

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I dedicate this work to my cousin Nils.
His entire life is an inspiration to me.

Contents

Chapter 1

Introduction

As far as is known, all phenomena and processes in nature are governed by the four fundamental forces of nature – gravitation, electromagnetism, weak and strong interactions. For the latter three a description in terms of quantized gauge fields proves to be highly successful – culminating in the standard model of particle physics – while gravitation still cannot be properly understood in terms of a quantum field theory.

The spectrum of fundamental particles in the standard model consists of two different species – on the one hand fermionic matter fields with spin $\frac{1}{2}$ governed by the Dirac equation and on the other hand bosons, the description of which follows the Klein-Gordon equation. The bosons belong to two classes: one class consists of gauge bosons with helicity 1. The Higgs boson – a hermitian scalar field – is the only representative of the other class. The fermions appear in three families – identical copies of each other – except that their masses differ vastly. The origin and nature of the fermionic masses are still not understood and pose one of the most important problems in particle physics (hierarchy problem). In a gauge symmetry realized in the Wigner-Weyl mode gauge invariance demands gauge bosons to be massless. Since experiment proves the gauge bosons of the weak interactions to be massive, a spontaneous breakdown of the corresponding gauge symmetry could explain the masses. This can be accomplished in terms of the Higgs formalism, even though experimental evidence for the necessary scalar Higgs boson does not exist yet.

The symmetry group of the standard model is $SU(N_c)\times SU(2)_W\times U(1)_Y$, where c stands for colour, W stands for weak and Y for hypercharge. Experimental evidence indicates that the number of colours is $N_c = 3$. But from a theoretical point of view, it is useful to retain the number of colours as a parameter. The gauge bosons directly correspond to the generators of the group. The subgroup $SU(2)_W \times U(1)_Y$ is spontaneously broken to $U(1)_{\text{e.m.}}$ by a non-vanishing vacuum expectation value (VEV) of the complex $SU(2)_W$-doublet of the Higgs field. $U(1)_{\text{e.m.}}$ is the gauge group of the well-known electromagnetic interaction, the description of which is called quantum electrodynamics (QED). Three out of the four degrees of freedom of the Higgs doublet appear as massless Goldstone bosons, which are absorbed as longitudinal degrees of freedom of the now-massive gauge bosons ($W^{\pm}, Z^0$) of the $SU(2)_W$-gauge group, while the fourth gauge boson, the photon (γ) of $U(1)_{\text{e.m.}}$, remains massless. The last remaining scalar degree of freedom corresponds to the scalar Higgs boson.

The interaction between matter fields and gauge fields is induced by minimal coupling. The fermion spectrum is divided into quarks taking part in strong interactions and leptons, which do not interact strongly.

The gauge group of the strong interactions is $SU(N_c)$, whose gauge bosons are named gluons. Since this is a non-Abelian group, self-interactions of the gluon field arise, which are responsible for a vast amount of new effects[1]. The restriction of the standard model to $SU(N_c)$ is called *quantum chromodynamics* (QCD, from Greek $\chi\rho o\mu o\sigma$: colour), which is the topic of this work.

The usual perturbative approach for solving quantum field theories by expanding the generating functional in the coupling constant is only successful at high energies in the case of QCD (*asymptotic freedom*), whereas the approach does not converge at low energies. Even worse, the renormalized coupling constant at the scale of the nucleon mass is $\alpha_s \gtrsim 1$. This gives rise to an increasing amount of gluonic self-interactions and prevents that quarks are observable as free particles. They appear only in bound states (*infrared slavery*). These colour-neutral bound states are called *hadrons* and are the observable asymptotic states. This phenomenon is called *confinement* and its proof is one of the millenium problems.

Hence, in the low-energy regime a non-perturbative approach to QCD is one option (e.g., lattice QCD). The second option is the use of phenomenological models imitating the behaviour of QCD. A third option is a $\frac{1}{N_c}$ expansion, where $N_c \to \infty$, but $g^2 N_c$ is kept fixed.

A fourth option is an *effective field theory* (EFT). Here *chiral perturbation theory* (ChPT) is the natural choice. Due to a theorem of Weinberg [Wei 79], an EFT yields the same S-matrix as its fundamental theory in a regime, where it is valid.

However, all possible terms satisfying the original symmetry constraints must be included. This generally requires an infinite number of structures, each with an a priori independent *low-energy coupling constant* (LEC). Therefore such an EFT is not renormalizable in the usual sense. Each diagram is assigned a so-called *chiral order D* by rescaling of external momenta and quark masses. This power counting scheme which was invented

[1]Because the electromagnetic subgroup of the standard model is Abelian, no self-interactions of the photon field exist on tree level. It is the part of the standard model, which is known best. The $SU(2)_W$-gauge group does have self-interactions of the gauge fields, too.

by Weinberg establishes a connection between the loop expansion and the chiral expansion.

The infinities arising from loop diagrams can be absorbed by a renormalisation of the LECs order by order. Therefore, the number of LECs, which must be renormalized at an arbitrary chiral order D, is still finite.

ChPT for N_f flavours is based on the accidental global chiral symmetry $U(N_f)_L \times U(N_f)$ of the QCD Lagrangian density for N_f light flavours in the limit of vanishing masses of the light quarks, which is broken to $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$ by the anomaly of QCD. Chiral transformations are independent rotations of the left- and right-handed (light) quark fields in flavour space. Since the scalar quark condensate has a non-vanishing vacuum expectation value [CGL 01], chiral symmetry is subject to further dynamical[2] symmetry breakdown to $SU(N_f)_V \times U(1)_V$. The corresponding effective Lagrangian density is written in terms of the hadronic degrees of freedom. The pseudoscalar Goldstone bosons of the dynamically broken symmetry are of utmost importance here. Since the quark masses of the real world are non-zero but small, chiral symmetry is only almost realized and therefore the Goldstone bosons are only almost massless.

The hadron spectrum shows that the pseudoscalar nonet ($M_\pi \approx 140\,\mathrm{MeV}$, $M_K \approx 500\,\mathrm{MeV}$, $M_\eta \approx 550\,\mathrm{MeV}$, $M_{\eta'} \approx 960, \mathrm{MeV}$), with the notable exception[3] of the η', is much lighter than the next lightest nonet (vector mesons, $M_\rho \approx 770\,\mathrm{MeV}$, $M_K \approx 900\,\mathrm{MeV}$, $M_\omega \approx 780\,\mathrm{MeV}$, $M_\phi \approx 1020\,\mathrm{MeV}$). This indicates that the pseudoscalar octet consists of the near-massless Goldstone bosons.

The roots of ChPT originate in current algebra in the 1960s [AD 68]. The leading order Lagrangian has a very tight connection to current algebra. It was extended to incorporate the effects of next-to-leading order corrections in an expansion in $\frac{1}{N_c}$ due to the θ term in 1980 [VV 80, Wit 80].

A systematic next-to-leading-order expansion was performed in 1984 by Gasser and Leutwyler [GL 85]. They introduced the method of using *symmetry relations* to achieve a reduction of the number of independent monomials. The effects of the anomaly were included in the Wess-Zumino-Witten action [WZ 71, Wit 83], which contributes at next-to-leading order in the chiral expansion.

It took another ten years, until a systematic extension to the next-to-next-to-leading order was performed by Fearing and Scherer in [FS 96]. Due to the complexity of the vector space of monomials at chiral order six, some symmetry relations between the monomials had been missed at that time. Therefore, the next-to-next-to-leading order Lagrangian was revisited by several authors in recent years [BCE 00, BGT 02, EFS 02].

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[2]Dynamical symmetry breaking is different from spontaneous symmetry breaking in the electroweak sector, due to the fact that there is no need for the introduction of additional fields.

[3]This exception is due to the anomaly of QCD.

When [BGT 02, EFS 02] were both published nearly at the same time, the resulting Lagrangians appeared to be very similar, but because different bases were used and different monomials were kept as the independent ones, it was unclear for a long time, if there were a contradiction or not. The first purpose of this thesis is a proof of their equivalence. Since the number of LECs in the *anomalous sector*, which is necessary for renormalization of loops from the anomalous WZW action, is fairly small, it seems reasonable to expect that in principle measurable processes could be found for a measurement of most of the corresponding LECs [Hac 08].

Any further simplification of the Lagrangian, which could be achieved by application of previously unknown symmetry relations reduces the number of monomials, which represent operators for measurements of the LECs. If fewer LECs are measurable in theory, the demand for different experiments, which could determine these parameters, is lowered, too. It is demonstrated that the Lagrangian in [BCE 00] can be simplified in the two-flavour case.

The following procedure is applied order by order:

1. Construct all a priori independent chirally invariant monomials satisfying the constraining symmetries up to a given order in power counting,

2. determine all symmetry relations among these monomials, due to the properties of the monomials,

3. and use these relations to reduce the number of independent operators to as few monomials as possible.

The second part of this thesis is concerned with the anomaly, too, and with the θ term of QCD. In its simplest form, the anomaly is a quantum effect due to a fermionic triangle loop, which is coupled to one axial-vector and two vector currents. It was investigated for the first time in [Adl 69]. The outcome is that the Noether current corresponding to axial-transformations of the fermionic fields is not conserved on the quantum level. Measurable effects due to the anomaly require the existence of local axial-vector currents. Such axial-vector current operators are proportional to the field operators of the pseudoscalar mesons. Therefore, QCD exhibits anomalous effects.

The inclusion of the pseudoscalar flavour singlet necessitates that the anomalous contribution to the mass of the singlet is taken into account. Further, a discussion of the mixing of the octet η and the singlet η is performed. Due to the properties of the singlet, an excursion into large-N_c-counting cannot be avoided.

Parity is conserved in QCD if and only if the sum of the vacuum angle θ, the strength of the θ term, and the phase of the quark mass matrix, which originates in the elctroweak sector, is constrained to certain values. There exists a class of parity violating mesonic processes due to a new interaction term, which is induced by deviations of this vacuum angle from 0 or π.

Two such processes, the decays $\eta \to 2\pi^0$ and $\eta \to 4\pi^0$, are investigated. The decay $\eta \to 4\pi^0$ has a very clear signature in a detector. If it could be expected from theory to occur at a reasonable rate for deviations of θ from 0, it would be an efficient tool for setting further constraints on the vaccum angle.

Chapter 2

Foundations of QCD and Chiral Perturbation Theory

This introduction into QCD and ChPT is inspired by [EFS 02, Sch 03].

2.1 The QCD Lagrangian and its symmetries

In this chapter properties of QCD as the foundation of mesonic ChPT are discussed. Mesonic ChPT is normally understood as the theory of the pseudoscalar octet $(\pi^0, \pi^+, \pi^-, K^+, K^-, K^0, \overline{K^0}, \eta_8)$.

The gauge group of QCD is $SU(N_c)$. The hadron spectrum, the cross-section $e^+e^- \to f\overline{f}$ and the decay width of the Z^0 indicate that N_c equals three. The quark fields transform in the fundamental representation (N_c fields), while the massless gluon fields mediating the interactions between quarks, transform in the adjoint representation $\big((N_c^2-1)$ fields$\big)$. The three different quark fields are labeled by so-called colour *red, blue* and *green.* Besides their behaviour under $SU(N_c)$, they are identical copies of each other. A further property of the quark fields is their flavour. Six different flavours exist, which differ only in their masses and in their coupling to the electroweak sector. They are listed in table 2.1. QCD itself is flavour-blind, if masses are neglected.

Comparison with the scale of typical low-lying hadronic systems ($\Lambda \approx 1\,\mathrm{GeV}$) shows that the masses of the three *light* flavours are small[1], whereas the masses of the *heavy* flavours are large. Therefore, a suppression of the heavy degrees of freedom can be expected at the typical hadronic scale. Considering the light flavours only, after the heavy flavours are integrated out, seems to be reasonable. Furthermore, the relevant masses of the light quarks are small in comparison to this scale, thus a treatment of the masses

[1]Furthermore, the masses of up and down quarks are notably smaller than the mass of the strange quark.

Table 2.1: Quark flavours [PDG 2008]

flavour	mass	electromagnetic charge [$e > 0$]
u (up)	1.5 to 3.0 MeV	$+\frac{2}{3}$
d (down)	3 to 7 MeV	$-\frac{1}{3}$
s (strange)	95 ± 25 MeV	$-\frac{1}{3}$
c (charm)	1.25 ± 0.09 GeV	$+\frac{2}{3}$
b (bottom)	4.20 ± 0.07 GeV	$-\frac{1}{3}$
t (top)	174.2 ± 3.3 GeV	$+\frac{2}{3}$

as a perturbation to a massless theory is a reasonable approach. The limit of neglected light quark masses is called the *chiral limit*.

The QCD Lagrangian for N_f light flavours and N_c colours is given by the massless fermionic Dirac Lagrangian, where a minimal coupling to the gluon field ensures local gauge invariance under $SU(N_c)$. A kinetic term of the gluon field is added:

$$\mathcal{L}^0_{\text{QCD}} = \sum_{f=1}^{N_f} \sum_{c=1}^{N_c} \overline{q}_{f,c}\, i\not{D}\, q_{f,c} - \frac{1}{2} \langle G_{\alpha\beta} G^{\alpha\beta} \rangle_c . \tag{2.1}$$

The gluon field is parametrised in colour space by the Gell-Mann matrices divided by 2:

$$\sum_{a=1}^{N_c^2-1} G_{\alpha\beta} = G^a_{\alpha\beta} \tfrac{\lambda_a}{2} . \tag{2.2}$$

A trace is denoted by enclosing left- and right-angled brackets. In most cases the trace is taken in flavour space. Whenever the trace corresponds to another space, it is indicated explicitly[2].

The full QCD Lagrangian actually contains three additional terms:

1. the mass term for the light flavours,
2. the heavy flavours
3. and possibly the so-called θ term.

Among these, the mass term for the light flavours is included by the external field method. It explicitly selects certain directions in flavour space and

[2] c indicates colour space.

breaks chiral symmetry explicitly. The anomaly, which is responsible for the breakdown of the $U(1)_A$-symmetry, is treated in detail in chapter 4. The θ term, which is related to the anomaly, is also responsible for the possibility of parity violation in QCD.

The Lagrangian of Eq. (2.1) satisfies the usual symmetries of most field theories, namely Lorentz invariance, parity invariance, charge conjugation invariance and, of course, CPT symmetry. Using chiral projection operators

$$\begin{aligned} P_R = \tfrac{1}{2}(1+\gamma_5) \quad \text{and} \quad P_L = \tfrac{1}{2}(1-\gamma_5), \\ q_R = P_R q \quad \text{and} \quad q_L = P_L q, \end{aligned} \tag{2.3}$$

this Lagrangian splits into parts, where only either left- or right-handed quark fields appear. The result is invariant under global $U(N_f)$ chiral rotations of the left-and the right-handed quarks with independent but constant $U(N_f)$ matrices:

$$q_R \mapsto V_R q_R, \quad q_L \mapsto V_L q_L. \tag{2.4}$$

This invariance is called *chiral symmetry.* In addition to the already existent structure, colour-neutral external fields can be introduced. The total Lagrangian density remains invariant, if they transform in a certain way. Later, these are used to model other species of non-QCD interactions.

$$\begin{aligned} \mathcal{L}^{ext} = \; & \overline{q}_R(\not{v} + \not{a} + \tfrac{1}{N_c}(\not{v}^{(s)} + \not{a}^{(s)}))q_R + \overline{q}_L(\not{v} - \not{a} + \tfrac{1}{N_c}(\not{v}^{(s)} - \not{a}^{(s)}))q_L \\ & - \overline{q}_R(s+ip)q_L - \overline{q}_L(s-ip)q_R, \end{aligned} \tag{2.5}$$

where the external fields are parameterized by the generators[3] of $SU(N_f)$, $\lambda_a,\ a = 1, \ldots, (N_f^2 - 1)$, in addition to $\lambda_0 = \sqrt{\frac{2}{N_f}}\,1_{N_f \times N_f}$

$$\begin{aligned} v_\alpha = \sum_{a=1}^{N_f^2-1} \frac{\lambda^a}{2} v_\alpha^a, \quad a_\alpha = \sum_{a=1}^{N_f^2-1} \frac{\lambda^a}{2} a_\alpha^a, \\ s = \sum_{a=0}^{N_f^2-1} \lambda^a s^a, \quad p = \sum_{a=0}^{N_f^2-1} \lambda^a p^a. \end{aligned} \tag{2.6}$$

The external fields are applied as sources to the QCD action functional. Often the linear combinations of the isovector sources are denoted as

$$r_\alpha^a = (v+a)_\alpha^a, \quad l_\alpha^a = (v-a)_\alpha^a, \tag{2.7}$$

and the scalar and pseudoscalar sources are collected[4] in

$$\chi \equiv 2B_0(s+ip) \text{ and } \chi^\dagger \equiv 2B_0(s-ip). \tag{2.8}$$

[3] In $SU(2)$ these are the Pauli matrices, whereas in $SU(3)$ these are the Gell-Mann matrices.

[4] The nature of the constant B_0 is clarified in section 3.2.2.

The transformation rules of the external fields result from the invariance of the Lagrangian:

$$\begin{aligned} r_\alpha \mapsto V_R r_\alpha V_R^\dagger, \quad l_\alpha \mapsto V_L l_\alpha V_L^\dagger, \\ v_\alpha^{(s)} \mapsto v_\alpha^{(s)}, \quad a_\alpha^{(s)} \mapsto a_\alpha^{(s)}, \\ (s+ip) \mapsto V_R(s+ip)V_L^\dagger, \quad (s-ip) \mapsto V_L(s-ip)V_R^\dagger. \end{aligned} \tag{2.9}$$

Chiral symmetry is manifested in the chiral Ward identities satisfied by the Green's functions of QCD. In an effort to get all Green's functions at once, the generating functional of QCD is considered and chiral symmetry is gauged. Thus it is promoted from a global symmetry to a local one. Then the partial derivatives operate on the local transformation matrices, too. This must be compensated in the usual manner of a gauge theory by introducing covariant derivatives and a more complicated transformation prescription for the external sources. The way the external sources are implemented satisfies the criterion for minimal coupling. Their transformations compensate the contributions of the partial derivative operating on the local transformation matrix:

$$\begin{aligned} r_\alpha &\mapsto V_R r_\alpha V_R^\dagger + i V_R \partial_\alpha V_R^\dagger, \\ l_\alpha &\mapsto V_L l_\alpha V_L^\dagger + i V_L \partial_\alpha V_L^\dagger. \end{aligned} \tag{2.10}$$

Under local $U(1)_V$-transformations $q \mapsto \exp\left(i\frac{1}{N_c}(\theta_V + \theta_A\gamma_5)\right)q$, the flavour singlet sources transform as

$$\begin{aligned} v_\alpha^{(s)} &\mapsto v_\alpha^{(s)} + \partial_\alpha\theta_V, \\ a_\alpha^{(s)} &\mapsto a_\alpha^{(s)} + \partial_\alpha\theta_A. \end{aligned} \tag{2.11}$$

2.2 Chiral symmetry breaking

Beautiful as it is, chiral symmetry is not realized in nature. The axial anomaly appears, when the transition from classical to quantum theory is performed. A transformation with

$$\exp(i\frac{1}{N_c}\theta_A) := A = V_R = V_L^\dagger \tag{2.12}$$

does not leave the generating functional invariant. Therefore, the transition to a quantum field theory reduces the symmetry to $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$. The $U(1)_V$ part can be identified as the conservation of baryon number and always remains valid. The anomaly is discussed in more detail later on (chapter 4).

2.2.1 Explicit symmetry breaking

Independent of the anomaly, chiral symmetry is broken further. An explicit source of chiral symmetry breaking lies in the non-vanishing quark masses of the light quarks. A second consequence of the quark masses is a further symmetry breakdown from $SU(3)_L \times SU(3)_R$ to $SU(2)_L \times SU(2)_R$. This is due to a treatment of the up and down quarks as massless fermions on the one hand, but the strange quark as a massive degree of freedom on the other hand. Since the strange quark mass is much larger than the up and down quark masses ($m_s \approx 25\,\hat{m}$, where $\hat{m} \approx m_u \approx m_d$), chiral symmetry is much better realized, if only up and down quarks are treated as massless. Furthermore, at higher chiral orders the number of independent LECs is far smaller in the two-flavour case.

Matters turn out even more interesting, since QCD exhibits dynamical symmetry breaking $SU(N_f)_L \times SU(N_f)_R \mapsto SU(N_f)_V$ even in the chiral limit. This can be concluded from at least two different experimental facts.

2.2.2 Hadron spectrum

The first fact is the lack of parity doubling in the spectrum of QCD. If no dynamical symmetry breaking existed, two baryon octets with the same masses would have to exist, but with opposite behaviour under parity transformations. This has not been found in any experiment. Since the generators of axial transformations Q_A^a commute with the QCD Hamiltonian without θ term,

$$[H^0_{QCD}, Q_A^a] = 0, \tag{2.13}$$

for every energy eigenstate $|i,+\rangle$ with energy eigenvalue E_i and even parity, there exists another energy eigenstate $Q_A^a|i,+\rangle$ with equal energy, but opposite parity.

A system is constrained to baryon number one. The commutation relation of axial generators Q_A,

$$[Q_A^a, a_i^\dagger] = -t^a_{ij} b_j^\dagger, \tag{2.14}$$

where $a_i^\dagger$ creates quanta of positive parity and $b_j^\dagger$ of negative parity, connects baryonic states of opposite parity and equal energy (since the Hamiltonian commutes with Q_A^a, Eq. (2.13)):

$$Q_A^a|i,+\rangle = Q_A^a a_i^\dagger|0\rangle = [Q_A^a, a_i^\dagger]|0\rangle + a_i^\dagger \underbrace{Q_A^a|0\rangle}_{\overset{!}{=}0} = -t^a_{ij} b_j^\dagger|0\rangle = -t^a_{ij}|j,-\rangle. \tag{2.15}$$

Another multiplet of the same energy and of opposite parity must exist, but only under the assumption $Q_A^a|0\rangle = 0$. If the axial generators do not annihilate the vacuum, the reasoning is invalid.

Comparison with experimental results shows only one lowest lying baryonic octet, which has even parity. It can be concluded that $SU(3)_V$ instead of $SU(3)_L \times SU(3)_R$ is realized. Furthermore, the pseudoscalar meson octet is much lighter than the next octet of mesonic particles (1^- vector mesons). Their number as well as space-time attributes match the number and space-time attributes of the axial generators, which must not annihilate the vacuum. This is a hint of considering them as the Goldstone bosons of the dynamically broken symmetry. It has been proved that in the chiral limit the vacuum is necessarily invariant under $SU(3)_V \times U(1)_V$. Following Coleman's theorem [Col 66], the Hamiltonian and all other states must have at least the same symmetry. While the algebra of the vector generators closes, the algebra of axial generators does not:

$$\begin{aligned}
[Q_V^a, Q_V^b] &= i f_{abc} Q_V^c, \\
[Q_A^a, Q_A^b] &= i f_{abc} Q_V^c, \\
[Q_V^a, Q_A^b] &= i f_{abc} Q_A^c.
\end{aligned} \tag{2.16}$$

If the pseudoscalar mesons are the Goldstone bosons, they must transform like the axial generators of the broken symmetry. They must be a multiplet in the adjoint representation of $SU(N_f)_V$:

$$[Q_V^a, \phi^b(x)] = i f_{abc} \phi^c(x). \tag{2.17}$$

2.2.3 The scalar quark condensate

The existence of a non-vanishing scalar quark condensate (in the chiral limit) is a sufficient, but not necessary, condition for chiral symmetry breaking. Consider the scalar and pseudoscalar quark densities

$$S_a(x) = \overline{q}(x)\lambda_a q(x), \tag{2.18}$$

$$P_a(x) = \overline{q}(x)\gamma_5\lambda_a q(x), \tag{2.19}$$

where the range of a is given by $a = 0, 1, \ldots, (N_f^2 - 1)$. In addition, the explicit forms of the vector and axial-vector charges in terms of the quark currents

$$Q_V^a(t) = \int d^3x\, q^\dagger(\vec{x}, t)\frac{\lambda_a}{2}q(\vec{x}, t), \tag{2.20}$$

$$Q_A^a(t) = \int d^3x\, q^\dagger(\vec{x}, t)\gamma_5\frac{\lambda_a}{2}q(\vec{x}, t), \tag{2.21}$$

are used. The equal time commutation relation (summation over colour indices is implied)

$$[q^\dagger(\vec{x}, t)A_1 q(\vec{x}, t), q^\dagger(\vec{y}, t)A_2 q(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y})\, q^\dagger(\vec{x}, t)[A_1, A_2]q(\vec{x}, t) \tag{2.22}$$

is applied, where the A_i are arbitrary products of Dirac and flavour matrices. Evaluation of equal time commutators yields

$$[Q_V^a(t), S_b(x)] = i f_{abc} S_c(x). \tag{2.23}$$

The structure constant f_{abc} vanishes, if any of the three indices equals zero, $0 \in \{a, b, c\}$. Use of

$$\sum_{a,b=1}^{N_f^2-1} f_{abc} f_{abd} = N_f \delta_{cd} \tag{2.24}$$

leads from Eq. (2.23) to

$$S_a(x) = -\frac{i}{N_f} \sum_{b,c=1}^{N_f^2-1} f_{abc} [Q_V^b(t), S_c(x)]. \tag{2.25}$$

Since the generators Q_V^a annihilate the vacuum, taking the vacuum expectation value and using translational invariance provides:

$$\langle 0|S_a(x)|0\rangle = \langle 0|S_a(0)|0\rangle = \langle S_a\rangle = 0. \tag{2.26}$$

Specialization of Eq. (2.26) to $N_f = 3$ and the choices $a = 3$ and $a = 8$ results in

$$0 = \langle S_3\rangle = \langle \overline{u}u\rangle - \langle \overline{d}d\rangle, \tag{2.27}$$

$$0 = \langle S_8\rangle = \frac{1}{3}(\langle \overline{u}u\rangle + \langle \overline{d}d\rangle - 2\langle \overline{s}s\rangle). \tag{2.28}$$

Thus, in the chiral limit all flavours have the same scalar condensate:

$$\langle \overline{q}q\rangle = \langle \overline{u}u\rangle + \langle \overline{d}d\rangle + \langle \overline{s}s\rangle = 3\langle \overline{u}u\rangle. \tag{2.29}$$

On the other hand, the isoscalar ETCR $\big($Eq. (2.23) for $b = 0\big)$ does not provide any such condition. A non-vanishing quark condensate is a valid assumption:

$$\langle S_0\rangle = \langle \overline{q}q\rangle \neq 0. \tag{2.30}$$

When considering the ETCR of the axial-vector charges and the pseudoscalar quark densities,

$$i[Q_A^a(t), P_a(x)] = \begin{cases} \overline{u}u + \overline{d}d & a = 1,2,3 \\ \overline{u}u + \overline{s}s & a = 4,5 \\ \overline{d}d + \overline{s}s & a = 6,7 \\ \frac{1}{3}(\overline{u}u + \overline{d}d + 4\overline{s}s) & a = 8, \end{cases} \tag{2.31}$$

the general result is:

$$\langle 0|i[Q_A^a(t), P_a(x)]|0\rangle = \frac{2}{3}\langle \overline{q}q\rangle, \quad a = 1, \dots, 8. \tag{2.32}$$

When a complete set of states is inserted into the commutator, the vacuum drops out of the summation. Therefore, non-vanishing matrix elements between vacuum states and one-particle states must exist for the axial-vector charge and for the pseudoscalar quark density. The translational invariance of the right-hand side finally demands that these states must be massless[5].

2.3 Non-linear realization of chiral symmetry

A realization of the chiral group is required, which allows the description of the dynamically broken symmetry group[6] $G = SU(N_f)_L \times SU(N_f)_R$ as well as the leftover symmetry group $H = SU(N_f)_V$. The generators, which do not correspond to $SU(N_f)_V$, are associated with the Goldstone bosons. A non-linear realization is necessary in the case of QCD.

2.3.1 General problem

Every Goldstone boson corresponds to a hermitian field ϕ_i. These fields are collected in an n-component vector Φ. These vectors span a vector space M_1:

$$M_1 \equiv \{\Phi : M^4 \to \mathbb{R}^n | \phi_i : M^4 \to \mathbb{R},\ i = 1, \dots, n\}. \tag{2.33}$$

An action $\varphi(g, \Phi)$ of a symmetry group G on this vector space is defined by:

- the identity leaves any field configuration invariant

$$\varphi(e, \Phi) = \Phi, \tag{2.34}$$

- a homomorphism property is satisfied

$$\varphi(g_1, \varphi(g_2, \Phi)) = \varphi(g_1 g_2, \Phi), \tag{2.35}$$

where e is the neutral element of G, g_1, g_2 are arbitrary elements of G and Φ is an arbitrary field configuration of M_1. Since there is no demand for linearity

$$\varphi(g, \lambda\Phi) = \lambda\varphi(g, \Phi), \tag{2.36}$$

the realization is in general non-linear. Consequently, it cannot be called a representation.

[5]A cancellation of multiple states could be possible. Then there would be multiple possibly massive states.

[6]The subgroup $U(1)_V$ from the direct product is not treated here.

There must be a subgroup H of G, which maps this origin of the vector space ($\Phi = 0$) onto itself. Due to the first aforementioned property of G, the identity lies in that subgroup. For $h_1, h_2 \in H$, $0 = \varphi(h_1, \varphi(h_2, 0)) = \varphi(h_1 h_2, 0)$ is easily obtained from the homomorphism property. For $h_2 = h_1^{-1}$, the last necessary group axiom is verified[7].

Because H is a subgroup of G, G is split into (left) cosets gH of the subgroup H. The quotient G/H is the set of these cosets. In general such cosets are either disjoint or overlap completely and all elements of the same coset map the origin on the same field configuration. The proofs are short exercises:

$$\varphi(g, 0) = \varphi(g, \varphi(h, 0)) = \varphi(gh, 0), \ \forall g \in G, h \in H. \tag{2.37}$$

For $g' \notin gH$, but $\varphi(g, 0) = \varphi(g', 0)$, it follows immediately

$$\begin{aligned} 0 &= \varphi(e, 0) = \varphi(g^{-1}g, 0) = \varphi(g^{-1}, \varphi(g, 0)) \\ &= \varphi(g^{-1}, \varphi(g', 0)) = \varphi(g^{-1}g', 0); \end{aligned} \tag{2.38}$$

this requires $g^{-1}g' \in H$ or accordingly $g' \in gH$, in contradiction to the beforehand assumption.

That is why there exists an isomorphic mapping between field configurations and cosets. The transformation of a field configuration under the action of the symmetry group then follows directly. The field configuration $\Phi = \varphi(f, 0), \ f = gh \in gH$ is transformed to

$$\varphi(\widetilde{g}, \Phi) = \varphi(\widetilde{g}, \varphi(gh, 0)) = \varphi(\widetilde{g}gh, 0) = \varphi(\widetilde{f}, 0), \ \widetilde{f} \in \widetilde{g}(gH). \tag{2.39}$$

That said, it is obvious that the transformed field configuration is simply given by the coset transformed with the appropriate group element.

2.3.2 Specialization to QCD

In the case of QCD, the relevant groups are $G = SU(N_f)_L \times SU(N_f)_R$ and $H = SU(N_f)_V$. Therefore an arbitrary group element is characterized by (V_L, V_R). The coset structure of the chiral group is clearly demonstrated by $(V_L, V_R) = \underbrace{(1, V_R V_L^\dagger)}_{\equiv(1,U)} \underbrace{(V_L, V_L)}_{\in H}$. Subjecting this group element[8] $(1, U)$ to a chiral transformation yields

$$(V_L, V_R)(1, U) = (V_L, V_R U) = (1, V_R U V_L^\dagger) \underbrace{(V_L, V_L)}_{\in H}. \tag{2.40}$$

[7] Since G is associative, $H \subseteq G$ must be associative, too.

[8] Some authors use $(U, 1)$ instead, where $U = V_R^\dagger V_L$. An example is found in [Bij 91]. A different realization naturally must be equivalent, but specific transformation properties differ.

Thus, the coset transforms as $(1,U) \mapsto (1, V_R U V_L^\dagger)$. This must be treated as a local prescription, therefore U, V_R and V_L may depend on Minkowski space-time coordinates. From this coset, as was demonstrated in the previous section, an isomorphic mapping to the Goldstone boson fields exists.

In a manner similar to the preceding general considerations, the sets M_1 and M_2 are defined[9]

$$M_1 \equiv \{\Phi : M^4 \to \mathbb{R}^{N_f} | \phi_i : M^4 \to \mathbb{R} \text{ continuous}\}, \tag{2.41}$$

$$M_2 \equiv \{\phi : M^4 \to \widetilde{\mathcal{H}}_{N_f} | \phi = \sum_{f=1}^{N_f^2-1} \lambda_f \phi_f,\ \phi_f \text{ continuous } \forall f\}. \tag{2.42}$$

$\widetilde{\mathcal{H}}_{N_f}$ is the set of traceless hermitian $N_f \times N_f$ matrices[10]:

$$\widetilde{\mathcal{H}}_{N_f} \equiv \{A \in \mathrm{gl}(N_f, \mathbb{C}) | A = A^\dagger, \langle A \rangle = 0\}. \tag{2.43}$$

In general, the latter matrix representation M_2 in terms of a parametrisation by the generators of $SU(N_f)$ proves to be more useful. Finally, another set

$$M_3 \equiv \Big\{U : M^4 \mapsto SU(N_f) | U(x) = \exp{(i\frac{\phi(x)}{F_0})}, \phi(x) \in M_2\Big\} \tag{2.44}$$

is introduced[11].

This set M_3 is acted upon by the chiral group: $\varphi((V_L, V_R), U)$. The conditions for the action of a group can be verified quickly:

- Closure: Let be $U \in M_3$. Zhen

$$V_R U V_L^\dagger \in M_3, \tag{2.45}$$

- Neutral element:

$$\varphi((1,1),U) = 1U(x)1^\dagger = U, \tag{2.46}$$

- Homomorphism property:

$$\begin{aligned}
&\varphi\Big((V_L, V_R), \varphi\big((\widetilde{V_L}, \widetilde{V_R}), U\big)\Big) \\
&= \varphi\Big((V_L, V_R), \widetilde{V_R} U \widetilde{V_L}^\dagger\Big) = V_R \widetilde{V_R} U \widetilde{V_L}^\dagger V_L^\dagger \\
&= (V_R \widetilde{V_R}) U (V_L \widetilde{V_L})^\dagger = (V_R \widetilde{V_R}) U (V_L \widetilde{V_L})^\dagger \\
&= \varphi\Big(\big((V_L \widetilde{V_L}), (V_R \widetilde{V_R})\big), U\Big).
\end{aligned} \tag{2.47}$$

[9]Inclusion of the singlet is done by addition of $\frac{1}{N_c}\phi_0$: $\phi_{\text{nonet}} = \phi_{\text{octet}} + \frac{1}{N_c}\phi_0$. Then the matrix $U(\phi_{\text{nonet}}) \in U(N_f)$, but $U(\phi_{\text{nonet}}) \notin SU(N_f)$.

[10]These matrices consist of linear combinations of $SU(N_f)$-generator with hermitian coefficient functions.

[11]M_3 must be a non-linear realization, since $\det{(U_1 + U_2)} \neq 1$ for arbitrary $U_1, U_2 \in M_3$.

This definiton of U in terms of ϕ yields an origin

$$U_0 \equiv \exp{(i\frac{\phi}{F_0})}|_{\phi=0} = 1, \tag{2.48}$$

which is left invariant by the subgroup H,

$$U_0 \overset{(V,V)}{\mapsto} VU_0V^\dagger = U_0 \tag{2.49}$$

but transformed to another state by axial transformations ($A = V_R = V_L^\dagger$):

$$U_0 \overset{(A^\dagger,A)}{\mapsto} AU_0A = AU_0A \neq U_0. \tag{2.50}$$

When this transformation behaviour of $U \in M_3$ is projected to $\phi \in M_2$. It is discovered by inserting $1 = V^\dagger V$ between every pair of fields ϕ,

$$U = 1+i\frac{\phi}{F_0}-\frac{1}{2}\frac{\phi^2}{F_0^2}+\dots \overset{(V,V)}{\mapsto} U = 1+i\frac{V\phi V^\dagger}{F_0}-\frac{1}{2}\frac{V\phi V^\dagger V\phi V^\dagger}{F_0^2}+\dots. \tag{2.51}$$

Therefore, ϕ transforms as a linear representation under the subgroup H:

$$(V\phi V^\dagger)^\dagger = V\phi V^\dagger,\ \langle V\phi V^\dagger\rangle = \langle\phi\rangle = 0 \tag{2.52}$$

and

$$V_1(V_2\phi V_2^\dagger)V_1^\dagger = (V_1V_2)\phi(V_1V_2)^\dagger. \tag{2.53}$$

If V is parameterized as $V = \exp{(-i\theta_V^a\frac{\lambda_a}{2})}$, where the λ_a are the generators of $SU(N_f)$ and satisfy the normalization condition $\langle\lambda_a\lambda_b\rangle = 2\delta_{ab}$, the result of an expansion to first order in θ_V^a is

$$\begin{aligned} V\phi V^\dagger &= \phi - i\theta_V^a[\frac{\lambda_a}{2},\lambda_b]\phi_b + \dots \\ &= \phi + f_{abc}\theta_V^a\phi_b\lambda_c + \dots. \end{aligned} \tag{2.54}$$

For the Goldstone bosons, this is just the required transformation behaviour in the adjoint representation.

2.4 Chirally invariant structures

2.4.1 L-R basis

Up to now, different types of structures with clearly defined behaviour under chiral transformations (V_L, V_R) have been collected. Moreover, traceless field strength tensors for the external fields are introduced:

$$F^{\substack{R\\L}}_{\alpha\beta} = \partial_\alpha\binom{r}{l}_\beta - \partial_\beta\binom{r}{l}_\alpha - i[\binom{r}{l}_\alpha, \binom{r}{l}_\beta]. \tag{2.55}$$

Table 2.2: Building blocks in L-R basis

building block	transformation behaviour	chiral order
U	$V_R U V_L^\dagger$	0
$D_\alpha U$	$V_R D_\alpha U V_L^\dagger$	1
χ	$V_R \chi V_L^\dagger$	2
$F_{\alpha\beta}^{R/L}$	$V_{R/L} F_{\alpha\beta}^{R/L} V_{R/L}^\dagger$	2

The set of structures in the L-R basis is summarized in table 2.2.

The transformation of adjoint quantities yields exactly the adjoints of the transformed quantities in accord with naive expectations. Chiral order is a measure of the importance of a structure in the low-energy regime (section 2.4.5).

It is important to note a certain ambiguity in the counting of chiral orders. If B_0 were small enough, the structure χ would have to be treated as chiral order 1 instead (generalized ChPT). In that case the quadratic Goldstone boson masses would be given by non-linear terms in the quark masses. Comparison with experimental data was necessary to prove that the structure χ must be treated as chiral order two.

Different structures with different transformation behaviour require different kinds of covariant derivatives (see Eq. (2.56). These covariant derivatives satisfy the Leibniz rule.

$$
\begin{aligned}
A \overset{G}{\mapsto} V_R A V_L^\dagger : &\quad D_\alpha A = \partial_\alpha A - i r_\alpha A + i A l_\alpha \\
B \overset{G}{\mapsto} V_L B V_R^\dagger : &\quad D_\alpha B = \partial_\alpha B - i l_\alpha B + i B r_\alpha \\
C \overset{G}{\mapsto} V_R C V_R^\dagger : &\quad D_\alpha C = \partial_\alpha C - i r_\alpha C + i C r_\alpha \\
D \overset{G}{\mapsto} V_L D V_L^\dagger : &\quad D_\alpha D = \partial_\alpha D - i l_\alpha D + i D l_\alpha \\
E \overset{G}{\mapsto} E : &\quad D_\alpha E = \partial_\alpha E
\end{aligned}
\tag{2.56}
$$

By multiplication of one structure of the first three species from table 2.2 to an adjoint of one of these, a product is generated that transforms as

$$
(\text{product}) \mapsto V_R (\text{product}) V_R^\dagger. \tag{2.57}
$$

These products transform like $F_{\alpha\beta}^R$. Any such product does have the same kind of covariant derivative. In the case of $F_{\alpha\beta}^L$, left-multiplication by a structure of the aforementioned species and right-multiplication by an adjoint of one of these yields the same transformation behaviour. Taking the trace and using the cyclic property, it is clear that the result is invariant under any chiral transformation. In this way, chirally invariant Lagrangian densities can be constructed.

2.4.2 R-R basis

Due to the desired chiral invariance, the need arose to combine structures with different transformation properties. Among these, two structures, U and $U^\dagger$, had chiral order 0. It is a reasonable approach to construct linear combinations of products with these. These combinations automatically have the transformation behaviour of Eq. (2.57). This approach can be extended further, if linear combinationsare constructed, which are automatically hermitian or antihermitian. Since the Lagrangian must be hermitian, any allowed hermitian structure must be equivalent to a linear combination of products of these. The construction prescription for these R-R blocks is

$$[A]_\pm = \frac{1}{2}[AU^\dagger \pm UA^\dagger], \tag{2.58}$$

where A is taken from table 2.3.

At this stage, the possibility of so-called *contact terms* has to be pointed out. These are structures, where no U matrix appears. That is why no interactions of Goldstone bosons can result from these, but only interactions of external fields[12]. The dynamics of external fields, if physical fields are inserted, are governed by their own quantum theories. But the LECs cannot be simply taken from these quantum field theories, since the quark loops are dressed in gluon clouds in accordance with the rules of large-N_c counting. An interpretation of the leading order in large-N_c counting in terms of an infinite sum of diagrams, with different numbers of internal gluon lines is discussed in [Wit 79a].

Contact terms cannot be measured directly, so their LECs are not observable.In the calculation of any process, it is advantageous to incorporate these and reduce the number of observable parameters. Use of these contact terms in the construction of a Lagrangian density requires leaving the basic set of monomials, where the construction principles are illuminated in the clearest way.

Using partial integration techniques requires messy calculations, since

$$D_\alpha[A]_\pm = [D_\alpha A]_\pm - \frac{1}{2}\Big\{[A]_\mp, [D_\alpha U]_-\Big\} - \frac{1}{2}\Big[[A]_\pm, [D_\alpha U]_-\Big] \neq [D_\alpha A]_\pm. \tag{2.59}$$

Many such structures, where covariant derivatives operate on external fields, appear for the first time in the construction of the Lagrangian at chiral order six. This, among other aspects, is the reason, why another set of basic building blocks is introduced.

[12]The interpretation of flavour traces in terms of interacting fields is clarified in section

Table 2.3: Input into the R-R blocks

structure	chiral order
$D_\alpha U$	1
χ	2
$F^R_{\alpha\beta} U$	2
$U^\dagger F^L_{\alpha\beta}$	2

2.4.3 u basis

In baryonic ChPT (and in ChPT, where vector mesons or any other species of non-Goldstone bosons are included), the interaction of other particles with Goldstone bosons has a description in terms of the so-called *chiral connection* Γ_α and the *chiral vielbein* u_α. In this basis, the Goldstone boson matrix is expressed in terms of the square root of the previously used one[13]:

$$u(x) = \sqrt{U(x)}. \tag{2.60}$$

The transformation behaviour of the new matrix is derived from the transformation behaviour of U:

$$\begin{aligned} u \overset{(V_L,V_R)}{\mapsto} u' &= V_R u K^\dagger(V_L, V_R, u) \\ &= K(V_L, V_R, u) u V_L^\dagger. \end{aligned} \tag{2.61}$$

The so-called *compensator field*

$$K(V_L, V_R, u) = \sqrt{V_R^\dagger u^\dagger \sqrt{V_L} V_R u} \tag{2.62}$$

is a $SU(N_f)$ matrix and depends explicitly on the Goldstone boson field configuration. Some authors use the terminology $u(\varphi), g(g_R, g_L)$ and $h(g, \varphi)$ instead. Using this Goldstone boson matrix u, structures X transforming as

$$X \overset{(V_L,V_R)}{\mapsto} K(V_L, V_R, u) X K^\dagger(V_L, V_R, u), \tag{2.63}$$

are constructed and flavour traces of products of these are taken.

The aforementioned quantities are defined as[14]

$$u_\alpha = i\{u^\dagger(\partial_\alpha - ir_\alpha)u - u(\partial_\alpha - il_\alpha)u^\dagger\}, \tag{2.64}$$

$$\Gamma_\alpha = \tfrac{1}{2}\{u^\dagger(\partial_\alpha - ir_\alpha)u + u(\partial_\alpha - il_\alpha)u^\dagger\}. \tag{2.65}$$

2.5. In principle, contact terms are quark loop corrections for the external fields.

[13] Only even total powers of u and $u^\dagger$ appear in the Lagrangian. Thus the sign of u is actually irrelevant.

[14] In the external field free case, u_α is odd and Γ_α is even in the fields.

Furthermore, occurrences of the external fields besides the covariant derivatives are introduced:

$$\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u, \quad (2.66)$$

$$f_\pm^{\alpha\beta} = u F_L^{\alpha\beta} u^\dagger \pm u^\dagger F_R^{\alpha\beta} u. \quad (2.67)$$

The covariant derivative of any such structure is given by

$$\nabla_\alpha X = \partial_\alpha X + [\Gamma_\alpha, X]. \quad (2.68)$$

Partial integration in terms of this covariant derivative is thus exceptionally easy. The difference between a partial derivative and a covariant derivative can only be found in the introduction of an additional commutator, where the initial structures can be kept as they are. Furthermore, traces of commutators vanish. This is a serious simplification of the construction of partial-integration-induced relations.

As a remark, some authors [BCE 00] use a structure

$$\chi_{\pm\alpha} u^\dagger D_\alpha \chi u^\dagger \pm u D_\alpha \chi^\dagger u = \nabla_\alpha \chi_\pm - \frac{i}{2}\{\chi_\mp, u_\alpha\} \quad (2.69)$$

instead of $\nabla_\alpha \chi_\pm$. This is (in a similar manner to some previous remarks concerning the contact terms) useful in calculations in the case of pure QCD, where the quark mass matrix (times $2B_0$) is substituted for χ, since it is a constant quantity. Otherwise, it does not offer any obvious advantage over the standard covariant derivative. Therefore, the standard version is used in this work.

Next, a "symmetrized covariant derivative of the Goldstone boson matrix" is introduced:

$$h_{\alpha\beta} = \nabla_\alpha u_\beta + \nabla_\beta u_\alpha, \quad (2.70)$$

$$\nabla_\alpha u_\beta = \frac{1}{2}(h_{\alpha\beta} - f_{-\alpha\beta}). \quad (2.71)$$

This name is explained later, when the comparison of the different bases is performed.

The final constituent to the assembly is the field strength tensor of the chiral connection. It is introduced in terms of a commutator of two covariant derivatives

$$[\nabla_\alpha, \nabla_\beta]X = [\Gamma_{\alpha\beta}, X] \quad (2.72)$$

and has the following shape

$$\begin{aligned}\Gamma_{\alpha\beta} &= \partial_\alpha \Gamma_\beta - \partial_\beta \Gamma_\alpha - [\Gamma_\alpha, \Gamma_\beta] \\ &= \frac{1}{4}[u_\alpha, u_\beta] - \frac{i}{2} f_{+\alpha\beta}. \end{aligned} \quad (2.73)$$

The Eqs. (2.72) and (2.73) are the reason, why any pair of covariant derivatives can be treated as a symmetrized covariant derivative plus some additive structures that has been included elsewhere in an approach, where the most general Lagrangian is constructed.

2.4.4 Comparison of different bases

As different authors use different bases, a translation prescription is necessary to compare their Lagrangians. The most important structures are summarized in the following table:

Table 2.4: Comparison of different bases

u basis	R-R basis
u_α	$iu^\dagger[D_\alpha U]_- u$
$h_{\alpha\beta}$	$2iu^\dagger[D_\alpha D_\beta U]^s_- u$
$\chi_\pm$	$2u^\dagger[\chi]_\pm u$
$\nabla_\alpha\chi_\pm$	$2u^\dagger([D_\alpha\chi]_\pm - \frac{1}{4}\{[D_\alpha U]_-,[\chi]_\mp\})u$
$f_\pm^{\alpha\beta}$	$u^\dagger[\,{}^{\;G^{\alpha\beta}}_{-H^{\alpha\beta}}]_+ u$
$\nabla_\gamma f_\pm^{\alpha\beta}$	$u^\dagger[D_\gamma\,{}^{\;G^{\alpha\beta}}_{-H^{\alpha\beta}}]_+ u$
u basis	L-R basis
u_α	$iu^\dagger D_\alpha U U^\dagger u$
$h_{\alpha\beta}$	$\frac{1}{4}u^\dagger\Big(\{D_\alpha,D_\beta\}UU^\dagger - U\{D_\alpha,D_\beta\}U^\dagger\Big)u$
$\chi_\pm$	$u^\dagger\Big(\chi U^\dagger \pm U\chi^\dagger\Big)u$
$\nabla_\alpha\chi_\pm$	$\frac{1}{2}u^\dagger\Big[(UD_\alpha U^\dagger\chi U^\dagger + \chi D_\alpha U^\dagger + 2D_\alpha\chi U^\dagger)$
	$\pm(U\chi^\dagger D_\alpha U U^\dagger + D_\alpha U\chi^\dagger + 2UD_\alpha\chi^\dagger)\Big]u$
$f_\pm^{\alpha\beta}$	$\pm u^\dagger\Big(F_R^{\alpha\beta} \pm UF_L^{\alpha\beta}U^\dagger\Big)u$
$\nabla_\gamma f_\pm^{\alpha\beta}$	$u^\dagger\Big[(D_\gamma F_R^{\alpha\beta} + UD_\gamma F_L^{\alpha\beta}U^\dagger)$
	$+\frac{1}{2}([F_R^{\alpha\beta}, D_\gamma UU^\dagger] + U[F_L^{\alpha\beta}, D_\gamma U^\dagger U]U^\dagger)\Big]u$
u basis	Ref. [EFS 02]
u_α	$\frac{i}{2}(D_\alpha U)_-$
$h_{\alpha\beta}$	$i(D_\alpha D_\beta U)^s_-$
$\chi_\pm$	$(\chi)_\pm$
$\nabla_\alpha\chi_\pm$	$(D_\alpha\chi)_\pm - \frac{1}{2}\{(D_\alpha U),(\chi)_\mp\}$
$f_\pm^{\alpha\beta}$	$\frac{1}{2}(\,{}^{\;G^{\alpha\beta}}_{-H^{\alpha\beta}})_+$
$\nabla_\gamma f_\pm^{\alpha\beta}$	$\frac{1}{2}(D_\gamma\,{}^{\;G^{\alpha\beta}}_{-H^{\alpha\beta}})_+$
u basis	Ref. [BCE 00]
u_α	u_α
$h_{\alpha\beta}$	$h_{\alpha\beta}$
$\chi_\pm$	$\chi_\pm$
$\nabla_\alpha\chi_\pm$	$\chi_{\pm\,\alpha} + \frac{i}{2}\{u_\alpha,\chi_\mp\}$
$f_\pm^{\alpha\beta}$	$f_\pm^{\alpha\beta}$
$\nabla_\gamma f_\pm^{\alpha\beta}$	$\nabla_\gamma f_\pm^{\alpha\beta}$

This comparison obviously shows that the L-R basis, even though it is closest to the physical input into the theory, requires very complicated structures, when generating a hermitian and chirally invariant function. The L-R basis clearly demonstrates that the chiral vielbein is, in principle, just a covariant derivative of the Goldstone boson matrix.

On the other hand, the u basis (and the basis used in [BCE 00]) have a nice geometric interpretation. Last but not least, the u basis is the most compact notation.

2.4.5 Chiral order and power counting

Chiral order is assigned to a given diagram (and thereby to the corresponding Lagrangian) by a rescaling of *external* momenta and of the quark masses. Even though a rescaling of external pionic momenta may be done in the experiment to some degree, a rescaling of the quark masses is clearly a theorist's approach. This mass rescaling can be understood as a closer approximation to the chiral limit. Quadratic rescaling of the quark masses is a feature of ChPT, in contrast to linear rescaling in generalized ChPT. The chiral dimension of a diagram is given by

$$\mathcal{M}(tp_i, t^2 m_q) = t^D \mathcal{M}(p_i, m_q). \tag{2.74}$$

In the case $t \ll 1$, diagrams with small values of D clearly dominate.

The chiral order D can be related to the number of loops N_L by the following formula:

$$D = 2 + \sum_{n=1}^{\infty} 2(n-1)N_{2n} + 2N_L. \tag{2.75}$$

The validity of this formula is demonstrated in multiple steps. d-dimensional integrations over internal momenta and the propagators of internal particles are rescaled:

$$\begin{aligned}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - M^2 + i\epsilon} &\overset{M^2 \mapsto t^2 M^2}{\mapsto} t^{-2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{(\frac{k}{t})^2 - M^2 + i\epsilon} \\ &\overset{l:=\frac{k}{t}}{=} t^{d-2}\int \frac{d^dl}{(2\pi)^d}\frac{i}{l^2 - M^2 + i\epsilon}.\end{aligned} \tag{2.76}$$

Vertices of the Lagrangian $\mathcal{L}_{2n}$ can be treated as proportional to p^{2n}, but also generate momentum conserving δ functions:

$$\delta^d(p+k) \overset{p\mapsto tp,\ k=tl}{\mapsto} t^{-d}\delta^d(p+l), \tag{2.77}$$

$$p^{2n-m}k^m \overset{p\mapsto tp,\ k=tl}{\mapsto} t^{2n}p^{2n-m}l^m. \tag{2.78}$$

When performing the transition from S-matrix to invariant amplitude, an overall momentum conserving δ function is factored out, thus

$$t^{D_S} = t^{-d}t^D.$$

Up to that point, the chiral order is given by

$$D = d + (d-2)N_I + \sum_{n=1}^{\infty} 2(n - \frac{d}{2})N_{2n}, \tag{2.79}$$

where N_I is the number of internal lines. This number is connected with the number of independent[15] loops N_L and with the total number of vertices N_V by

$$N_L = N_I - (N_V - 1) = 1 + N_I - \sum_{n=1}^{\infty} N_{2n}. \tag{2.80}$$

Insertion in Eq. (2.79) yields:

$$\begin{aligned} D &= d + (d-2)(N_L + N_V - 1) + \sum_{n=1}^{\infty} 2(n - \frac{d}{2})N_{2n} \\ &= 2 + (d-2)N_L + \sum_{n=1}^{\infty} 2(n-1)N_{2n}. \end{aligned} \tag{2.81}$$

In the physical case of $d = 4$, Eq. (2.75) is reproduced.

2.4.6 Additional invariant structures

Two alternative approaches could be employed to generate invariant structures. One additional approach for the construction of invariant traces could be proposed, which is inspired by the usual scalar products. Besides the usual Casimir operator known from general $SU(N_f)$,

$$C_1 = \sum_{a=1}^{N_f^2-1} \lambda_a \lambda_a, \tag{2.82}$$

another independent Casimir operator for $SU(3)$ exists:

$$C_2 = \sum_{a,b,c=1}^{8} d_{abc} \lambda_a \lambda_b \lambda_c = \frac{80}{9} 1_{3\times3}. \tag{2.83}$$

Both can be used in the construction of invariant structures. In the following cases, the flavour matrices S, T and U are such that traces of products of them are invariant (e.g.: structures of the u basis). This invariance is broken by explicit insertion of generators of the chiral groups into the traces. The Casimir operators demonstrate how to construct invariants from

[15] The argument is simply that each internal line has a momentum integration and each vertex has a momentum conserving δ function, one of which is factored out. The difference is the number of independent momentum integrations.

these. The zeroth generator (unit matrix with proper orthonormalization condition) can be included without problems.

The Casimir operator C_1 allows one product:

$$\sum_{a=0}^{N_f^2-1} \langle S\lambda_a\rangle\langle T\lambda_a\rangle = 2\langle ST\rangle. \tag{2.84}$$

Due to the Casimir operator C_2 (for $SU(3)$), three additional invariant products can be constructed:

$$\begin{aligned}
\sum_{a,b,c=0}^{8} d_{abc}\langle S\lambda_a\rangle\langle T\lambda_b\rangle\langle U\lambda_c\rangle &= 2\langle\{S,T\}U\rangle - \frac{8}{3}\langle ST\rangle\langle U\rangle, &(2.85)\\
\sum_{a,b,c=0}^{8} d_{abc}\langle S\lambda_a\rangle\langle T\{\lambda_b,\lambda_c\}\rangle &= \frac{10}{3}\langle ST\rangle, &(2.86)\\
\sum_{a,b,c=0}^{8} d_{abc}\langle S\lambda_a\lambda_b\lambda_c\rangle &= \frac{80}{9}\langle S\rangle. &(2.87)
\end{aligned}$$

Since any of these invariants can be expressed in terms of linear combinations of invariant traces, they do not have to be considered.

Another approach would be the inclusion of determinants of $SU(N_f)$-matrices. These would be invariant under $SU(N_f)$ and under $U(1)_V$, but not under $U(1)_A$. The Cayley-Hamilton formula expresses a determinant of an $GL(N, \mathbb{C})$ matrix in terms of a linear combination of products of traces. Even though the principal ideas of Cayley-Hamilton formulae for different N are exactly the same, the specific shapes are very different. The two relevant flavour groups are $SU(2)$ and $SU(3)$[16]:

$$\begin{aligned}
SU(2):\quad \det(A)1 &= -A^2 + \langle A\rangle A,\\
SU(3):\quad \det(A)1 &= A^3 - \langle A\rangle A^2 + \tfrac{1}{2}(\langle A\rangle^2 - \langle A^2\rangle)A.
\end{aligned} \tag{2.88}$$

That is why determinants do not appear[17] in the construction of the most general Lagrangian.

[16] The very useful assumption that $\langle u_\alpha\rangle = 0$ is not valid anymore, when the singlet η is taken into account (and thus the flavour group is $U(3)$ instead of $SU(3)$). The singlet η can always be factored out. Therefore, the shape of the trace relations is different, if the singlet η is included.

[17] For pure octet interactions, they can be used in the construction of contact terms (section 3.5.4). When the anomaly and the singlet η are considered, determinants turn out to be the preferable choice (section 4.3).

2.5 Large-N_c counting in chiral perturbation theory

Due to the work of 't Hooft in [Hoo 74], an expansion of QCD structures in terms of $\frac{1}{N_c}$ can be discussed. The hadron spectrum implies that nature has chosen $N_c = 3$, so $\frac{1}{N_c}$ is not very small. Even more, the number of diagrams of a given order in $\frac{1}{N_c}$ contributing to any given process is infinite. However, the expansion in $\frac{1}{N_c}$ is very attractive from a conceptual point of view, because it can explain important aspects of phenomenology of the strong interactions and it is the only expansion parameter QCD actually has. For a detailed discussion the reader is referred to [Wit 79a].

Actually, ChPT does not bother about quarks being coloured objects. On the other hand, considering a $\frac{1}{N_c}$ expansion without considering the colour degrees of freedom does not make sense. The question to be answered here is: How does colour enter into ChPT?

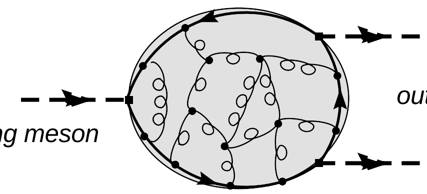

Figure 2.1: A typical leading order diagram, which describes tree level interactions of mesons. The shaded area contributes to a three meson ChPT interaction vertex with one flavour trace.

Hadrons are colour singlets. Baryons are created by three valence quarks of different colours contracted with an antisymmetric ϵ tensor in colour space. In contrast, mesons contain a quark of a given colour and an anti-quark of the respective anti-colour. This pair's colour index is then summed. In colour space an arbitrary meson $|\phi\rangle$ of flavour contents $|q_{f_1}\overline{q}_{f_2}\rangle$ is represented by

$$|\phi\rangle = \sum_{c=1}^{N_c} |q^c_{f_1}\overline{q}^c_{f_2}\rangle. \tag{2.89}$$

According to the $\frac{1}{N_c}$ expansion rules, tree level graphs, where mesons are exchanged, correspond to an infinite number of diagrams. Each consists of a quark loop, to which the meson sources are coupled. Its edges are quark lines. An arbitrary number of gluons are allowed, which all lie inside the loop (to leading order in $\frac{1}{N_c}$). Figure 2.1 shows a typical example.

The quark line going around the loop has two free choices: the colour of the quark line originating in one of the mesons and the flavour originating in the same meson (at least, if the meson is a linear combination in flavour space). In the monomials for a chiral Lagrangian, the arbitrariness of the choice of flavour is taken care of by the parameterisation of the meson fields via generators of the flavour group. The arbitrariness of the choice of the colour degree of freedom is taken care of by multiplication with a unit matrix $1_{N_c \times N_c}$ in colour space. Therefore, a physical meson looks like

$$\phi = \sum_{f=0}^{N_f^2-1} \phi_f \; \lambda_f \; 1_{N_c \times N_c}. \tag{2.90}$$

A product of such matrices is still a unit matrix in colour space. Taking the trace in flavour and in colour space generates a factor N_c corresponding to the quark loop and ensures the invariance under chiral and colour transformations.

The appearance of multiple traces ensures that the flavour and colour contents of each of the associated quark loops are independent. Thus, they can interact strongly only by at least twofold quark-antiquark annihilation into a gluon in one of the loops. These gluons form an intermediate (purely gluonic) state, which couples in the same way to the second quark loop. This intermediate gluonic state is suppressed by a factor of $\frac{1}{N_c}$. This is the OZI rule in action. An example is given in figure 2.2.

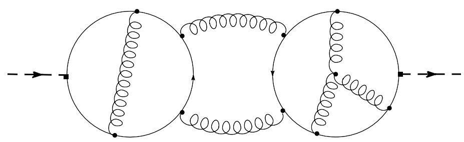

Figure 2.2: Two flavour traces implicate two quark loops, which interact by quark-antiquark annihilation into several gluons.

When the Cayley-Hamilton formulae of section 2.4.6 are applied, they require $GL(N_f, \mathbb{C})$ matrices instead of $GL(N_f + N_c, \mathbb{C})$ matrices. In other words, either the trace relations would have to be derived from $GL(N_f + N_c, \mathbb{C})$ Cayley-Hamilton formulae, or the traces of the colour matrices had to be taken before trace relations are applied.

Simply put, the entire N_c dependence can be absorbed into the LECs, when the colour summations are explicitly done and $\frac{1}{N_c}$ suppressions for each necessary gluonic intermediate state are taken into account:

$$LEC \propto \mathcal{O}(N_c^{(\#\text{ f.t.})-2((\#\text{ f.t.})-1)}) = \mathcal{O}(N_c^{2-(\#\text{ f.t.})}), \tag{2.91}$$

where # f.t. abbreviates number of flavour traces. The left-over flavour traces are N_c independent.

However, large-N_c-counting works in such a naive way only for the LECs of the set of a priori independent monomials. When symmetry relations are applied, which connect monomials with different numbers of flavour traces (this means leading-order-equation-of-motion-induced relations and trace relations), the *linear combinations of LECs* multiplied to the kept monomials must consist of *multiple orders in* $\frac{1}{N_c}$. Therefore, the orders of the redefined LECs for the kept monomials are enhanced accordingly, if the eliminated monomial was leading order in the symmetry relation[18].

[18] This phenomenon is discussed in detail in section 3.4.3. In the next-to-next-to-leading order, the argument is exactly the same, but not discussed in detail. A symmetry relation linking multiple non-leading-order monomials to any leading-order monomial, increases the number of leading-order LECs, if a leading-order monomial is eliminated. Therefore, leading-order-equation-of-motion-induced relations, where $\langle \chi_- \rangle$ is relevant, do not change the number of leading-order contributions, no matter which monomial is eliminated. It is demonstrated at next-to-leading order that even though trace relations may change the number of leading-order LECs, it is non-trivial to find out, how many leading-order LECs are left, after all trace relations have been applied.

Chapter 3

Construction of the Most General Mesonic Lagrangian up to Chiral Order 6

In this chapter, only the interactions of the pseudoscalar octet are treated. The structure is organized in the following way:

- First, the symmetry requirements of the theory and the symmetry properties of the building blocks are discussed.
- In a second part, the leading-order Lagrangian $\mathcal{L}_2$ is and the nature of the mass term and the associated constant B_0 is clarified. The leading-order equation of motion is derived, since it is necessary for a reduction of the absolute number of independent monomials at higher chiral orders.
- In a third part, the different species of relations among monomials are discussed.
- In a fourth part, the next-to-leading-order Lagrangian $\mathcal{L}_4$ is constructed. A comparison to the Lagrangian of Gasser and Leutwyler [GL 85] is done. The calculation of relations among redundant monomials can be demonstrated explicitly. The elimination paradigm is discussed. In particular it is demonstrated, why the approach of first generating all possible relations and then finding the minimal set of independent monomials via computer algebra systems avoids losing information.
- In the fifth part, the next-to-next-to-leading-order Lagrangian $\mathcal{L}_6$ for even intrinsic parity is constructed.
- In the sixth part, the anomalous Lagrangian $\mathcal{L}_6^\epsilon$ at next-to-next-to-leading order is constructed.

3.1 Symmetries in ChPT

Due to Weinberg, the symmetries of the effective Lagrangian must be the same as the symmetries of the fundamental theory. Recapitulating Eq. (2.1),

$$\mathcal{L}^0_{QCD} = \sum_{f=1}^{N_f} \sum_{c=1}^{N_c} \overline{q}_{f,\,c}\, i\not{D}\, q_{f,\,c} - \frac{1}{2}\big\langle G_{\alpha\beta} G^{\alpha\beta} \big\rangle_c,$$

shows unbroken chiral symmetry, Lorentz invariance, hermiticity and P-, C- and T-invariance. Later, in chapter 4, when the θ term is introduced, $U(1)_A$-symmetry is violated in a controlled manner and parity becomes parameter dependent.

The construction of an effective Lagrangian with invariance under chiral transformations is performed by taking traces of blocks transforming in the u basis. Lorentz invariance is automatically satisfied, since $\mathcal{L}$ is a scalar. As all blocks in the u basis are hermitian except χ_- (and $\nabla_\alpha \chi_-$), an overall hermitian conjugation of the argument of a trace inverts the order and picks up a minus sign for each occurrence of these structures. The inversion of order produces also a minus sign, if the argument of the trace is antisymmetric in two exchanged quantities.

Parity transformations

Table 3.1: Dirac matrices subject to parity transformations

Γ	1	γ_α	$\sigma_{\alpha\beta}$	γ_5	$\gamma_\alpha\gamma_5$
$\gamma_0\Gamma\gamma_0$	1	$\varepsilon(\alpha)\gamma_\alpha$	$\varepsilon(\alpha)\varepsilon(\beta)\sigma_{\alpha\beta}$	$-\gamma_5$	$-\varepsilon(\alpha)\gamma_\alpha\gamma_5$

The transformation behaviour of the fields under a parity transformation must be derived from the QCD-Lagrangian itself. Since quark fields (each flavour treated separately) transform as

$$q(t,\vec{x}) \overset{P}{\mapsto} \gamma_0 q(t,-\vec{x}), \tag{3.1}$$

the properties of the Dirac matrices from table 3.1 ensure invariance of the Lagrangian under parity transformations, $\mathcal{L}(t,\vec{x}) \overset{P}{\mapsto} \mathcal{L}(t,-\vec{x})$, where a function

$$\varepsilon(\alpha = 0) = -\varepsilon(\alpha \neq 0) = 1 \tag{3.2}$$

is introduced, which differentiates between temporal and spatial indices[1]. This induces the changes of table 3.2 in the physical quantities and the

[1]Some authors prefer interchanging covariant and contravariant Lorentz indices under

changes of table 3.3, where the space-time coordinates are suppressed, in the building blocks of the u basis.

Table 3.2: Parity rules for the physical fields

$$\begin{array}{|rl|}\hline \phi(t,\vec{x}) \overset{P}{\mapsto} -\phi(t,-\vec{x}), & u(t,\vec{x}) \overset{P}{\mapsto} u^\dagger(t,-\vec{x}), \\ r_\alpha(t,\vec{x}) \overset{P}{\mapsto} \varepsilon(\alpha) l_\alpha(t,-\vec{x}), & l_\alpha(t,\vec{x}) \overset{P}{\mapsto} \varepsilon(\alpha) r_\alpha(t,-\vec{x}) \ , \\ s(t,\vec{x}) \overset{P}{\mapsto} s(t,-\vec{x}), & p(t,\vec{x}) \overset{P}{\mapsto} -p(t,-\vec{x}). \\ \hline \end{array}$$

Table 3.3: Parity rules for the u basis

$$\begin{array}{|rl|}\hline u_\alpha \overset{P}{\mapsto} -\varepsilon(\alpha) u_\alpha, & h_{\alpha\beta} \overset{P}{\mapsto} -\varepsilon(\alpha)\varepsilon(\beta) h_{\alpha\beta}, \\ f_\pm^{\alpha\beta} \overset{P}{\mapsto} \pm\varepsilon(\alpha)\varepsilon(\beta) f_\pm^{\alpha\beta}, & \chi_\pm \overset{P}{\mapsto} \pm\chi_\pm \ , \\ \hline \end{array}$$

Covariant derivatives pick up an additional factor of the function $\varepsilon(\alpha)$:

$$\nabla_\alpha X \overset{P}{\mapsto} \varepsilon(\alpha) \nabla_\alpha P X. \tag{3.3}$$

Charge conjugations

Table 3.4: Dirac matrices subject to charge conjugations

Γ	1	γ_α	$\sigma_{\alpha\beta}$	γ_5	$\gamma_\alpha\gamma_5$
$C^{-1}\Gamma C$	1	γ_α^T	$-\sigma_{\alpha\beta}^T$	$-\gamma_5$	$(\gamma_\alpha\gamma_5)^T$

The transformation properties under charge conjugation are derived in the same manner. The quark fields transform like

$$q \overset{C}{\mapsto} C\overline{q}^T, \quad \overline{q} \overset{C}{\mapsto} -q^T C^{-1}, \tag{3.4}$$

where the transformation matrix C is given by

$$C = i\gamma^2\gamma^0 = -C^\dagger = -C^{-1} = -C^T. \tag{3.5}$$

parity transformations. This works fine as long as these are contracted with metric tensors. When ϵ tensors appear, each ϵ tensor produces an additional sign under parity transformations. Because contraction of all structures with the same ϵ tensor is convenient, the metric sign function $\varepsilon(\alpha)$ is preferred here.

According to table 3.4, the entire Dirac structure of the Lagrangian is subject to transposition and the Lagrangian itself is invariant. The behaviour of the physical quantities under charge conjugation is presented in table 3.5, while the transformation properties of the u basis are presented in table 3.6.

Table 3.5: Charge conjugation rules for the physical fields

$$\begin{array}{cc} \phi \overset{C}{\mapsto} \phi^T, & u \overset{C}{\mapsto} u^T, \\ r_\alpha \overset{C}{\mapsto} -l_\alpha^T, & l_\alpha \overset{C}{\mapsto} -r_\alpha^T\ , \\ s \overset{C}{\mapsto} s^T, & p \overset{C}{\mapsto} p^T. \end{array}$$

Table 3.6: Charge conjugation rules for the u basis

$$\begin{array}{cc} u_\alpha \overset{C}{\mapsto} u_\alpha^T, & h_{\alpha\beta} \overset{C}{\mapsto} h_{\alpha\beta}^T, \\ f_\pm^{\alpha\beta} \overset{C}{\mapsto} \mp f_\pm^{\alpha\beta}, & \chi_\pm \overset{C}{\mapsto} \chi_\pm^T, \end{array}$$

Covariant derivatives of any structure transform like the structure itself. Finally, the transpositions are taken out of the flavour traces by inverting their internal ordering.

Summary of transformation properties

The isoscalar source $v_\alpha^{(s)}$ transforms as

$$\begin{aligned} v_\alpha^{(s)} &\overset{P}{\mapsto} \varepsilon(\alpha) v_\alpha^{(s)}, \\ v_\alpha^{(s)} &\overset{C}{\mapsto} -v_\alpha^{(s)\,T}. \end{aligned} \tag{3.6}$$

The complete set of invariance properties is fulfilled by adding the transformed structures to the Lagrangian. A more compact notation is achieved by introduction of *"generalized (anti-)commutators"*:

$$[A_1, A_2, \ldots, A_n] = A_1 A_2 \ldots A_n - A_n \ldots A_2 A_1, \tag{3.7}$$

$$\{A_1, A_2, \ldots, A_n\} = A_1 A_2 \ldots A_n + A_n \ldots A_2 A_1. \tag{3.8}$$

The transformation properties of the basic building blocks are summarized for quick reference in table 3.7.

Table 3.7: Complete transformation rules for the u basis

building block	P	C	h.c.
u_α	$-\varepsilon(\alpha)u_\alpha$	u_α^T	u_α
$h_{\alpha\beta}$	$-\varepsilon(\alpha)\varepsilon(\beta)h_{\alpha\beta}$	$h_{\alpha\beta}^T$	$h_{\alpha\beta}$
$\chi_\pm$	$\pm\chi_\pm$	$\chi_\pm^T$	$\pm\chi_\pm$
$f_\pm^{\alpha\beta}$	$\pm\varepsilon(\alpha)\varepsilon(\beta)f_\pm^{\alpha\beta}$	$\mp(f_\pm^{\alpha\beta})^T$	$f_\pm^{\alpha\beta}$
$\nabla_\gamma X$	$\varepsilon(\gamma)\nabla_\gamma PX$	$\nabla_\gamma CX$	$\nabla_\gamma(X)^\dagger$

Intrinsic parity

Another property of mesonic Lagrangians for Goldstone bosons only is called intrinsic parity. The transformation prescription is similar to that of a parity transformation, except that the spatial coordinates are left unchanged. Table 3.8 demonstrates the intrinsic parity of physical quantities. The building blocks transform in the same way as under parity (if the space-time coordinates are ignored), except that no metric sign functions $\varepsilon(\alpha)$ appear. Even intrinsic parity implies that the number of Goldstone bosons is always even, while odd intrinsic parity means that the respective number is odd. Every monomial in the anomalous sector automatically has odd intrinsic parity.

Table 3.8: Intrinsic parity rules for the physical fields

$$\begin{array}{rlrl} \phi(t,\vec{x}) & \stackrel{\text{i.p.}}{\mapsto} -\phi(t,\vec{x}), & u(t,\vec{x}) & \stackrel{\text{i.p.}}{\mapsto} u^\dagger(t,\vec{x}), \\ r_\alpha(t,\vec{x}) & \stackrel{\text{i.p.}}{\mapsto} l_\alpha(t,\vec{x}), & l_\alpha(t,\vec{x}) & \stackrel{\text{i.p.}}{\mapsto} r_\alpha(t,\vec{x}) , \\ s(t,\vec{x}) & \stackrel{\text{i.p.}}{\mapsto} s(t,\vec{x}), & p(t,\vec{x}) & \stackrel{\text{i.p.}}{\mapsto} -p(t,\vec{x}). \end{array}$$

3.2 Leading-order Lagrangian $\mathcal{L}_2$

3.2.1 Monomials

At chiral order two Lorentz invariance and parity invariance allow only three different chirally invariant structures (see table 3.9). The third column denotes the leading contribution in a field expansion.

Table 3.9: Monomials at chiral order two

monomial	shape	contributes to
(1)	$\langle u_\alpha u^\alpha \rangle$	2ϕ
(2)	$\langle u_\alpha \rangle \langle u^\alpha \rangle$	2ϕ
(3)	$\langle \chi_+ \rangle$	contact term

Recurring to Eq. (2.64) and including only the octet fields, the tracelessness of u_α is obvious. Cyclicity of the trace and the parametrisations of the external fields (Eq. (2.6)) and of the Goldstone boson matrix (Eq. (2.60)) are used:

$$\begin{aligned}
\langle u_\alpha \rangle &= i\langle u^\dagger(\partial_\alpha - ir_\alpha)u - u(\partial_\alpha - il_\alpha)u^\dagger \rangle \\
&\overset{\text{cyclicity}}{=} i(\langle u^\dagger \partial_\alpha u \rangle - \underbrace{\langle u \partial_\alpha u^\dagger \rangle}_{\overset{u\,\in\,SU(N_f)}{=} -\langle u^\dagger \partial_\alpha u \rangle}) + \underbrace{\langle r_\alpha \rangle}_{\overset{(2.6)}{=}0} - \underbrace{\langle l_\alpha \rangle}_{\overset{(2.6)}{=}0} \\
&= 2i\langle u^\dagger \partial_\alpha u \rangle \\
&\overset{(2.60)}{=} -\frac{1}{F_0}\sum_{k,l=0}^{\infty} \frac{i^{(l-k)}}{(2F_0)^{k+l}k!l!}\langle \phi^k \partial_\alpha \phi \phi^l \rangle \\
&\overset{m=k+l}{=} -\frac{1}{F_0}\sum_{m=0}^{\infty} \frac{i^{(m)}}{(2F_0)^m m!} \underbrace{\sum_{k=0}^{m} i^{2k}\frac{m!}{k!(m-k)!}}_{=\sum_{k=0}^{m}(-1)^k\binom{m}{k}=\delta_{m0}} \langle \phi^{(m)}\partial_\alpha \phi \rangle \\
&= -\frac{1}{F_0}\langle \partial_\alpha \phi \rangle \overset{(2.42)}{=} 0.
\end{aligned}$$

Thus, the structure (2) of table 3.9 vanishes by itself. Similar structures are not considered at higher orders. Two invariant structures are left, allowing two different LECs accompanying these. The first is absorbed by an overall multiplicative prefactor, which ensures that the lowest order in the field expansion of (1) of table 3.9 matches the kinetic term of the Klein-Gordon equation. The prefactor is quadratic in the decay constant[2].

$$m_1^{(2)} \equiv m_1 = \frac{F_0^2}{4}. \tag{3.9}$$

[2] The decay constant in this case is the decay constant in the chiral limit F_0. It must be of order $\mathcal{O}(\sqrt{N_c})$. At higher chiral orders, it obtains different corrections for the different fields (and it is necessary to differentiate between physical decay constants: F_π, F_K, F_8 and F_1.). If the physical values are inserted in a calculation, the error is always of higher chiral order, therefore the calculation is correct up to the given chiral order. Proper large-N_c-counting is not that simple.

The second LEC is the aforementioned (Eq. (2.8)) constant B_0. This LEC has been absorbed into the external field χ, which can be used to produce a mass term at leading order. Therefore, the leading-order Lagrangian has the shape

$$\mathcal{L}_2 = \frac{F_0^2}{4}(\langle u_\alpha u^\alpha \rangle + \langle \chi_+ \rangle). \tag{3.10}$$

In this thesis, LECs at chiral order $2n$ are denoted by $m_i^{(2n)}$ in the general case. If it is obvious, to which chiral order the LECs belong, the corresponding label is not mentioned.

3.2.2 The nature of B_0

Looking back at the relevant part of the QCD Lagrangian in the $SU(3)$-case,

$$\mathcal{L}^{s,p} = -\overline{q}_R(s + ip)q_L - \overline{q}_L(s - ip)q_R, \tag{3.11}$$

and choosing

$$s = M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \quad p = 0, \tag{3.12}$$

the effect of the usual quark masses in the QCD Lagrangian can be reproduced. Taking the vacuum expectation value in the chiral limit of the derivative of $-\mathcal{L}^{s,p}$ with respect to a quark mass (here: m_u) yields

$$\langle -\frac{\partial \mathcal{L}^{s,p}}{\partial m_u} \rangle_0 = \langle \overline{u}_R u_L + \overline{u}_L u_R \rangle_0 = \langle \overline{u} u \rangle_0 = \frac{1}{3}\langle \overline{q} q \rangle_0. \tag{3.13}$$

The Hellmann-Feynman theorem is applicable here. It states a relation for a hermitian operator $\hat{A}(\lambda)$, its expectation value $A_\alpha(\lambda)$ and the state $|\alpha(\lambda)\rangle$, where λ is an arbitrary parameter[3]:

$$\frac{\partial A(\lambda)}{\partial \lambda} = \langle \alpha(\lambda) | \frac{\partial \hat{A}(\lambda)}{\partial \lambda} | \alpha(\lambda) \rangle. \tag{3.14}$$

The QCD Hamiltonian (or the fraction of the QCD Lagrangian discussed above) is such a quantity. The chosen state is the QCD vacuum in the chiral limit. The parameter that is investigated is a quark mass. After taking the derivative, the quark masses are set to zero, to ensure the chiral limit for the vacuum state (in the Hamiltonian no quark mass dependence is left).

Since observable quantities derived in the effective field theory must not deviate from the same quantities derived in the fundamental theory, the

[3]Smooth λ dependence is a natural requirement.

same procedure can be applied to the effective Hamiltonian. The replacement by the mass matrix of Eq. (3.12) is done in the relevant part of $\mathcal{L}_2$,

$$\frac{F_0^2}{4}\langle\chi_+\rangle \overset{s=M,p=0}{\mapsto} \frac{F_0^2 B_0}{2}\langle M(u^2+u^{\dagger\,2})\rangle, \tag{3.15}$$

evaluated at zeroth order in the fields[4] (higher orders vanish, when taking the vacuum expectation value),

$$F_0^2 B_0\langle M\rangle. \tag{3.16}$$

The derivative with respect to the u-quark mass yields after evaluation of the remaining trivial trace a constant

$$\langle -\frac{\partial \mathcal{L}_2}{\partial m_u}|^{\phi^0}_{s=M,p=0}\rangle_0 = -F_0^2 B_0. \tag{3.17}$$

Both expectation values must match:

$$-F_0^2 B_0 \overset{!}{=} \frac{1}{3}\langle\overline{q}q\rangle_0. \tag{3.18}$$

This fixes the numerical value of B_0 in terms of the decay constant F_0 and the scalar quark condensate in the chiral limit $\langle\overline{q}q\rangle_0$.

3.2.3 Leading-order equations of motion

The leading-order equations of motion (abbreviated as *eom*) are derived by a variation of the field ϕ_a. Since the field $\phi \in M_2$ appears in the Lagrangian only in terms of the Goldstone boson matrix $u(x)$, this is the quantity to be subjected to variation

$$u(x) \mapsto u'(x) = u(x) + \delta u(x) = [1 + i\varepsilon v(x)]u(x), \tag{3.19}$$

where

$$v(x \mapsto \infty) = 0 \tag{3.20}$$

and

$$[v(x), u(x)] = 0 \quad \Rightarrow u(x)v(x)u^\dagger(x) = u^\dagger(x)v(x)u(x) = v(x). \tag{3.21}$$

In the ensuing calculation the x dependencies are suppressed.

$$\frac{\partial}{\partial\varepsilon}u_\alpha(u') = -\Big(u^\dagger\partial_\alpha v u + u\partial_\alpha v u^\dagger\Big) + i\Big(u^\dagger[r_\alpha, v]u + u[l_\alpha, v]u^\dagger\Big) \tag{3.22}$$

[4]This naive procedure requires a vacuum expectation value of $U_0 = 1$ (or $u_{\mathrm{VEV}} = \pm 1$). Inclusion of the singlet η and the anomaly necessitates another vacuum state.

Eq. (3.22) is inserted into the kinetic term

$$
\begin{aligned}
k &\equiv \frac{\partial}{\partial\varepsilon}\langle u_\alpha u^\alpha\rangle|_{\varepsilon=0} = 2\langle\frac{\partial}{\partial\varepsilon}u_\alpha(u')u^\alpha\rangle \\
&= 2\langle -\Big(u^\dagger\partial_\alpha v u + u\partial_\alpha v u^\dagger\Big)u^\alpha + i\Big(u^\dagger[r_\alpha,v]u + u[l_\alpha,v]u^\dagger\Big)u^\alpha\rangle. \quad (3.23)
\end{aligned}
$$

The partial derivative is removed from $v(x)$ via partial integration in Eq. (3.23), where the boundary terms vanish due to Eq. (3.20):

$$
\begin{aligned}
k &= 2\langle v\partial_\alpha\Big(uu^\alpha u^\dagger + u^\dagger u^\alpha u\Big) + iv\Big([uu^\alpha u^\dagger, r_\alpha] + [u^\dagger u^\alpha u, l_\alpha]\Big)\rangle \\
&= 2\langle v\ u\Big(\partial_\alpha u^\alpha - i[u^\dagger r_\alpha u, u^\alpha] + [u^\dagger\partial_\alpha u, u^\alpha]\Big)u^\dagger\rangle \\
&+ 2\langle v\ u^\dagger\Big(\partial_\alpha u^\alpha - i[ul_\alpha u^\dagger, u^\alpha] + [u\partial_\alpha u^\dagger, u^\alpha]\Big)u\rangle, \\
&\overset{(3.21)}{=} \langle 4v\partial_\alpha u^\alpha\rangle \\
&\quad +\langle v[u^\dagger\partial_\alpha u + u\partial_\alpha u^\dagger, u^\alpha] - iv[u^\dagger r_\alpha u + ul_\alpha u^\dagger, u^\alpha]\rangle \qquad (3.24)
\end{aligned}
$$

Afterwards, Eqs. (2.68) and (2.70) are applied:

$$
\begin{aligned}
k &= 2\langle 2v\Big(\partial_\alpha u^\alpha + [\Gamma_\alpha, u^\alpha]\Big)\rangle \\
&= 2\langle 2v\nabla_\alpha u^\alpha\rangle = 2\langle v h^\alpha_\alpha\rangle. \qquad (3.25)
\end{aligned}
$$

The external field term yields after a short calculation

$$
\begin{aligned}
\frac{\partial}{\partial\varepsilon}\langle\chi_+\rangle &= \langle\chi\frac{\partial}{\partial\varepsilon}(u^\dagger)^2 + \chi^\dagger\frac{\partial}{\partial\varepsilon}u^2\rangle \overset{(3.21)}{=} -2i\langle\chi u^\dagger v u^\dagger - \chi^\dagger u v u\rangle \\
&= -2i\langle v(u^\dagger\chi u^\dagger - u\chi^\dagger u)\rangle \;=\; -2i\langle v\chi_-\rangle. \qquad (3.26)
\end{aligned}
$$

Thus, since $v(x) \in \widetilde{\mathcal{H}}_{N_f}$, the variation of the Lagrangian produces the leading-order equations of motion

$$
\frac{1}{2}\frac{\delta}{\delta\varepsilon v_a(x)}\mathcal{L}_2|_{\varepsilon=0} = \langle\lambda_a(h^\alpha_\alpha - i\chi_-)\rangle \overset{!}{=} 0. \qquad (3.27)
$$

These are eight independent equations, which can be transferred into matrix form

$$
\mathcal{O}_2^{\text{eom}} = h^\alpha_\alpha - i(\chi_- - \frac{1}{N_f}\langle\chi_-\rangle) = 0. \qquad (3.28)
$$

Here the trace of χ_- has to be included, since h^α_α is traceless, but χ_- is not (for general χ). This equation is consistent in large-N_c-counting, due to the fact that the trace is only a trace in flavour space[5].

In the case of nonet symmetry, where the singlet η is included in the theory, this artificial construction is no more necessary, but an additional mass term for the singlet field arises. This topic is discussed in more detail in chapter 4.

[5]In the leading-order-equation-of-motion-induced relations, $\langle\chi_-\rangle$ complicates large-N_c-counting (section 2.5).

3.3 Relations among monomials

Five different species of relations among monomials exist. Each different species links different groups of monomials, though not all relations are independent. Even worse, the exact shape of the relations is N_f dependent. Even though $SU(2)$ is a subgroup of $SU(3)$, it is better to construct all relations for general N_f, if possible, and derive the entirely N_f-specific trace relations separately, since the absolute number of $SU(2)$ trace relations is much larger than the fraction of these, which could be transferred from $SU(3)$.

3.3.1 Partial-integration-induced relations

The equations of motion derived by variation of the action do not change if a constant is added to the action. If a total derivative is added to the Lagrangian, a constant is added to the action.

The covariant derivative is subject to the Leibniz rule. When the covariant derivative of the entire argument of a flavour trace is taken, it is identical to its partial derivative (Eq. (2.68)).

- If more than half of the covariant derivatives operate on the same structure, their number can always be reduced to a maximum of half of the covariant derivatives operating on the same structure via partial integration. Therefore, these structures can always be eliminated and need not be considered at all in the construction of the most general Lagrangian, even though they may be necessary as intermediate steps in the derivation of relations or they may be more useful for calculations of processes.
- If covariant derivatives meet chiral vielbeins, they are subjected to the symmetrisation procedure (Eq. (2.70)).
- If covariant derivatives are commuted, chiral field strength tensors arise (Eq. (2.72)).
- If covariant derivatives meet symmetrized covariant derivatives of the Goldstone boson matrix, different orderings produce different structures, so that many different commutation relations have to be evaluated.
- When covariant derivatives meet chiral field strength tensors, Bianchi identities may be applicable.

3.3.2 Bianchi identities

The first Bianchi identity

$$dF = d \circ dA = 0 \tag{3.29}$$

can be applied in some cases. In the geometric picture of the u basis, the connection A is represented by the components of the chiral connection Γ_α, while the chiral field strength tensor $\Gamma_{\alpha\beta}$ corresponds to F:

$$\sum_{\{\alpha,\beta,\gamma\}} \nabla_\gamma \Gamma_{\alpha\beta} = 0, \tag{3.30}$$

where the sum runs over all permutations of the set $\{\alpha, \beta, \gamma\}$[6].

On the other hand, because the field strength tensors $f_\pm^{\alpha\beta}$ appear in the Lagrangian, the Bianchi identity can be formulated in terms of these:

$$\sum_{\{\alpha,\beta,\gamma\}} \nabla_\gamma f_{\pm\,\alpha\beta} = \sum_{\{\alpha,\beta,\gamma\}} -\frac{i}{2}[u_\gamma, f_{\mp\,\alpha\beta}]. \tag{3.31}$$

The equivalence of the Bianchi identities in terms of $\Gamma_{\alpha\beta}$ and $f_{+\,\alpha\beta}$ can be deduced from Eq. (2.73). On the other hand, any Bianchi identity for $f_{-\,\alpha\beta}$ can be expressed in terms of multiple covariant derivatives and their commutators operating on Goldstone boson matrices via Eq. (2.70). Therefore, these are not independent from the complete set of partial-integration-induced relations.

3.3.3 Leading-order-equation-of-motion-induced relations

Concerning this topic two different interpretations shall be considered.

The first is an interpretation in terms of a constraint implemented via a Lagrangian multiplier. A leading-order-equation-of-motion-induced relation (abbreviated as eom relation) is given by a flavour trace of an arbitrary tensor structure A of chiral order $D-2$ multiplied by the equation of motion and vanishes at chiral order D (due to Eq. (3.28)), but gives further contributions at chiral order $D' \geq D+2$. The Lagrangian multiplier can be chosen to exactly cancel one LEC, while modifying up to two others, which must be redefined[7]. Effectively, one degree of freedom is eliminated. This aspect is clarified in the discussion of $\mathcal{L}_4$.

$$\langle A \times \mathcal{O}_2^{\text{eom}} \rangle = 0 + (\text{ structures of chiral order} D' \geq D+2) \tag{3.32}$$

The first interpretation has a drawback. In an effective Lagrangian at multiple chiral orders, where the LECs are fixed, the transformation of the LECs of higher orders are not correct, if eom relations are applied as constraints in accord with the first interpretation.

[6]When the rest of the tensor structure is symmetric in two of the indices, it is impossible to derive a Bianchi identity. This is demonstrated in Eq. (3.52).

[7]The usual choice is elimination of the monomial $\langle A h_\alpha^\alpha \rangle$. The LECs of both monomials $\langle A\chi_- \rangle$ and $\langle A \rangle \langle \chi_- \rangle$ are modified. This redefinition of the LECs forces a new contribution to the LEC of $\langle A \rangle \langle \chi_- \rangle$, which is of the same order in large-N_c counting as the LEC of $\langle A\chi_- \rangle$ (see section 3.4.3).

The second interpretation [SF 95] is advantageous in this case. An eom relation is obtained via a field redefinition in the lower orders by a unitary transformation of the Goldstone boson matrix with a hermitian matrix $A[u(x)]$ of chiral order $D-2$:

$$u(x) \mapsto u(x) = \exp\,(i\frac{\kappa}{2}A[u'(x)])u'(x), \tag{3.33}$$

The hermitian matrix must satisfy the condition (compare Eq. (3.21)):

$$\Big[A[u'(x)], u'(x)\Big] = 0. \tag{3.34}$$

When considering the leading order[8] in κ, the results of section 3.2.3 are reproduced, except that $\varepsilon v(x)$ is replaced by $\kappa A[u'(x)]$, and the Lagrangian is expressed in terms of a transformed Goldstone boson matrix $u'(x)$.

In a similar way as in the first interpretation, complications arise at higher chiral orders. As an example, when $D-2=2$, at chiral order six the additional contributions to the equation of motion at chiral order four and the structures of order $(\kappa\mathcal{O}_2^{\text{eom}})^2$ for the expansion of the exponential, both contribute to corrections. It is exactly this contribution, which is lost, if the eom relations are treated as simple constraints.

Ultimately the matrix $\kappa A[u'(x)]$ must be a linear combination of multiple chiral orders, where the coefficients κ of higher chiral orders in $A[u'(x)]$ must absorb the higher-order (this means higher order in the coefficients) contributions of the lower-order structures. Thus, the coefficients of the field redefinitions do not coincide with the corresponding Lagrangian multipliers of the first interpretation, if higher-order corrections are necessary.

In the case when the most general Lagrangian is constructed, no new structures appear. If, on the other hand, a given multi-order chiral Lagrangian is transformed, new structures arise, a fact which ultimately necessitates redefinitions of many LECs.

3.3.4 Trace relations

Trace relations are derived from the Cayley-Hamilton formulae (Eq. (2.88)). The trace is taken of these and the determinant is replaced by repeated application of Eq. (2.88). The result is

$$\begin{aligned} SU(2): \quad 0 = &\quad A^2 - \langle A\rangle A - \tfrac{1}{2}(\langle A^2\rangle - \langle A\rangle^2), \\ SU(3): \quad 0 = &\quad A^3 - \langle A\rangle A^2 + \tfrac{1}{2}(\langle A\rangle^2 - \langle A^2\rangle)A \\ &\quad -\tfrac{1}{3}\langle A^3\rangle + \tfrac{1}{2}\langle A^2\rangle\langle A\rangle - \tfrac{1}{6}\langle A\rangle^3. \end{aligned} \tag{3.35}$$

[8] Leading order in κ is also leading chiral order.

These formulae are (in the approach of [FS 96]) multiplied by the matrix A^X and the trace of the product is taken. The power X satisfies $X \leq N_f$. The matrix A is expressed as

$$A = \sum_{i=1}^{X+N_f} \kappa_i A_i \tag{3.36}$$

and sorted in terms of the products of the κ_i. Relations involving $X + N_f$ different matrices are the result of this procedure.

For $SU(2)$ this procedure yields

$$\begin{aligned} 0 &= \sum_{2perm.} \langle A_1 A_2 A_3 \rangle - \sum_{3perm.} \langle A_1 \rangle \langle A_2 A_3 \rangle - \langle A_1 \rangle \langle A_2 \rangle \langle A_3 \rangle, \quad (3.37) \\ 0 &= \sum_{6perm.} \langle A_1 A_2 A_3 A_4 \rangle - \frac{3}{4} \sum_{8perm.} \langle A_1 \rangle \langle A_2 A_3 A_4 \rangle \\ &- \sum_{3perm.} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle + \frac{1}{4} \sum_{12perm.} \langle A_1 \rangle \langle A_2 \rangle \langle A_3 A_4 \rangle, \end{aligned}$$

while it generates for $SU(3)$

$$\begin{aligned} 0 &= \sum_{6perm.} \langle A_1 A_2 A_3 A_4 \rangle - \sum_{8perm.} \langle A_1 \rangle \langle A_2 A_3 A_4 \rangle \\ &- \sum_{3perm.} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle + \sum_{6perm.} \langle A_1 \rangle \langle A_2 \rangle \langle A_3 A_4 \rangle \\ &- \langle A_1 \rangle \langle A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle, \quad (3.38) \\ 0 &= \sum_{24perm.} \langle A_1 A_2 A_3 A_4 A_5 \rangle - \sum_{20perm.} \langle A_1 A_2 \rangle \langle A_3 A_4 A_5 \rangle \\ &- \sum_{20perm.} \langle A_1 \rangle \langle A_2 \rangle \langle A_3 A_4 A_5 \rangle \\ &+ 2 \sum_{10perm.} \langle A_1 \rangle \langle A_2 \rangle \langle A_3 \rangle \langle A_4 A_5 \rangle \\ &- 4 \langle A_1 \rangle \langle A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle \langle A_5 \rangle, \\ 0 &= \sum_{120perm.} \langle A_1 A_2 A_3 A_4 A_5 A_6 \rangle - \frac{2}{3} \sum_{90perm.} \langle A_1 A_2 \rangle \langle A_3 A_4 A_5 A_6 \rangle \\ &- \sum_{40perm.} \langle A_1 A_2 A_3 \rangle \langle A_4 A_5 A_6 \rangle. \end{aligned}$$

Important special cases include relations, where some matrices are traceless or where some matrices are antisymmetric in a certain way. The relation for six matrices is valid only for traceless matrices. This is due to the fact that six independent matrices must be traceless vielbeins in any case. The trace relations are totally symmetric under interchange of any of the matrices, so antisymmetry can only be included, if some of the matrices are antisymmetric by themselves[9].

A further possibility is establishing a trace relation only in certain parts of monomials, which are then multiplied by an additional trace[10]. In the even sector of the next-to-next-to-leading-order Lagrangian twenty trace relations of this kind were constructed, but none was independent of the rest of the set of relations. They do not appear in the lists.

No independent additional relations for $X > 1$ were found. Whether there cannot be any independent relations for $X > 1$ in general, is not yet proved. Only a small number of $X > 1$ trace relations for $SU(3)$ were generated. These are not included in the presented list of relations.

3.3.5 ϵ relations

ϵ relations are an intrinsic property of Minkowski space-time and the associated ϵ tensor. They appear only in the anomalous sector, because every monomial is contracted with an ϵ tensor. That is why the anomalous sector is also called ϵ sector. The product of a metric tensor times an ϵ tensor satisfies the following relation:

$$0 = g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta} - g_{\mu\alpha}\epsilon_{\nu\beta\gamma\delta} - g_{\mu\beta}\epsilon_{\alpha\nu\gamma\delta} - g_{\mu\gamma}\epsilon_{\alpha\beta\nu\delta} - g_{\mu\delta}\epsilon_{\alpha\beta\gamma\nu}. \tag{3.39}$$

This has to be contracted with a chirally invariant six tensor. Then in all but the first part indices are renamed and permuted, until every term has the same tensor structure $(g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta})$ like the first. Since all monomials constructed for the most general Lagrangian are made to have the same tensor structure, the corresponding relations can be extracted. Any further symmetry or antisymmetry of the chirally invariant six tensor has great effect on the shape of the ϵ relation.

3.4 Next-to-leading-order Lagrangian $\mathcal{L}_4$

3.4.1 Monomials

At chiral order four, there are 17 non-trivial monomials, which are a priori independent (see table 3.10). The third column denotes the potential leading contribution in a field expansion, where the external fields model electromagnetic processes[11]. The fourth column denotes the number of light flavours, for which the corresponding monomial is still linearly independent.

---

[9] It seems that the key to discovering new trace relations was introducing antisymmetry by taking commutators of basic building blocks as some of the A_j. In the instance, where a new relation could be clearly identified, it was constructed in this manner.

[10] Example: Relation 5 from section 3.4 (Eq. (3.44)) multiplied by $\langle u_\gamma u^\gamma \rangle$ yields a relation at chiral order six: $4 \times (6) + 2 \times (7) - (13) - 2 \times (14) = 0$.

[11] Symmetry properties of the monomials can cancel these leading contributions. The monomial (11) demonstrates this effect (see Eq. (3.41)).

Table 3.10: Monomials at chiral order four

monomial	shape	contributes to	# of flavours
(1)	$\langle u_\alpha u^\alpha u_\beta u^\beta \rangle$	4ϕ	3
(2)	$\langle u_\alpha u_\beta u^\alpha u^\beta \rangle$	4ϕ	N_f
(3)	$\langle u_\alpha u^\alpha \rangle \langle u_\beta u^\beta \rangle$	4ϕ	2
(4)	$\langle u_\alpha u_\beta \rangle \langle u^\alpha u^\beta \rangle$	4ϕ	2
(5)	$\langle h^\alpha_\alpha h^\beta_\beta \rangle$	2ϕ	
(6)	$\langle h_{\alpha\beta} h^{\alpha\beta} \rangle$	2ϕ	
(7)	$\langle u_\alpha u^\alpha \chi_+ \rangle$	2ϕ	3
(8)	$\langle u_\alpha u^\alpha \rangle \langle \chi_+ \rangle$	2ϕ	2
(9)	$i\langle u^\alpha \nabla_\alpha \chi_- \rangle$	2ϕ	
(10)	$i\langle u_\alpha u_\beta f_+^{\alpha\beta} \rangle$	$2\phi + \gamma$	2
(11)	$\langle u_\alpha \nabla_\beta f_-^{\alpha\beta} \rangle$	$2\phi + \gamma$	
(12)	$\langle \chi_+ \chi_+ \rangle$	contact term	2
(13)	$\langle \chi_+ \rangle \langle \chi_+ \rangle$	contact term	2
(14)	$\langle \chi_- \chi_- \rangle$	2ϕ	2
(15)	$\langle \chi_- \rangle \langle \chi_- \rangle$	2ϕ	2
(16)	$\langle f_{+\,\alpha\beta} f_+^{\alpha\beta} \rangle$	contact term	2
(17)	$\langle f_{-\,\alpha\beta} f_-^{\alpha\beta} \rangle$	$2\phi + 2\gamma$	2

3.4.2 Explicit construction of relations at chiral order four

At chiral order four, the explicit construction of relations is still sufficiently simple to be presented here, and complicated enough, to show some of the techniques and difficulties arising at higher chiral orders. Therefore, they are constructed here.

Partial-integration-induced relations

$$
\begin{aligned}
\langle h^\alpha_\alpha h^\beta_\beta \rangle &= 2\langle \nabla_\alpha u^\alpha h^\beta_\beta \rangle = -2\langle u^\alpha \nabla_\alpha h^\beta_\beta \rangle \\
&= -4\langle u^\alpha \underbrace{\nabla_\alpha \nabla_\beta}_{=\nabla_\beta\nabla_\alpha + [\nabla_\alpha,\nabla_\beta]} u^\beta \rangle \\
&= -4\langle u^\alpha \nabla_\beta \underbrace{\nabla_\alpha u^\beta}_{=\frac{1}{2}(h^\beta_\alpha - f_{-\,\alpha}{}^\beta)} \rangle - 4\langle u^\alpha \underbrace{[\nabla_\alpha, \nabla_\beta] u^\beta}_{=[\Gamma_{\alpha\beta}, u^\beta]} \rangle \\
&= 2\langle \underbrace{\nabla_\beta u^\alpha}_{=\frac{1}{2}(h^\alpha_\beta - f_{-\,\beta}{}^\alpha)} h^\beta_\alpha \rangle + 2\langle u^\alpha \nabla_\beta f_{-\,\alpha}{}^\beta \rangle \underbrace{-4\langle [u^\beta, u^\alpha] \Gamma_{\alpha\beta} \rangle}_{=8\langle u_\alpha u_\beta \Gamma^{\alpha\beta} \rangle},
\end{aligned}
$$

The product ${f_-}_\beta{}^\alpha h_\alpha^\beta$ vanishes, because ${f_-}_\beta{}^\alpha$ is antiysmmetric and h_α^β is symmetric under interchange of Lorentz indices.

$$\begin{aligned}
\langle h_\alpha^\alpha h_\beta^\beta \rangle &= \langle h_\alpha^\beta h_\beta^\alpha \rangle + 2\langle u_\alpha \nabla_\beta f_-^{\alpha\beta} \rangle + 8\langle u_\alpha u_\beta \underbrace{\Gamma^{\alpha\beta}}_{=\frac{1}{4}[u^\alpha,u^\beta]-\frac{i}{2}f_+^{\alpha\beta}} \rangle \\
&= \langle h_\alpha^\beta h_\beta^\alpha \rangle + 2\langle u_\alpha \nabla_\beta f_-^{\alpha\beta} \rangle + 2\langle u_\alpha u_\beta [u^\alpha, u^\beta] \rangle - 4i\langle u_\alpha u_\beta f_+^{\alpha\beta} \rangle \\
&= 2\Big(\langle u_\alpha u_\beta u^\alpha u^\beta \rangle - \langle u_\alpha u^\alpha u_\beta u^\beta \rangle \Big) + \langle h_\alpha^\beta h_\beta^\alpha \rangle \\
&\quad -4i\langle u_\alpha u_\beta f_+^{\alpha\beta} \rangle + 2\langle u_\alpha \nabla_\beta f_-^{\alpha\beta} \rangle .
\end{aligned}$$

Thus:

$$(1) - (2) + \frac{1}{2}\Big((5) - (6)\Big) + 2 \times (10) - (11) = 0. \tag{3.40}$$

$$\langle u_\alpha \nabla_\beta f_-^{\alpha\beta} \rangle = -\langle \underbrace{\nabla_\beta u_\alpha}_{\frac{1}{2}(h_{\beta\alpha}-f_{-\,\beta\alpha})} f_-^{\alpha\beta} \rangle = \frac{1}{2}\langle f_{-\,\beta\alpha} f_-^{\alpha\beta} \rangle = -\frac{1}{2}\langle f_{-\,\alpha\beta} f_-^{\alpha\beta} \rangle .$$

Therefore:

$$2 \times (11) + (17) = 0. \tag{3.41}$$

Considering the third column of table 3.10, Eq. (3.41) seems to contradict the table. This is an example, where the naive leading order contribution is forbidden by the symmetry.

Leading-order-equation-of-motion-induced relations

$$\langle h_\alpha^\alpha h_\beta^\beta \rangle = i\langle h_\alpha^\alpha \chi_- \rangle - \frac{i}{N_f} \underbrace{\langle h_\alpha^\alpha \rangle}_{=0} \langle \chi_- \rangle = 2i\langle \nabla^\alpha u_\alpha \chi_- \rangle = -2i\langle u_\alpha \nabla^\alpha \chi_- \rangle .$$

So:

$$(5) + 2 \times (9) = 0. \tag{3.42}$$

$$i\langle u_\alpha \nabla^\alpha \chi_- \rangle = -i\langle \nabla^\alpha u_\alpha \chi_- \rangle = -\frac{i}{2}\langle h_\alpha^\alpha \chi_- \rangle = \frac{1}{2}\langle \chi_- \chi_- \rangle - \frac{1}{2N_f}\langle \chi_- \rangle \langle \chi_- \rangle .$$

Consequently:

$$2 \times (9) - (14) + \frac{1}{N_f} \times (15) = 0. \tag{3.43}$$

Here the monomial $\langle h_\alpha^\alpha \chi_- \rangle$ appears in an intermediate step, even though it belongs to a category, which does not require consideration, since more than half of all covariant derivatves operate on the same structure, namely the Goldstone boson matrix. Furthermore, the derivation of Eq. (3.42) demonstrates why some eom relations link only two structures: this occurs, when $A[u]$ is traceless.

Trace relations

One $SU(3)$ trace relation

$$4 \times (1) + 2 \times (2) - (3) - 2 \times (4) = 0, \tag{3.44}$$

and three $SU(2)$ trace relations

$$(1) + (2) - (4) = 0, \tag{3.45}$$

$$2 \times (1) - (3) = 0, \tag{3.46}$$

$$2 \times (7) - (8) = 0 \tag{3.47}$$

can be generated. The $SU(3)$ trace relation expressed in terms of the $SU(2)$ trace relations is (3.44) $= 2 \times$ (3.45) + (3.46). It is visible here that $SU(2)$ trace relations are in general much shorter than $SU(3)$ trace relations, and the effort of comparison of given $SU(3)$ trace relations with already derived $SU(2)$ trace relations is by no means smaller than the effort of deriving the corresponding $SU(2)$ trace relations from Eq. (3.37).

Complete set of relations

Table 3.11: Relations at chiral order four

relation	Eq.	shape	eliminated monomial
1	(3.40)	$(1) - (2) + \frac{1}{2} \times ((5) - (6)) + 2 \times (10) - (11) = 0$	(6)
2	(3.41)	$2 \times (11) + (17) = 0$	(11)
3	(3.42)	$(5) + 2 \times (9) = 0$	(5)
4	(3.43)	$2 \times (9) - (14) + \frac{1}{N_f} \times (15) = 0$	(9)
		this section is $SU(3)$ specific	
5	(3.44)	$2 \times (1) + (2) - \frac{1}{2} \times (3) - (4) = 0$	(2)
		this section is $SU(2)$ specific	
6	(3.45)	$(1) + (2) - (4) = 0$	(2)
7	(3.46)	$2 \times (1) - (3) = 0$	(1)
8	(3.47)	$2 \times (7) - (8) = 0$	(7)

This list visualizes another aspect, which most authors do not mention (e.g.: [GL 85, FS 96, BCE 00]). If relations are applied as they are derived, it could happen that relation *1* would be used to eliminate monomial (5) and relation *4* would be used to eliminate monomial (9). This would mean that both monomials appearing in relation *3* already would have been eliminated. Thus, the relation would not have been found and the maximal set of independent relations would not have been constructed.

Most authors (e.g.: see above) use leading order eom induced relations to always eliminate the monomial, where h_α^α appears. This approach probably is not ideal, since at chiral order six some relations between monomials can be constructed, all of which would have been eliminated due to that elimination paradigm. Even though in this thesis no additional *independent* relations due to this effect have been found, this phenomenon is worth further investigation.

3.4.3 Connection to $\mathcal{L}_4$ of Gasser and Leutwyler

As the usual form of $\mathcal{L}_4$ derived by Gasser and Leutwyler in [GL 85] is in the L-R basis, table 2.4 can be used to reformulate the derived Lagrangian and achieve a matching of the LECs. Two contact terms have to be constructed.

$$\begin{aligned}
\langle f_{\pm\,\alpha\beta} f_\pm^{\alpha\beta}\rangle &= \langle (uF_{L\,\alpha\beta}u^\dagger \pm u^\dagger F_{R\,\alpha\beta}u)(uF_L^{\alpha\beta}u^\dagger \pm u^\dagger F_R^{\alpha\beta}u)\rangle \\
&= \langle F_{R\,\alpha\beta}F_R^{\alpha\beta} + F_{L\,\alpha\beta}F_L^{\alpha\beta}\rangle \pm 2\langle F_{R\,\alpha\beta}u^2F_L^{\alpha\beta}(u^\dagger)^2\rangle \\
\langle \chi_\pm\chi_\pm\rangle &= \langle (u^\dagger\chi u^\dagger \pm u\chi u)(u^\dagger\chi u^\dagger \pm u\chi u)\rangle \\
&= \pm 2\langle \chi\chi^\dagger\rangle + \langle \chi(u^\dagger)^2\chi(u^\dagger)^2 + \chi^\dagger u^2\chi^\dagger u^2\rangle.
\end{aligned} \tag{3.48}$$

Thus, the contact terms are

$$\langle F_{R\,\alpha\beta}F_R^{\alpha\beta} + F_{L\,\alpha\beta}F_L^{\alpha\beta}\rangle = \frac{1}{2}(\langle f_{+\,\alpha\beta}f_+^{\alpha\beta}\rangle + \langle f_{-\,\alpha\beta}f_-^{\alpha\beta}\rangle), \tag{3.49}$$

$$\langle \chi\chi^\dagger\rangle = \frac{1}{4}(\langle \chi_+\chi_+\rangle - \langle \chi_-\chi_-\rangle). \tag{3.50}$$

The Lagrangian of Gasser and Leutwyler (three-flavour case) can be transferred into the u basis with these tools:

$$\begin{aligned}
\mathcal{L}_4 = {} & L_1\langle u_\alpha u^\alpha\rangle\langle u_\beta u^\beta\rangle && +L_2\langle u_\alpha u_\beta\rangle\langle u^\alpha u^\beta\rangle \\
& +L_3\langle u_\alpha u^\alpha u_\beta u^\beta\rangle && +L_4\langle u_\alpha u^\alpha\rangle\langle \chi_+\rangle \\
& +L_5\langle u_\alpha u^\alpha \chi_+\rangle && +L_6\langle \chi_+\rangle\langle \chi_+\rangle \\
& +L_7\langle \chi_-\rangle\langle \chi_-\rangle && +L_8\,\tfrac{1}{2}\Big(\langle \chi_+\chi_+\rangle + \langle \chi_-\chi_-\rangle\Big) \\
& -L_9\, i\langle u_\alpha u_\beta f_+^{\alpha\beta}\rangle && +L_{10}\,\tfrac{1}{4}\Big(\langle f_{+\,\alpha\beta}f_+^{\alpha\beta}\rangle + \langle f_{-\,\alpha\beta}f_-^{\alpha\beta}\rangle\Big) \\
& +H_1\,\tfrac{1}{2}\Big(\langle f_{+\,\alpha\beta}f_+^{\alpha\beta}\rangle + \langle f_{-\,\alpha\beta}f_-^{\alpha\beta}\rangle\Big) \\
& +H_2\,\tfrac{1}{4}\Big(\langle \chi_+\chi_+\rangle - \langle \chi_-\chi_-\rangle\Big).
\end{aligned} \tag{3.51}$$

Due to the eom relation *4* of Eq. (3.43), the LEC $L_7 = (m_{15} - \frac{1}{2}\frac{1}{N_f}m_9)$ is enhanced in order from $\mathcal{O}(1)$ to $\mathcal{O}(N_c)$ for arbitrary N_f. Gasser and Leutwyler pointed out that the main contribution to L_7 originates in an exchange of an η' meson. The propagator of the η' breaks down to $M_{\eta'}^{-2}$ for momenta, which are small in comparison to the η' mass. Because the main contribution to the mass of the η' which is due to the anomaly is of order $\mathcal{O}(\frac{1}{N_c})$, this propagator enhances the original LEC m_{15} by a factor of order $\mathcal{O}(N_c^2)$. This enhancement is not relevant any more, if the momenta are large enough to prevent a breakdown of the η' propagator.

Furthermore, due to the trace relation *5* of Eq. (3.44), the LECs $L_1 = (m_3 + \frac{1}{2}m_2)$, $L_2 = (m_4 + m_2)$ and $L_3 = (m_1 - 2m_2)$ are of order $\mathcal{O}(N_c)$, even though L_1 and L_2 would be of order $\mathcal{O}(1)$, if no trace relations had been applied. Additionally, usually L_3[12] and $L_5 = m_7$ are eliminated (by use of relations *7*[13] and *8*) from the three-flavour Lagrangian in the two-flavour case[14]. The elimination of L_5 enhances the order of $L_4 = (m_8 + \frac{1}{2}m_7)$ from $\mathcal{O}(1)$ to $\mathcal{O}(N_c)$.

3.5 Next-to-next-to-leading-order Lagrangian $\mathcal{L}_6$ with even intrinsic parity

The extension to chiral order six creates a very large increase of complexity for the construction of the Lagrangian. The number of a priori independent monomials and the number of, after application of relations, independent monomials becomes approximately tenfold. It seems that there is no possibility of matching most of the corresponding LECs to values obtained from experiments. The construction process has been done before in [FS 96, BCE 00].

This section is organized in the following way:

1. the complete list of a priori independent monomials is presented,

2. the complete list of all derived relations is summarized for general N_f and for three- and two-flavour cases,

3. the list is reduced to a maximal set of independent relations,

4. the list of monomials is reduced to a list of independent monomials and two elimination paradigms are discussed,

5. the extension of the chiral group in terms of $U(1)_V$ is discussed by inclusion of the flavour singlet field strength tensor,

[12] If relation *5* had not been used to eliminate (3) or (4), the remaining LEC (either L_1 or L_2) would not have been enhanced. Later use of a second trace relation would either eliminate the remaining multiple trace LEC or enhance it by elimination of a single trace LEC. In each of these cases, the total number of leading order LECs in the two-flavour case is the same. This demonstrates, why determining the number of leading order LECs is non-trivial (see last footnote in section 2.5).

[13] Relation *6* is treated here as corresponding to relation *5*.

[14] This is a further modification of the LECs L_1 and L_2. Since it does not change the order anymore, details are not considered here.

6. a comparison with the results of [BCE 00] is performed.

Essential segments of the source code used for finding the maximal set and for comparing the Lagrangians are part of the appendix A.4.

3.5.1 Monomials in the even sector at chiral order six

At chiral order six 191 a priori independent monomials of even intrinsic parity exist. The third column of table 3.12 denotes the leading contribution in a field expansion, where the external fields model electromagnetic processes. The fourth column denotes the number of light flavours, for which the corresponding monomial is still linearly independent. The elimination scheme applied here prefers multiple traces. This is explained in section 3.5.4.

Table 3.12: Monomials at chiral order six in the even sector

monomial	shape	contributes to	# of flavours
(1)	$\langle u_\alpha u^\alpha u_\beta u^\beta u_\gamma u^\gamma \rangle$	6ϕ	
(2)	$\langle u_\alpha u^\alpha u_\beta u_\gamma u^\beta u^\gamma \rangle$	6ϕ	
(3)	$\langle u_\alpha u^\alpha u_\beta u_\gamma u^\gamma u^\beta \rangle$	6ϕ	
(4)	$\langle u_\alpha u_\beta u^\alpha u_\gamma u^\beta u^\gamma \rangle$	6ϕ	N_f
(5)	$\langle u_\alpha u_\beta u_\gamma u^\alpha u^\beta u^\gamma \rangle$	6ϕ	N_f
(6)	$\langle u_\alpha u^\alpha \rangle \langle u_\beta u^\beta u_\gamma u^\gamma \rangle$	6ϕ	
(7)	$\langle u_\alpha u^\alpha \rangle \langle u_\beta u_\gamma u^\beta u^\gamma \rangle$	6ϕ	N_f
(8)	$\langle u_\alpha u_\beta \rangle \langle u^\alpha u^\beta u_\gamma u^\gamma \rangle$	6ϕ	
(9)	$\langle u_\alpha u_\beta \rangle \langle u^\alpha u_\gamma u^\beta u^\gamma \rangle$	6ϕ	N_f
(10)	$\langle u_\alpha u^\alpha u_\beta \rangle \langle u^\beta u_\gamma u^\gamma \rangle$	6ϕ	3
(11)	$\langle u_\alpha u_\beta u_\gamma \rangle \langle u^\alpha u^\beta u^\gamma \rangle$	6ϕ	N_f
(12)	$\langle u_\alpha u_\beta u_\gamma \rangle \langle u^\alpha u^\gamma u^\beta \rangle$	6ϕ	N_f
(13)	$\langle u_\alpha u^\alpha \rangle \langle u_\beta u^\beta \rangle \langle u_\gamma u^\gamma \rangle$	6ϕ	3
(14)	$\langle u_\alpha u^\alpha \rangle \langle u_\beta u_\gamma \rangle \langle u^\beta u^\gamma \rangle$	6ϕ	3
(15)	$\langle u_\alpha u_\beta \rangle \langle u^\alpha u_\gamma \rangle \langle u^\beta u^\gamma \rangle$	6ϕ	2
(16)	$\langle \{u_\alpha u^\alpha, u_\beta\} \nabla^\beta h^\gamma_\gamma \rangle$	4ϕ	
(17)	$\langle u_\alpha u_\beta u^\alpha \nabla^\beta h^\gamma_\gamma \rangle$	4ϕ	
(18)	$\langle \{u_\alpha u^\alpha, u_\beta\} \nabla^\gamma h^\beta_\gamma \rangle$	4ϕ	
(19)	$\langle u_\alpha u_\beta u^\alpha \nabla^\gamma h^\beta_\gamma \rangle$	4ϕ	
(20)	$\langle \{u_\alpha, u_\beta u_\gamma\} \nabla^\alpha h^{\beta\gamma} \rangle$	4ϕ	
(21)	$\langle u_\alpha u_\beta u_\gamma \nabla^\beta h^{\alpha\gamma} \rangle$	4ϕ	
(22)	$\langle u_\alpha u^\alpha \rangle \langle u_\beta \nabla^\beta h^\gamma_\gamma \rangle$	4ϕ	
(23)	$\langle u_\alpha u_\beta \rangle \langle u^\alpha \nabla^\beta h^\gamma_\gamma \rangle$	4ϕ	
(24)	$\langle u_\alpha u^\alpha \rangle \langle u_\beta \nabla^\gamma h^\beta_\gamma \rangle$	4ϕ	
(25)	$\langle u_\alpha u_\beta \rangle \langle u^\alpha \nabla^\gamma h^\beta_\gamma \rangle$	4ϕ	
(26)	$\langle u_\alpha u_\beta \rangle \langle u_\gamma \nabla^\alpha h^{\beta\gamma} \rangle$	4ϕ	
(27)	$\langle u_\alpha u_\beta \rangle \langle u_\gamma \nabla^\gamma h^{\alpha\beta} \rangle$	4ϕ	

monomial	shape	contributes to	# of flavours
(28)	$\langle u_\alpha u^\alpha h^\beta_\beta h^\gamma_\gamma \rangle$	4ϕ	
(29)	$\langle \{u_\alpha u_\beta, h^{\alpha\beta}\} h^\gamma_\gamma \rangle$	4ϕ	
(30)	$\langle u_\alpha u^\alpha h_{\beta\gamma} h^{\beta\gamma} \rangle$	4ϕ	
(31)	$\langle u_\alpha u_\beta h^\alpha_\gamma h^{\beta\gamma} \rangle$	4ϕ	
(32)	$\langle u_\alpha u_\beta h^{\beta\gamma} h^\alpha_\gamma \rangle$	4ϕ	
(33)	$\langle u_\alpha h^\beta_\beta u^\alpha h^\gamma_\gamma \rangle$	4ϕ	
(34)	$\langle u_\alpha h^\alpha_\beta u^\beta h^\gamma_\gamma \rangle$	4ϕ	
(35)	$\langle \{u_\alpha, h^\alpha_\beta, u_\gamma\} h^{\beta\gamma} \rangle$	4ϕ	N_f
(36)	$\langle u_\alpha h_{\beta\gamma} u^\alpha h^{\beta\gamma} \rangle$	4ϕ	N_f
(37)	$\langle u_\alpha u^\alpha \rangle \langle h^\beta_\beta h^\gamma_\gamma \rangle$	4ϕ	
(38)	$\langle u_\alpha u_\beta \rangle \langle h^{\alpha\beta} h^\gamma_\gamma \rangle$	4ϕ	
(39)	$\langle u_\alpha u^\alpha \rangle \langle h_{\beta\gamma} h^{\beta\gamma} \rangle$	4ϕ	2
(40)	$\langle u_\alpha u_\beta \rangle \langle h^\alpha_\gamma h^{\beta\gamma} \rangle$	4ϕ	2
(41)	$\langle u_\alpha h^\beta_\beta \rangle \langle u^\alpha h^\gamma_\gamma \rangle$	4ϕ	
(42)	$\langle u_\alpha h^{\alpha\beta} \rangle \langle u_\beta h^\gamma_\gamma \rangle$	4ϕ	
(43)	$\langle u_\alpha h^{\alpha\beta} \rangle \langle u^\gamma h_{\beta\gamma} \rangle$	4ϕ	
(44)	$\langle u_\alpha h_{\beta\gamma} \rangle \langle u^\alpha h^{\beta\gamma} \rangle$	4ϕ	2
(45)	$\langle u_\alpha h_{\beta\gamma} \rangle \langle u^\beta h^{\alpha\gamma} \rangle$	4ϕ	2
(46)	$\langle \nabla_\alpha h^\beta_\beta \rangle \langle \nabla^\alpha h^\gamma_\gamma \rangle$	2ϕ	
(47)	$\langle \nabla_\alpha h^{\alpha\beta} \rangle \langle \nabla_\beta h^\gamma_\gamma \rangle$	2ϕ	
(48)	$\langle \nabla_\alpha h^{\alpha\beta} \rangle \langle \nabla^\gamma h_{\beta\gamma} \rangle$	2ϕ	3
(49)	$\langle \nabla_\alpha h_{\beta\gamma} \rangle \langle \nabla^\alpha h^{\beta\gamma} \rangle$	2ϕ	3
(50)	$\langle \nabla_\alpha h_{\beta\gamma} \rangle \langle \nabla^\beta h^{\alpha\gamma} \rangle$	2ϕ	3
(51)	$\langle u_\alpha u^\alpha u_\beta u^\beta \chi_+ \rangle$	4ϕ	
(52)	$\langle u_\alpha u_\beta u^\alpha u^\beta \chi_+ \rangle$	4ϕ	N_f
(53)	$\langle u_\alpha u_\beta u^\beta u^\alpha \chi_+ \rangle$	4ϕ	N_f
(54)	$\langle u_\alpha u^\alpha u_\beta u^\beta \rangle \langle \chi_+ \rangle$	4ϕ	
(55)	$\langle u_\alpha u_\beta u^\alpha u^\beta \rangle \langle \chi_+ \rangle$	4ϕ	N_f
(56)	$\langle u_\alpha u^\alpha u_\beta \rangle \langle u^\beta \chi_+ \rangle$	4ϕ	3
(57)	$\langle u_\alpha u^\alpha \rangle \langle u_\beta u^\beta \chi_+ \rangle$	4ϕ	3
(58)	$\langle u_\alpha u_\beta \rangle \langle u^\alpha u^\beta \chi_+ \rangle$	4ϕ	3
(59)	$\langle u_\alpha u^\alpha \rangle \langle u_\beta u^\beta \rangle \langle \chi_+ \rangle$	4ϕ	3
(60)	$\langle u_\alpha u_\beta \rangle \langle u^\alpha u^\beta \rangle \langle \chi_+ \rangle$	4ϕ	2
(61)	$i\langle \{u_\alpha u^\alpha, h^\beta_\beta\} \chi_- \rangle$	4ϕ	
(62)	$i\langle \{u_\alpha u_\beta, h^{\alpha\beta}\} \chi_- \rangle$	4ϕ	
(63)	$i\langle u_\alpha h^\beta_\beta u^\alpha \chi_- \rangle$	4ϕ	
(64)	$i\langle u_\alpha h^{\alpha\beta} u_\beta \chi_- \rangle$	4ϕ	
(65)	$i\langle u_\alpha u^\alpha h^\beta_\beta \rangle \langle \chi_- \rangle$	4ϕ	

monomial	shape	contributes to	# of flavours
(66)	$i\langle u_\alpha u_\beta h^{\alpha\beta}\rangle\langle\chi_-\rangle$	4ϕ	
(67)	$i\langle u_\alpha u^\alpha\rangle\langle h^\beta_\beta\chi_-\rangle$	4ϕ	
(68)	$i\langle u_\alpha u_\beta\rangle\langle h^{\alpha\beta}\chi_-\rangle$	4ϕ	
(69)	$i\langle u_\alpha h^\beta_\beta\rangle\langle u^\alpha\chi_-\rangle$	4ϕ	
(70)	$i\langle u_\alpha h^{\alpha\beta}\rangle\langle u_\beta\chi_-\rangle$	4ϕ	
(71)	$\langle h^\alpha_\alpha h^\beta_\beta\chi_+\rangle$	2ϕ	
(72)	$\langle h_{\alpha\beta} h^{\alpha\beta}\chi_+\rangle$	2ϕ	3
(73)	$\langle h^\alpha_\alpha h^\beta_\beta\rangle\langle\chi_+\rangle$	2ϕ	
(74)	$\langle h_{\alpha\beta} h^{\alpha\beta}\rangle\langle\chi_+\rangle$	2ϕ	2
(75)	$i\langle\{u_\alpha u^\alpha, u_\beta\}\nabla^\beta\chi_-\rangle$	4ϕ	
(76)	$i\langle u_\alpha u_\beta u^\alpha\nabla^\beta\chi_-\rangle$	4ϕ	N_f
(77)	$i\langle u_\alpha u^\alpha u_\beta\rangle\langle\nabla^\beta\chi_-\rangle$	4ϕ	3
(78)	$i\langle u_\alpha u^\alpha\rangle\langle u_\beta\nabla^\beta\chi_-\rangle$	4ϕ	3
(79)	$i\langle u_\alpha u_\beta\rangle\langle u^\alpha\nabla^\beta\chi_-\rangle$	4ϕ	2
(80)	$\langle\{u_\alpha, h^\beta_\beta\}\nabla^\alpha\chi_+\rangle$	4ϕ	
(81)	$\langle\{u_\alpha, h^{\alpha\beta}\}\nabla_\beta\chi_+\rangle$	4ϕ	3
(82)	$\langle u_\alpha h^\beta_\beta\rangle\langle\nabla^\alpha\chi_+\rangle$	4ϕ	
(83)	$\langle u_\alpha h^{\alpha\beta}\rangle\langle\nabla_\beta\chi_+\rangle$	4ϕ	2
(84)	$i\langle h^\alpha_\alpha\nabla_\beta\nabla^\beta\chi_-\rangle$	2ϕ	
(85)	$i\langle h_{\alpha\beta}\nabla^\alpha\nabla^\beta\chi_-\rangle$	2ϕ	2
(86)	$i\langle\{u_\alpha u^\alpha, u_\beta u_\gamma\}f_+^{\beta\gamma}\rangle$	$4\phi+\gamma$	
(87)	$i\langle[u_\alpha u_\beta u^\alpha, u_\gamma]f_+^{\beta\gamma}\rangle$	$4\phi+\gamma$	
(88)	$i\langle u_\alpha u_\beta u_\gamma u^\alpha f_+^{\beta\gamma}\rangle$	$4\phi+\gamma$	N_f
(89)	$i\langle u_\alpha u_\beta u^\beta u_\gamma f_+^{\alpha\gamma}\rangle$	$4\phi+\gamma$	N_f
(90)	$i\langle u_\alpha u_\beta u_\gamma\rangle\langle u^\alpha f_+^{\beta\gamma}\rangle$	$4\phi+\gamma$	3
(91)	$i\langle u_\alpha u^\alpha\rangle\langle u_\beta u_\gamma f_+^{\beta\gamma}\rangle$	$4\phi+\gamma$	3
(92)	$i\langle u_\alpha u_\beta\rangle\langle[u^\alpha, u_\gamma]f_+^{\beta\gamma}\rangle$	$4\phi+\gamma$	3
(93)	$\langle[u_\alpha u_\beta, h^\gamma_\gamma]f_-^{\alpha\beta}\rangle$	$4\phi+\gamma$	
(94)	$\langle\{u_\alpha, u_\beta, h^\alpha_\gamma\}f_-^{\beta\gamma}\rangle$	$4\phi+\gamma$	
(95)	$\langle\{u_\alpha, u_\beta, h^\beta_\gamma\}f_-^{\alpha\gamma}\rangle$	$4\phi+\gamma$	
(96)	$\langle\{u_\alpha, h^\alpha_\beta, u_\gamma\}f_-^{\beta\gamma}\rangle$	$4\phi+\gamma$	
(97)	$\langle u_\alpha u_\beta\rangle\langle h^\alpha_\gamma f_-^{\beta\gamma}\rangle$	$4\phi+\gamma$	
(98)	$\langle u_\alpha h^\beta_\beta\rangle\langle u_\gamma f_-^{\alpha\gamma}\rangle$	$4\phi+\gamma$	
(99)	$\langle u_\alpha h^\alpha_\beta\rangle\langle u_\gamma f_-^{\beta\gamma}\rangle$	$4\phi+\gamma$	
(100)	$\langle u_\alpha h_{\beta\gamma}\rangle\langle u^\beta f_-^{\alpha\gamma}\rangle$	$4\phi+\gamma$	
(101)	$i\langle h_{\alpha\beta} h^\alpha_\gamma f_+^{\beta\gamma}\rangle$	$2\phi+\gamma$	2
(102)	$\langle\{u_\alpha u^\alpha, u_\beta\}\nabla_\gamma f_-^{\beta\gamma}\rangle$	$4\phi+\gamma$	
(103)	$\langle u_\alpha u_\beta u^\alpha\nabla_\gamma f_-^{\beta\gamma}\rangle$	$4\phi+\gamma$	N_f

monomial	shape	contributes to	# of flavours
(104)	$\langle[u_\alpha, u_\beta u_\gamma]\nabla^\alpha f_-^{\beta\gamma}\rangle$	$4\phi+\gamma$	3
(105)	$\langle u_\alpha u^\alpha\rangle\langle u_\beta\nabla_\gamma f_-^{\beta\gamma}\rangle$	$4\phi+\gamma$	3
(106)	$\langle u_\alpha u_\beta\rangle\langle u^\alpha\nabla_\gamma f_-^{\beta\gamma}\rangle$	$4\phi+\gamma$	2
(107)	$\langle u_\alpha u_\beta\rangle\langle u_\gamma\nabla^\alpha f_-^{\beta\gamma}\rangle$	$4\phi+\gamma$	2
(108)	$i\langle[u_\alpha, h_\beta^\beta]\nabla_\gamma f_+^{\alpha\gamma}\rangle$	$2\phi+\gamma$	
(109)	$i\langle[u_\alpha, h_\beta^\alpha]\nabla_\gamma f_+^{\beta\gamma}\rangle$	$2\phi+\gamma$	3
(110)	$i\langle[u_\alpha, h_{\beta\gamma}]\nabla^\beta f_+^{\alpha\gamma}\rangle$	$2\phi+\gamma$	2
(111)	$\langle h_{\alpha\beta}\{\nabla_\gamma, \nabla^\alpha\}f_-^{\beta\gamma}\rangle$	$2\phi+\gamma$	2
(112)	$\langle u_\alpha u^\alpha\chi_+\chi_+\rangle$	2ϕ	N_f
(113)	$\langle u_\alpha\chi_+ u^\alpha\chi_+\rangle$	2ϕ	3
(114)	$\langle u_\alpha\chi_+\rangle\langle u^\alpha\chi_+\rangle$	2ϕ	2
(115)	$\langle u_\alpha u^\alpha\chi_+\rangle\langle\chi_+\rangle$	2ϕ	3
(116)	$\langle u_\alpha u^\alpha\rangle\langle\chi_+\chi_+\rangle$	2ϕ	2
(117)	$\langle u_\alpha u^\alpha\rangle\langle\chi_+\rangle\langle\chi_+\rangle$	2ϕ	2
(118)	$\langle u_\alpha u^\alpha\chi_-\chi_-\rangle$	4ϕ	N_f
(119)	$\langle u_\alpha\chi_- u^\alpha\chi_-\rangle$	4ϕ	3
(120)	$\langle u_\alpha\chi_-\rangle\langle u^\alpha\chi_-\rangle$	4ϕ	2
(121)	$\langle u_\alpha u^\alpha\chi_-\rangle\langle\chi_-\rangle$	4ϕ	3
(122)	$\langle u_\alpha u^\alpha\rangle\langle\chi_-\chi_-\rangle$	4ϕ	2
(123)	$\langle u_\alpha u^\alpha\rangle\langle\chi_-\rangle\langle\chi_-\rangle$	4ϕ	2
(124)	$i\langle\{u_\alpha, \nabla^\alpha\chi_-\}\chi_+\rangle$	2ϕ	
(125)	$i\langle\{u_\alpha, \nabla^\alpha\chi_+\}\chi_-\rangle$	2ϕ	3
(126)	$i\langle u_\alpha\nabla^\alpha\chi_-\rangle\langle\chi_+\rangle$	2ϕ	
(127)	$i\langle u_\alpha\nabla^\alpha\chi_+\rangle\langle\chi_-\rangle$	2ϕ	
(128)	$i\langle u_\alpha\chi_+\rangle\langle\nabla^\alpha\chi_-\rangle$	2ϕ	2
(129)	$i\langle u_\alpha\chi_-\rangle\langle\nabla^\alpha\chi_+\rangle$	2ϕ	2
(130)	$\langle\nabla_\alpha\chi_+\nabla^\alpha\chi_+\rangle$	contact term	2
(131)	$\langle\nabla_\alpha\chi_-\nabla^\alpha\chi_-\rangle$	2ϕ	2
(132)	$\langle\nabla_\alpha\chi_+\rangle\langle\nabla^\alpha\chi_+\rangle$	contact term	2
(133)	$\langle\nabla_\alpha\chi_-\rangle\langle\nabla^\alpha\chi_-\rangle$	2ϕ	2
(134)	$i\langle\{u_\alpha u_\beta, f_+^{\alpha\beta}\}\chi_+\rangle$	$2\phi+\gamma$	3
(135)	$i\langle u_\alpha f_+^{\alpha\beta}u_\beta\chi_+\rangle$	$2\phi+\gamma$	3
(136)	$i\langle u_\alpha u_\beta f_+^{\alpha\beta}\rangle\langle\chi_+\rangle$	$2\phi+\gamma$	2
(137)	$i\langle[u_\alpha u_\beta, f_-^{\alpha\beta}]\chi_-\rangle$	$4\phi+\gamma$	
(138)	$i\langle u_\alpha f_-^{\alpha\beta}\rangle\langle u_\beta\chi_-\rangle$	$2\phi+\gamma$	3
(139)	$\langle[u_\alpha, \nabla_\beta f_+^{\alpha\beta}]\chi_-\rangle$	$2\phi+\gamma$	
(140)	$\langle[u_\alpha, f_+^{\alpha\beta}]\nabla_\beta\chi_-\rangle$	$2\phi+\gamma$	2
(141)	$\langle\{u_\alpha, \nabla_\beta f_-^{\alpha\beta}\}\chi_+\rangle$	$2\phi+\gamma$	
(142)	$\langle\{u_\alpha, f_-^{\alpha\beta}\}\nabla_\beta\chi_+\rangle$	$2\phi+\gamma$	3

monomial	shape	contributes to	# of flavours
(143)	$\langle u_\alpha \nabla_\beta f_-^{\alpha\beta}\rangle\langle\chi_+\rangle$	$2\phi+\gamma$	2
(144)	$\langle u_\alpha f_-^{\alpha\beta}\rangle\langle\nabla_\beta\chi_+\rangle$	$2\phi+\gamma$	2
(145)	$i\langle\nabla_\alpha f_-^{\alpha\beta}\nabla_\beta\chi_-\rangle$	$2\phi+\gamma$	2
(146)	$\langle u_\alpha u^\alpha f_{+\,\beta\gamma} f_+^{\beta\gamma}\rangle$	$2\phi+2\gamma$	
(147)	$\langle u_\alpha u^\beta f_+^{\alpha\gamma} f_{+\,\beta\gamma}\rangle$	$2\phi+2\gamma$	
(148)	$\langle u_\alpha u^\beta f_{+\,\beta\gamma} f_+^{\alpha\gamma}\rangle$	$2\phi+2\gamma$	N_f
(149)	$\langle\{u^\alpha, f_{+\,\alpha\beta}, u_\gamma\} f_+^{\beta\gamma}\rangle$	$2\phi+2\gamma$	N_f
(150)	$\langle u_\alpha f_{+\,\beta\gamma} u^\alpha f_+^{\beta\gamma}\rangle$	$2\phi+2\gamma$	3
(151)	$\langle u_\alpha u^\alpha\rangle\langle f_{+\,\beta\gamma} f_+^{\beta\gamma}\rangle$	$2\phi+2\gamma$	2
(152)	$\langle u_\alpha u^\beta\rangle\langle f_+^{\alpha\gamma} f_{+\,\beta\gamma}\rangle$	$2\phi+2\gamma$	3
(153)	$\langle u^\alpha f_{+\,\alpha\beta}\rangle\langle u_\gamma f_+^{\beta\gamma}\rangle$	$2\phi+2\gamma$	3
(154)	$\langle u_\alpha f_{+\,\beta\gamma}\rangle\langle u^\alpha f_+^{\beta\gamma}\rangle$	$2\phi+2\gamma$	2
(155)	$\langle u_\alpha f_{+\,\beta\gamma}\rangle\langle u^\beta f_+^{\alpha\gamma}\rangle$	$2\phi+2\gamma$	2
(156)	$\langle u_\alpha u^\alpha f_{-\,\beta\gamma} f_-^{\beta\gamma}\rangle$	$4\phi+2\gamma$	
(157)	$\langle u_\alpha u^\beta f_-^{\alpha\gamma} f_{-\,\beta\gamma}\rangle$	$4\phi+2\gamma$	
(158)	$\langle u_\alpha u^\beta f_{-\,\beta\gamma} f_-^{\alpha\gamma}\rangle$	$4\phi+2\gamma$	N_f
(159)	$\langle\{u^\alpha, f_{-\,\alpha\beta}, u_\gamma\} f_-^{\beta\gamma}\rangle$	$4\phi+2\gamma$	N_f
(160)	$\langle u_\alpha f_{-\,\beta\gamma} u^\alpha f_-^{\beta\gamma}\rangle$	$4\phi+2\gamma$	3
(161)	$\langle u_\alpha u^\alpha\rangle\langle f_{-\,\beta\gamma} f_-^{\beta\gamma}\rangle$	$4\phi+2\gamma$	2
(162)	$\langle u_\alpha u^\beta\rangle\langle f_-^{\alpha\gamma} f_{-\,\beta\gamma}\rangle$	$4\phi+2\gamma$	3
(163)	$\langle u^\alpha f_{-\,\alpha\beta}\rangle\langle u_\gamma f_-^{\beta\gamma}\rangle$	$4\phi+2\gamma$	3
(164)	$\langle u_\alpha f_{-\,\beta\gamma}\rangle\langle u^\alpha f_-^{\beta\gamma}\rangle$	$4\phi+2\gamma$	2
(165)	$\langle u_\alpha f_{-\,\beta\gamma}\rangle\langle u^\beta f_-^{\alpha\gamma}\rangle$	$4\phi+2\gamma$	2
(166)	$i\langle[u^\alpha, f_{+\,\alpha\beta}]\nabla_\gamma f_-^{\beta\gamma}\rangle$	$2\phi+2\gamma$	
(167)	$i\langle[u_\alpha, f_{+\,\beta\gamma}\nabla^\alpha f_-^{\beta\gamma}\rangle$	$2\phi+2\gamma$	
(168)	$i\langle[u_\alpha, f_{+\,\beta\gamma}]\nabla^\beta f_-^{\alpha\gamma}\rangle$	$2\phi+2\gamma$	
(169)	$i\langle[u^\alpha, f_{-\,\alpha\beta}]\nabla_\gamma f_+^{\beta\gamma}\rangle$	$2\phi+2\gamma$	2
(170)	$i\langle[u_\alpha, f_{-\,\beta\gamma}]\nabla^\alpha f_+^{\beta\gamma}\rangle$	$2\phi+2\gamma$	
(171)	$i\langle[u_\alpha, f_{-\,\beta\gamma}]\nabla^\beta f_+^{\alpha\gamma}\rangle$	$2\phi+2\gamma$	2
(172)	$\langle\nabla^\alpha f_{+\,\alpha\beta}\nabla_\gamma f_+^{\beta\gamma}\rangle$	contact term	2
(173)	$\langle\nabla_\alpha f_{+\,\beta\gamma}\nabla^\alpha f_+^{\beta\gamma}\rangle$	contact term	2
(174)	$\langle\nabla_\alpha f_{+\,\beta\gamma}\nabla^\beta f_+^{\alpha\gamma}\rangle$	contact term	2
(175)	$\langle\nabla^\alpha f_{-\,\alpha\beta}\nabla_\gamma f_-^{\beta\gamma}\rangle$	$2\phi+2\gamma$	2
(176)	$\langle\nabla_\alpha f_{-\,\beta\gamma}\nabla^\alpha f_-^{\beta\gamma}\rangle$	$2\phi+2\gamma$	2
(177)	$\langle\nabla_\alpha f_{-\,\beta\gamma}\nabla^\beta f_-^{\alpha\gamma}\rangle$	$2\phi+2\gamma$	2
(178)	$\langle\chi_+\chi_+\chi_+\rangle$	contact term	3
(179)	$\langle\chi_+\chi_+\rangle\langle\chi_+\rangle$	contact term	2
(180)	$\langle\chi_+\rangle\langle\chi_+\rangle\langle\chi_+\rangle$	contact term	2
(181)	$\langle\chi_+\chi_-\chi_-\rangle$	2ϕ	3

monomial	shape	contributes to	# of flavours
(182)	$\langle\chi_+\chi_-\rangle\langle\chi_-\rangle$	2ϕ	2
(183)	$\langle\chi_+\rangle\langle\chi_-\chi_-\rangle$	2ϕ	2
(184)	$\langle\chi_+\rangle\langle\chi_-\rangle\langle\chi_-\rangle$	2ϕ	2
(185)	$\langle f_{+\,\alpha\beta}f_+^{\alpha\beta}\chi_+\rangle$	contact term	3
(186)	$\langle f_{+\,\alpha\beta}f_+^{\alpha\beta}\rangle\langle\chi_+\rangle$	contact term	2
(187)	$\langle f_{-\,\alpha\beta}f_-^{\alpha\beta}\chi_+\rangle$	$2\phi+2\gamma$	3
(188)	$\langle f_{-\,\alpha\beta}f_-^{\alpha\beta}\rangle\langle\chi_+\rangle$	$2\phi+2\gamma$	2
(189)	$\langle[f_{+\,\alpha\beta},f_-^{\alpha\beta}]\chi_-\rangle$	$2\phi+2\gamma$	2
(190)	$i\langle f_{+\,\alpha\beta}f_-^{\alpha\gamma}f_{-\,\gamma}^{\;\beta}\rangle$	$2\phi+3\gamma$	2
(191)	$i\langle f_{+\,\alpha\beta}f_+^{\alpha\gamma}f_{+\,\gamma}^{\;\beta}\rangle$	contact term	2

3.5.2 Relations in the even sector at chiral order six

The list of relations is organzized in accord with their derivation technique.

Table 3.13: Partial-integration-induced relations in the even sector

relation	shape
1	$(1)-2\times(2)+(3)-(16)+(18)+2\times(89)-(102)=0$
2	$-(2)+(3)-(4)+(5)-(20)+2\times(21)$
	$+(86)-2\times(88)+(104)=0$
3	$(2)-(4)-(17)+(19)+(87)-(103)=0$
4	$(6)-(7)-(22)+(24)+2\times(91)-(105)=0$
5	$(8)-(9)-(23)+(25)+(92)-(106)=0$
6	$(8)-(9)+(26)-(27)+(92)+(107)=0$
7	$(16)-2\times(17)-2\times(46)+2\times(47)-\frac{1}{2}\times(93)$
	$-2\times(108)+3\times(167)-3\times(170)=0$
8	$(16)+(28)+\frac{1}{2}(29)+(34)+\frac{1}{2}\times(93)=0$
9	$2\times(17)+(29)+(33)-(93)=0$
10	$(18)-2\times(19)-2\times(47)+2\times(48)+\frac{1}{4}\times(94)-\frac{1}{4}\times(95)-2\times(101)$
	$-2\times(110)+(111)+3\times(168)+3\times(169)-3\times(190)=0$
11	$(18)+(30)+(32)+\frac{1}{2}\times(35)+\frac{1}{2}\times(95)-\frac{1}{2}\times(96)=0$
12	$2\times(19)+2\times(31)+(36)+(94)=0$
13	$-(20)+2\times(21)+2\times(49)-2\times(50)-\frac{1}{4}\times(94)+\frac{1}{4}\times(95)-2\times(101)$
	$+2\times(109)+(111)-3\times(168)-3\times(169)+3\times(190)=0$
14	$2\times(20)+(29)+2\times(31)+(35)-(94)+(96)=0$
15	$2\times(21)+2\times(32)+(34)-(95)=0$
16	$(22)+\frac{1}{2}\times(37)+(42)-(98)=0$
17	$2\times(23)+(38)+(41)+(42)+(98)=0$

relation	shape
18	$(24)+\frac{1}{2}\times(39)+(43)-(99)=0$
19	$2\times(25)+(40)+(44)+(45)+(97)+(100)=0$
20	$1\times(26)+(40)+(42)+(43)-(97)+(99)=0$
21	$(27)+\frac{1}{2}\times(38)+(45)-(100)=0$
22	$-\frac{1}{2}\times(31)+\frac{1}{2}\times(32)+(48)-(50)+(101)=0$
23	$(51)-2\times(52)+(53)+(71)-(72)$
	$+(80)-(81)+(134)-2\times(135)-(141)=0$
24	$(54)-(55)+\frac{1}{2}(73)-\frac{1}{2}(74)+(82)-(83)+2\times(136)-(143)=0$
25	$(61)+(62)+2\times(64)+2\times(75)-(137)=0$
26	$(62)+(63)+2\times(76)+(137)=0$
27	$(65)+2\times(66)+2\times(77)=0$
28	$\frac{1}{2}\times(67)+(70)+(78)+(138)=0$
29	$(68)+(69)+(70)+2\times(79)+(138)=0$
30	$\frac{1}{2}(75)-(76)+(84)-(85)+(140)+(145)=0$
31	$-(166)-(167)+(168)+(169)+(170)-(190)=0$
32	$-\frac{1}{2}\times(86)+(88)+(101)-(109)$
	$-(146)+(150)+2\times(168)+(169)-(190)=0$
33	$-(87)+(88)+(89)+(108)-(109)-(110)-2\times(147)$
	$+2\times(148)+(169)+(170)-(171)=0$
34	$\frac{1}{2}\times(93)+\frac{1}{4}\times(94)-\frac{1}{4}\times(95)-(102)+2\times(103)+(111)+\frac{1}{2}\times(157)$
	$-\frac{1}{2}(158)+(166)-(167)+(170)-(171)-(175)+(177)=0$
35	$(93)-(94)-(96)-2\times(104)+2\times(157)+(159)=0$
36	$-\frac{1}{4}\times(94)+\frac{1}{4}\times(95)+(104)+(111)-\frac{1}{2}\times(157)+\frac{1}{2}\times(158)$
	$+(166)-(167)+(171)-(175)+(177)=0$
37	$-\frac{1}{4}\times(94)+\frac{1}{4}\times(95)+(104)+(111)+\frac{1}{2}\times(157)-\frac{1}{2}\times(158)$
	$-3\times(168)-(169)+(175)+2\times(176)-(177)=0$
38	$(94)+2\times(103)+2\times(157)+(160)=0$
39	$(95)-(96)+2\times(102)+2\times(156)+2\times(158)-(159)=0$
40	$(97)+(98)+(99)+2\times(107)-(162)-(163)=0$
41	$(97)+(100)+2\times(106)+(162)+(164)+(165)=0$
42	$(99)-(105)-\frac{1}{2}\times(161)+(163)=0$
43	$(137)-4\times(145)+(189)=0$
44	$(139)+(140)-\frac{1}{2}\times(189)=0$
45	$(141)+(142)+(187)=0$
46	$(143)+(144)+\frac{1}{2}\times(188)=0$
47	$\frac{1}{2}\times(147)-\frac{1}{2}\times(148)+(172)+(174)-(191)=0$
48	$\frac{1}{2}\times(157)-\frac{1}{2}\times(158)+(175)+(177)-(190)=0$
49	$(166)+(168)+(169)+(171)-2\times(190)=0$

Table 3.14: Leading-order-equation-of-motion-induced relations in the even sector

relation	shape
50	$(16) - (75) + \frac{2}{N_f} \times (77) = 0$
51	$(17) - (76) + \frac{1}{N_f} \times (77) = 0$
52	$(22) - (78) = 0$
53	$(23) - (79) = 0$
54	$(28) - \frac{1}{2} \times (61) + \frac{1}{N_f} \times (65) = 0$
55	$(29) - (62) + \frac{2}{N_f} \times (66) = 0$
56	$(33) - (63) + \frac{1}{N_f} \times (65) = 0$
57	$(34) - (64) + \frac{1}{N_f} \times (66) = 0$
58	$(37) - (67) = 0$
59	$(38) - (68) = 0$
60	$(41) - (69) = 0$
61	$(42) - (70) = 0$
62	$(46) + (84) = 0$
63	$(47) + (85) = 0$
64	$\frac{1}{2} \times (61) + (118) - \frac{1}{N_f} \times (121) = 0$
65	$(63) + (119) - \frac{1}{N_f} \times (121) = 0$
66	$(65) + (121 - \frac{1}{N_f} \times (123) = 0$
67	$(67) + (122) - \frac{1}{N_f} \times (123) = 0$
68	$(69) + (120) = 0$
69	$(71) + (124) + (125) - \frac{2}{N_f} \times (127) - \frac{2}{N_f} \times (128) = 0$
70	$(73) + 2 \times (126) + 2 \times (129) = 0$
71	$(80) - (125) + \frac{2}{N_f} \times (127) = 0$
72	$(82) - (129) = 0$
73	$(84) - (131) + \frac{1}{N_f} \times (133) = 0$
74	$(93) + (137) = 0$
75	$(98) + (138) = 0$
76	$(108) - (139) = 0$
77	$(124) + (125) - (181) + \frac{1}{N_f} \times (182) = 0$
78	$(126) + (129) - \frac{1}{2} \times (183) + \frac{1}{2N_f} \times (184) = 0$
79	$(127) + (128) - \frac{1}{2} \times (182) + \frac{1}{2N_f} \times (184) = 0$
80	$-(167) + (170) + \frac{1}{2} \times (189) = 0$

Table 3.15: Bianchi identities in the even sector

relation	shape
81	$-(146) + 2 \times (147) + (149) + (150) - (167) + 2 \times (168) = 0$
82	$-(156) + 2 \times (157) + (159) + (160) - (170) + 2 \times (171) = 0$
83	$\frac{1}{2} \times (167) - (168) + (176) - 2 \times (177) = 0$
84	$\frac{1}{2} \times (170) - (171) + (173) - 2 \times (174) = 0$

Naively,another Bianchi identity would have been expected:

$$\sum_{\{\alpha,\beta,\gamma\}} \big\langle [u_\alpha, h_{\beta\gamma}] \nabla^\beta f_+^{\alpha\gamma} \big\rangle = \sum_{\{\alpha,\beta,\gamma\}} -\frac{i}{2} \big\langle [u_\alpha, h_{\beta\gamma}][u^\beta, f_-^{\alpha\gamma}] \big\rangle. \qquad (3.52)$$

A relation between the arguments of the sums does not exist, since both sides of Eq. (3.52) vanish trivially due to the symmetry of $h_{\beta\gamma}$.

Table 3.16: $SU(3)$ trace relations in the even sector

relation	shape
85	$(1) + 3 \times (2) + 2 \times (3) - (6) - (7) - 3 \times (8) - 2 \times (10) + (14) = 0$
86	$(1) + (2) + (3) - \frac{1}{2}(6) - (8) - 2 \times (10) = 0$
87	$(1) + 3 \times (2) + (4) + (5) - 3 \times (8) - 2 \times (9) - (10) - (11) + (15) = 0$
88	$4 \times (1) + 2 \times (3) - 5 \times (6) - 2 \times (10) + (13) = 0$
89	$2 \times (2) + (3) + 2 \times (4) + (5) - 2 \times (8) - (9) - (11) - (12) = 0$
90	$2 \times (2) + (3) + 3 \times (4) - 2 \times (8) - 3 \times (9) - (10) - (12) + (15) = 0$
91	$2 \times (2) + (4) - \frac{1}{2} \times (7) - (9) - (10) = 0$
92	$4 \times (2) + 2 \times (5) - (7) - 4 \times (8) - 2 \times (11) + (14) = 0$
93	$4 \times (3) + 2 \times (4) - (6) - 4 \times (8) - 2 \times (12) + (14) = 0$
94	$(16) + (17) - \frac{1}{2} \times (22) - (23) = 0$
95	$(18) + (19) - \frac{1}{2} \times (24) - (25) = 0$
96	$(20) + (21) - (26) - \frac{1}{2} \times (27) = 0$
97	$2 \times (28) + (33) - \frac{1}{2} \times (37) - (41) = 0$
98	$(29) + (34) - \frac{1}{2} \times (38) - (42) = 0$
99	$2 \times (30) + (36) - \frac{1}{2} \times (39) - (44) = 0$
100	$2 \times (31) + 2 \times (32) + (35) - (40) - (43) - (45) = 0$
101	$(51) + 3 \times (52) + 2 \times (53) - (54) - (55) - 2 \times (56) - 3 \times (58) + (60) = 0$
102	$(51) + (52) + (53) - (56) - \frac{1}{2} \times (57) - (58) = 0$
103	$4 \times (51) + 2 \times (53) - 2 \times (54) - 2 \times (56) - 3 \times (57) + (59) = 0$
104	$(61) + (63) - (65) - \frac{1}{2} \times (67) - (69) = 0$
105	$(62) + (64) - (66) - \frac{1}{2} \times (68) - (70) = 0$
106	$(75) + (76) - (77) - \frac{1}{2} \times (78) - (79) = 0$

relation	shape
107	$(86) - (87) + 2 \times (88) - 2 \times (89) - 2 \times (90) + (92) = 0$
108	$(86) + (87) - (65) + 2 \times (89) - (91) - (92) = 0$
109	$(86) + (88) - (90) - \frac{1}{2} \times (91) = 0$
110	$(94) + (95) - (96) - (97) + (99) - (100) = 0$
111	$(102) + (103) - \frac{1}{2} \times (105) - (106) = 0$
112	$4 \times (112) + 2 \times (113) - 2 \times (114) - 4 \times (115) - (116) + (117) = 0$
113	$4 \times (118) + 2 \times (119) - 2 \times (120) - 4 \times (121) - (122) + (123) = 0$
114	$2 \times (146) + (150) - \frac{1}{2} \times (151) - (154) = 0$
115	$2 \times (147) + 2 \times (148) - (149) - (152) + (153) - (155) = 0$
116	$2 \times (156) + (160) - \frac{1}{2} \times (161) - (164) = 0$
117	$2 \times (157) + 2 \times (158) - (159) - (162) + (163) - (165) = 0$

Table 3.17: $SU(2)$ trace relations in the even sector

relation	shape
118	$(1) + (2) - (8) = 0$
119	$(1) + (2) - (8) - (10) = 0$
120	$(1) + (3) - (6) - (10) = 0$
121	$(1) + (5) - 3 \times (8) + (15) = 0$
122	$2 \times (1) - (6) = 0$
123	$2 \times (1) - 3 \times (6) + (13) = 0$
124	$(2) + (3) - (8) = 0$
125	$(2) + (4) - (8) - 2 \times (9) + (15) = 0$
126	$(2) + (4) - (9) = 0$
127	$(2) + (4) - (9) - (10) = 0$
128	$(2) + (5) - (8) - (11) = 0$
129	$2 \times (2) - (7) = 0$
130	$2 \times (2) - (7) - 2 \times (8) + (14) = 0$
131	$2 \times (2) - (7) - (10) = 0$
132	$2 \times (2) - (9) - (11) = 0$
133	$2 \times (1) - 3 \times (6) + (13) = 0$
134	$2 \times (3) - (6) = 0$
135	$2 \times (3) - (6) - 2 \times (8) + (14) = 0$
136	$(4) + (5) - (8) = 0$
137	$2 \times (4) - (9) - (12) = 0$
138	$(16) + 2 \times (17) - 2 \times (23) = 0$
139	$(16) - (22) = 0$
140	$(18) + 2 \times (19) - 2 \times (25) = 0$

relation	shape
141	$(18) - (24) = 0$
142	$(20) + 2 \times (21) - 2 \times (26) = 0$
143	$(20) - (27) = 0$
144	$(28) + (33) - (41) = 0$
145	$(2 \times (28) - (37) = 0$
146	$(29) + 2 \times (34) - 2 \times (42) = 0$
147	$(29) - (38) = 0$
148	$(30) + (36) - (44) = 0$
149	$2 \times (30) - (39) = 0$
150	$(31) + (32) - (40) = 0$
151	$2 \times (31) + (35) - 2 \times (45) = 0$
152	$2 \times (32) + (35) - 2 \times (43) = 0$
153	$(51) + (52) - (56) - (58) = 0$
154	$(51) + (53) - (54) - (56) = 0$
155	$(51) + (53) - (56) - (57) = 0$
156	$2 \times (51) - (54) - 2 \times (57) + (59) = 0$
157	$2 \times (51) - (57) = 0$
158	$(52) + (53) - (56) - (58) = 0$
159	$(52) + (53) - (58) = 0$
160	$2 \times (52) - (55) - (56) = 0$
161	$2 \times (52) - (55) - 2 \times (58) + (60) = 0$
162	$2 \times (53) - (54) - 2 \times (58) + (60) = 0$
163	$\frac{1}{2}(61) + (63) - (65) - (69) = 0$
164	$\frac{1}{2}(61) + (63) - (69) = 0$
165	$(61) - (65) - (67) = 0$
166	$(61) - (67) = 0$
167	$\frac{1}{2} \times (62) + (64) - (66) - (70) = 0$
168	$\frac{1}{2} \times (62) + (64) - (70) = 0$
169	$(62) - (66) - (68) = 0$
170	$(62) - (68) = 0$
171	$2 \times (71) - (73) = 0$
172	$2 \times (72) - (74) = 0$
173	$\frac{1}{2} \times (75) + (76) - (77) - (79) = 0$
174	$\frac{1}{2} \times (75) + (76) - (79) = 0$
175	$(75) - (77) - (78) = 0$
176	$(75) - (78) = 0$
177	$(80) - (82) = 0$
178	$(81) - (83) = 0$
179	$-(86) + 2 \times (87) - 2 \times (88) + 4 \times (90) = 0$

relation	shape
180	$(86)+2\times(87)+2\times(89)-2\times(92)=0$
181	$(86)+2\times(88)-2\times(90)=0$
182	$(86)-2\times(88)-2\times(92)=0$
183	$(86)-2\times(89)=0$
184	$(86)+2\times(89)-2\times(91)=0$
185	$(86)-(91)=0$
186	$(87)-(88)+(89)-(92)=0$
187	$(87)+(90)=0$
188	$(88)-(89)+(92)=0$
189	$(93)+2\times(98)=0$
190	$(94)+(95)-2\times(97)=0$
191	$-(94)+(96)+2\times(100)=0$
192	$(95)-(96)+2\times(99)=0$
193	$\frac{1}{2}\times(102)+(103)-(106)=0$
194	$(102)-(105)=0$
195	$(104)-2\times(107)=0$
196	$(112)+(113)-(114)-(115)=0$
197	$2\times(112)-2\times(115)-(116)+(117)=0$
198	$2\times(112)-(116)=0$
199	$(118)+(119)-(120)-(121)=0$
200	$2\times(118)-2\times(121)-(122)+(123)=0$
201	$2\times(118)-(122)=0$
202	$(124)-(126)-(128)=0$
203	$(125)-(127)-(129)=0$
204	$(134)+2\times(135)=0$
205	$(134)-(136)=0$
206	$(137)+2\times(138)=0$
207	$(141)-(143)=0$
208	$(142)-(144)=0$
209	$(146)+(150)-(154)=0$
210	$2\times(146)-(151)=0$
211	$(147)+(148)-(152)=0$
212	$-(147)+\frac{1}{2}(149)+(155)=0$
213	$(148)-\frac{1}{2}(149)+(153)=0$
214	$(156)+(160)-(164)=0$
215	$(2\times(156)-(161)=0$
216	$((157)+(158)-(162)=0$
217	$-(157)+\frac{1}{2}(159)+(165)=0$
218	$(158)-\frac{1}{2}(159)+(163)=0$

relation	shape
219	$2 \times (178) - 3 \times (179) + (180) = 0$
220	$2 \times (181) - 2 \times (182) - (183) + (184) = 0$
221	$2 \times (185) - (186) = 0$
222	$2 \times (187) - (188) = 0$

3.5.3 Reduction to maximal set of independent relations

Among this set of relations in 191 variables, namely the monomials, a notable fraction is not independent of the remaining maximal set. The dependencies are investigated by use of a mathematica program. Interpretation of each relation's coefficients as components of a 191-dimensional vector yields 191 equations

$$\sum_{j=1}^{\#\text{ of relations}} a_j \, j_\alpha = 0, \quad \alpha = 1, \ldots, 191. \tag{3.53}$$

Here, j_α is the coefficient of the monomial (α) in the symmetry relation j. The a_j are coefficients of a linear combination. The eliminated relations are chosen in such a manner that among the $SU(N_f)$ relations the same relations are dropped out for every N_f; furthermore, the $SU(3)$ trace relations are eliminated completely in the $SU(2)$ case. The decision, which relations are to be kept and which ones shall be eliminated is somewhat arbitrary. Other choices may allow for a simpler connection to the Cayley-Hamiltion relations of [BCE 00]. According to the applied paradigm, the following relations are dropped out:

Table 3.18: Linearly dependent relations in the even sector

# of flavours	eliminated relations
N_f	*7*, *25*, *27*, *28*, *29*, *31*, *34*, *36*
3	..., *92*, *93*, *98*, *100*, *104*, *105*, *106*, *109*, *110*, *111*, *113*, *117*
2	..., *85–117*, *118*, *127*, *131*, *132*, *134*, *135*, *136*, *137*, *147*, *152*, *159*, *162*, *163*, *165*, *166*, *167*, *169*, *170*, *173*, *174*, *175*, *176*, *183*, *185*, *187*, *188*, *189*, *190*, *191*, *192*, *193*, *194*, *195*, *199*, *200*, *201*, *202*, *203*, *206*, *207*, *216*, *217*, *218*, *220*, *222*

The number of independent monomials is therefore 115 for an arbitrary number of flavours, 94 for three flavours and 55 for two flavours.

3.5.4 Reduction to minimal set of independent monomials

Elimination paradigms

A mathematica program is used to solve the maximal system of independent coupled linear algebraic equations. The method of selection is explained in the appendix A.4.

Consideration of large-N_c counting shows a preference of the LECs of original single-trace monomials due to Eq. (2.91). Preferring single traces guarantees a minimal number of leading order LECs. That is why some authors prefer single traces.

On the other hand, the flavour structure of multiple traces is more simple. The example of the next-to-leading-order Lagrangian $\mathcal{L}_4$ demonstrated (see section 3.4.3) that the number of leading-order LECs does not necessarily increase, if multiple traces are preferred[15]. Therefore, in the fourth column of table 3.12 multiple traces were chosen[16]. An alternative elimination of multiple traces in preference of single traces is shown in the appendix A.1.

Contact terms

Three general-N_f contact terms exist and one, the construction of which requires use of the N_f-specific Cayley-Hamilton formulae.

$$i(\langle F_{R\,\alpha\beta}F_R^{\alpha\gamma}F_{R\,\gamma}^{\beta}\rangle + \langle F_{L\,\alpha\beta}F_L^{\alpha\gamma}F_{L\,\gamma}^{\beta}\rangle) = \frac{1}{4}\Big(3\times(190)+(191)\Big), \tag{3.54}$$

$$\begin{aligned} &\langle D_\alpha F_{R\,\beta\gamma}D^\alpha F_R^{\beta\gamma}\rangle + \langle D_\alpha F_{L\,\beta\gamma}D^\alpha F_L^{\beta\gamma}\rangle \\ = \;& \tfrac{1}{4}\Big((173)-(174)\Big) - \tfrac{1}{64}\times(189) - \tfrac{1}{8}\Big((171)-(176)\Big), \end{aligned} \tag{3.55}$$

$$\begin{aligned} &\langle D_\alpha\chi D^\alpha\chi^\dagger\rangle + h.c. \\ = \;& \tfrac{1}{2}\times(125) - \tfrac{1}{4}\Big((130)-(131)\Big) - \tfrac{1}{4}\Big((181)-\tfrac{1}{N_f}\times(182)\Big). \end{aligned} \tag{3.56}$$

In the two-flavour case the additional contact term is

$$\begin{aligned} &\langle D_\alpha\chi\tau_2 D^\alpha\chi^T\tau_2\rangle + h.c. \\ = \;& -\tfrac{1}{2}\Big((114)+(120)\Big) + \tfrac{1}{4}\Big((117)+(123)\Big) \\ & -2\Big((128)+(129)\Big) - (130) - (131) + (132) + (133), \end{aligned} \tag{3.57}$$

[15] Even more so, the leading-order contributions of the enhanced LECs are not independent. Thus, the number of independent LECs at leading order does not change.

[16] Even though all LECs from the minimal set are independent, when all orders are taken into account, some of them are equal up to a pure number at leading order. In the three-flavour case, all of the LECs except m_{132}, m_{179}, m_{180}, m_{182}, m_{183}, m_{184}, m_{186} and m_{188} have leading order contributions. Due to additional trace relations, m_{132} is the only LEC without leading-order contributions. This does not mean that each of the other LECs produces an independent leading-order LEC.

while the inclusion of a third flavour yields

$$\begin{aligned} & \det\chi + h.c. \\ = \; & \tfrac{1}{12}\Big((178) + 3\times(181)\Big) + \tfrac{1}{24}\Big((180) + 3\times(184)\Big) \\ & -\tfrac{1}{8}\Big((179) + 2\times(182) + (183)\Big). \end{aligned} \tag{3.58}$$

The $SU(2)$-relation is derived by using $\det u = 1$ in an intermediate step, when the Cayley-Hamilton formula is used to express the traces in terms of determinants. The determinant is eliminated again in favour of traces by taking the trace of the Cayley-Hamilton formula (see Eq. (2.88)).

For the purpose of use in the third contact term[17] of Eq. (3.56), the explicit expression for (125) in the two-flavour case in accordance with the above elimination paradigm is given here:

$$(125) \equiv \frac{1}{9}\Big(-6\times(128) + 9\times(129) + 3\times(182) - (184)\Big). \tag{3.59}$$

One of the monomials included in each of the contact-term relations can be eliminated in favour of the respective contact term.

3.5.5 Extension of the chiral group in the even sector

The number of monomials containing flavour singlet field strength tensors

$$f_{\alpha\beta} = \partial_\alpha v_\beta^{(s)} - \partial_\beta v_\alpha^{(s)} \tag{3.60}$$

is limited to eight. The field strength tensor $f_{\alpha\beta}$ transforms as $f_{+\alpha\beta}$ under all symmetry operations except chiral transformations. Being singlets in flavour and colour space, they can couple to each quark loop[18]. No additional suppressions in large-N_c counting are generated.

[17] The reader might wonder, why monomials cubic in the source χ appear in this relation. This is due to the application of an eom relation.

[18] The flavour singlet field strength tensor is just the field $f_{\alpha\beta} = f^0_{\alpha\beta}$. It explicitly is not multiplied to a unit matrix (or to λ_0). That definition would differ by a simple numerical factor, when replacement rules are considered. The advantage of this approach is that, the LECs of all these structures contribute at leading order in N_c. Of course, a flavour singlet field could couple to its own quark loop and connect to a second flavour trace via annihilation into a gluonic intermediate state.

Monomials

Table 3.19: Monomials of the extended chiral group in the even sector

monomial	shape
(1x)	$f_{\alpha\beta}\langle u^\alpha u^\beta u_\gamma u^\gamma \rangle$
(2x)	$f_{\alpha\beta}\langle u^\alpha u^\beta \chi_+ \rangle$
(3x)	$f_{\alpha\beta}\langle u_\gamma u^\gamma f_+^{\alpha\beta} \rangle$
(4x)	$f_{\alpha\beta}\langle \{u^\alpha, u_\gamma\} f_+^{\beta\gamma} \rangle$
(5x)	$f_{\alpha\beta}\langle f_+^{\alpha\beta} \chi_+ \rangle$
(6x)	$f_{\alpha\beta} f^{\alpha\beta}\langle u_\gamma u^\gamma \rangle$
(7x)	$f_{\alpha\beta} f^{\alpha\gamma}\langle u^\beta u_\gamma \rangle$
(8x)	$f_{\alpha\beta} f^{\alpha\beta}\langle \chi_+ \rangle$

Relations

In this work, the extension is treated only for the two-flavour case. Only three $SU(2)$ trace relations can be derived.

Table 3.20: Relations for the extended chiral group in the even sector

relation	shape
1x	(1x)=0
2x	(3x)=0
3x	(4x)=0

Replacement rules in the electromagnetic case

The remaining monomials could be generated from monomials of table 3.12 by certain replacements of the external fields. Normally, this is not allowed, since the external fields in the generating functional must all be treated as independent. But if they represent electromagnetic interactions[19], the replacement rules for the field strength tensors are exactly fulfilled.

Three monomials are proportional to monomials from the usual even sector, if the field strength tensors are replaced:

$$f^{\alpha\beta} \to \frac{1}{2} f_{+3}^{\alpha\beta}. \tag{3.61}$$

[19]Electromagnetic interactions: $f_{+3}^{\alpha\beta} = -e\mathcal{F}^{\alpha\beta}$, $f^{\alpha\beta} = -\frac{e}{2}\mathcal{F}^{\alpha\beta}$.

Therefore,

$$\begin{aligned}(6x) &= \tfrac{1}{8} \times (151),\\ (7x) &= \tfrac{1}{8} \times (152),\\ (8x) &= \tfrac{1}{8} \times (186).\end{aligned} \tag{3.62}$$

The remaining monomials require an additional replacement[20]:

$$\chi_{+\,3} \rightarrow \sqrt{2N_f}\chi_{+\,0}. \tag{3.63}$$

Both replacements identify the monomial $(2x)$ as a correction to (136) proportional to isospin breaking:

$$(2x) = \frac{1}{2\sqrt{2N_f}} \frac{\chi_3}{\chi_0} \times (136). \tag{3.64}$$

Undoing one of the field strength tensor replacements in $(8x)$ and performing the replacement of Eq. (3.63) makes the monomial $(5x)$ turn out as an isospin-breaking correction to (186):

$$(5x) = \frac{1}{2\sqrt{2N_f}} \frac{\chi_3}{\chi_0} \times (186). \tag{3.65}$$

Thus, in the electromagnetic case, two isospin breaking corrections to standard structures arise, but no fundamentally new structures follow from the inclusion of the singlet field strength tensor. This topic is discussed in further detail in the two-flavour case of the anomalous sector.

3.5.6 Comparison to the Lagrangian of Bijnens, Colangelo and Ecker

The minimal set of independent monomials in [BCE 00] had been generated with a preference for single traces. Furthermore, the construction of all possible contact terms had been done and some monomials were expressed in terms of structures, which do not appear in the list of a priori independent monomials. Reasons are the appearance of more than half of the covariant derivatives operating on the same structure, use of a different structure for the covariant derivative of scalar and pseudoscalar sources or that they are contact terms.

[20] χ_0 is the isospin-symmetric part of the two-flavour mass matrix, while χ_3 is the isospin breaking part.

The monomials of [BCE 00]

Table 3.21: Monomials of [BCE 00]

General N_f	structure	$N_f = 3$	$N_f = 2$
Y_1	(30)	Y_1	Y_1
Y_2	(39)	Y_2	
Y_3	(36)	Y_3	Y_2
Y_4	(44)		
Y_5	(35)	Y_4	Y_3
Y_6	(45)		
Y_7	(51)	Y_5	Y_4
Y_8	(54)	Y_6	
Y_9	(57)	Y_7	
Y_{10}	(59)		
Y_{11}	(53)	Y_8	
Y_{12}	(56)	Y_9	
Y_{13}	(52)	Y_{10}	Y_5
Y_{14}	(55)	Y_{11}	
Y_{15}	(58)		
Y_{16}	(60)		
Y_{17}	(72)	Y_{12}	Y_6
Y_{18}	(74)	Y_{13}	
Y_{19}	(112)	Y_{14}	Y_7
Y_{20}	(115)	Y_{15}	Y_8
Y_{21}	(116)	Y_{16}	
Y_{22}	(117)		
Y_{23}	(113)	Y_{17}	Y_9
Y_{24}	(114)	Y_{18}	
Y_{25}	(178)	Y_{19}	Y_{10}
Y_{26}	(179)	Y_{20}	Y_{11}
Y_{27}	(180)	Y_{21}	
Y_{28}	(62)	Y_{22}	Y_{12}
Y_{29}	(68)	Y_{23}	
Y_{30}	(66)	Y_{24}	
Y_{31}	(64)	Y_{25}	Y_{13}
Y_{32}	(70)		
Y_{33}	(118)	Y_{26}	Y_{14}
Y_{34}	(121)	Y_{27}	Y_{15}
Y_{35}	(122)	Y_{28}	

General N_f	structure	$N_f = 3$	$N_f = 2$
Y_{36}	(123)		
Y_{37}	(119)	Y_{29}	Y_{16}
Y_{38}	(120)	Y_{30}	
Y_{39}	(181)	Y_{31}	Y_{17}
Y_{40}	(183)	Y_{32}	Y_{18}
Y_{41}	(182)	Y_{33}	Y_{19}
Y_{42}	(184)		
Y_{43}	$(118) + (119) + (125)$	Y_{34}	Y_{20}
Y_{44}	$(121) + (127)$	Y_{35}	Y_{21}
Y_{45}	$(120) + (129)$	Y_{36}	
Y_{46}	$-(114) - 2 \times (128) + (133)$	Y_{37}	
Y_{47}	$-\frac{1}{2}\left((118) + (119)\right) - (125) + (130)$	Y_{38}	Y_{22}
Y_{48}	$-(120) - 2 \times (129) + (132)$	Y_{39}	Y_{23}
Y_{49}	(1)	Y_{40}	Y_{24}
Y_{50}	(6)	Y_{41}	
Y_{51}	(13)		
Y_{52}	(3)	Y_{42}	
Y_{53}	(10)	Y_{43}	
Y_{54}	(2)	Y_{44}	Y_{25}
Y_{55}	(8)		
Y_{56}	(14)		
Y_{57}	(7)	Y_{45}	
Y_{58}	(5)	Y_{46}	Y_{26}
Y_{59}	(11)	Y_{47}	
Y_{60}	(4)		
Y_{61}	(12)		
Y_{62}	(9)		
Y_{63}	(15)		
Y_{64}	(86)	Y_{48}	
Y_{65}	(91)	Y_{49}	
Y_{66}	(88)	Y_{50}	Y_{27}
Y_{67}	(89)	Y_{51}	Y_{28}
Y_{68}	(87)	Y_{52}	
Y_{69}	(90)		
Y_{70}	(92)		
Y_{71}	(146)	Y_{53}	Y_{29}
Y_{72}	(151)	Y_{54}	
Y_{73}	(150)	Y_{55}	Y_{30}

General N_f	structure	$N_f = 3$	$N_f = 2$
Y_{74}	(154)		
Y_{75}	(147)	Y_{56}	Y_{31}
Y_{76}	(148)	Y_{57}	Y_{32}
Y_{77}	(152)	Y_{58}	
Y_{78}	$-(149)$	Y_{59}	Y_{33}
Y_{79}	$-(153)$		
Y_{80}	(155)	Y_{60}	
Y_{81}	(185)	Y_{61}	Y_{34}
Y_{82}	(186)	Y_{62}	
Y_{83}	(134)	Y_{63}	
Y_{84}	(136)	Y_{64}	
Y_{85}	$-(135)$	Y_{65}	Y_{35}
Y_{86}	(95)	Y_{66}	Y_{36}
Y_{87}	(97)	Y_{67}	Y_{37}
Y_{88}	$-(99)$	Y_{68}	
Y_{89}	$-(96)$	Y_{69}	Y_{38}
Y_{90}	(156)	Y_{70}	Y_{39}
Y_{91}	(161)	Y_{71}	
Y_{92}	(160)	Y_{72}	Y_{40}
Y_{93}	(164)		
Y_{94}	(157)	Y_{73}	Y_{41}
Y_{95}	(158)	Y_{74}	Y_{42}
Y_{96}	(162)	Y_{75}	
Y_{97}	$-(159)$	Y_{76}	Y_{43}
Y_{98}	(165)	Y_{77}	
Y_{99}	$-(163)$		
Y_{100}	$2\left((166)+(171)+(190)\right)$	Y_{78}	Y_{44}
Y_{101}	$2\times(190)$	Y_{79}	Y_{45}
Y_{102}	(187)	Y_{80}	Y_{46}
Y_{103}	(188)	Y_{81}	
Y_{104}	(189)	Y_{82}	Y_{47}
Y_{105}	(137)	Y_{83}	Y_{48}
Y_{106}	$-(138)$	Y_{84}	
Y_{107}	$-\frac{1}{2}\times(137)-(142)$	Y_{85}	Y_{49}
Y_{108}	$(138)-(144)$	Y_{86}	
Y_{109}	(176)	Y_{87}	Y_{50}
Y_{110}	$-(110)$	Y_{88}	Y_{51}
Y_{111}	$-(169)$	Y_{89}	Y_{52}

General N_f	structure	$N_f = 3$	$N_f = 2$
Y_{112}	$-(109)$	Y_{90}	Y_{53}
Y_{113}	$\frac{1}{4}\Big((124) - (125) + (130) - (131)\Big)$	Y_{91}	Y_{54}
Y_{114}	$\frac{1}{4}\Big(3 \times (190) + (191)\Big)$	Y_{92}	Y_{55}
Y_{115}	$-\frac{1}{32}\Big((167) + (170)\Big) + \frac{1}{8}\Big((173) + (176)\Big)$	Y_{93}	Y_{56}
	$\frac{1}{12}\Big((178) + 3 \times (181)\Big) + \frac{1}{24}\Big((180) + 3 \times (184)\Big)$		
	$-\frac{1}{8}\Big((179) + 2 \times (182) + (183)\Big)$	Y_{94}	
	$-\frac{1}{2}\Big((114) + (120)\Big) + \frac{1}{4}\Big((117) + (123)\Big)$		
	$-2\Big((128) + (129)\Big) - (130) - (131) + (132) + (133)$		Y_{57}

Thus, the number of independent monomials is two less in the two-flavour case in this thesis than in [BCE 00]. Two additional relations exist. These relations can be identified by a comparison of the Cayley-Hamilton relations of [BCE 00] with the maximal set of independent relations in the two-flavour case.

The Cayley-Hamilton relations of [BCE 00]

The Cayley-Hamilton relations of [BCE 00] have the shape

$$Y_{j_0} = \sum_{j=1,\, j\neq j_0}^{j_{max}} \kappa_j Y_j. \tag{3.66}$$

They were constructed to express one eliminated monomial in terms of only kept monomials. Thus, they have not been derived in this way by hand, but are linear combinations of large numbers of relations, which have to be decomposed for a clarification of their origin[21].

They are transformed to the same shape as the relations used here by subtraction of the left hand side. The result is then described as a linear combination of the relations in section 3.5.2, where the reduction to the maximal linearly independent set of section 3.5.3 has already been performed. Their enumeration is in accord with their ordering of appearance in [BCE 00].

In most cases, the correspondence is unambiguous. In some cases only groups of relations can be linked to each other. In both cases, where a relation was missed in [BCE 00], five relations from [BCE 00] correspond to six from 3.5.2.

[21] Actually, the first 21 Cayley-Hamilton relations had been derived for the three-flavour case. N_f-dependent leading-order-equation-of-motion-induced relations are used in the linear combinations. These relations, where $N_f = 3$, are used in the two-flavour case, where $N_f = 2$. This artificially complicates the decomposition process.

Table 3.22: Comparison to Cayley-Hamilton relations

C.H.	linear combination of relations	corresp. to
CH1	$148+\frac{1}{2}\times 149$	*148*
CH2	$\Big[-\frac{2}{3}\times 1+\frac{2}{3}\times 2+\frac{4}{3}\times 3-\frac{2}{3}\times 6-\frac{1}{6}\times 8+\frac{11}{12}\times 9+\frac{5}{3}\times 11$	
	$-\frac{1}{6}\times 12+\frac{1}{6}\times 14-\frac{5}{6}\times 15-\frac{1}{4}\times 16-\frac{1}{4}\times 17-\frac{1}{2}\times 18-\frac{1}{2}\times 19$	
	$+\frac{1}{2}\times 20-\frac{1}{2}\times 21+\frac{1}{3}\times 35+\frac{2}{3}\times 38-\frac{1}{3}\times 39+\frac{1}{3}\times 40+\frac{1}{3}\times 54$	
	$-\frac{11}{12}\times 55-\frac{2}{3}\times 56+\frac{7}{6}\times 57+\frac{1}{2}\times 59-\frac{1}{6}\times 61+\frac{1}{3}\times 64-\frac{2}{3}\times 65$	
	$+\frac{2}{3}\times 74-\frac{1}{3}\times 75\Big]-\frac{1}{3}\times 122-\frac{2}{3}\times 124+\frac{2}{3}\times 126+\frac{1}{3}\times 129$	
	$-\frac{1}{4}\times 138-\frac{1}{4}\times 139-\frac{1}{2}\times 140-\frac{1}{2}\times 141+\frac{1}{6}\times 142+\frac{1}{6}\times 143$	
	$-\frac{1}{4}\times 144-\frac{1}{8}\times 145-\frac{1}{12}\times 146-\frac{1}{2}\times 148-\frac{1}{4}\times 149+\frac{1}{6}\times 168$	
	$-\frac{1}{3}\times 180-\frac{1}{3}\times 184$	*140*
CH3	$-154-155-156$	*156*
CH4	$\frac{1}{2}\times 157+158$	*158*
CH5	$-154+\frac{3}{2}\times 157+2\times 158-161$	*161*
CH6	$-2\times 196-197$	*197*
CH7	$\Big[8+\frac{1}{2}\times 9-\frac{1}{2}\times 16-\frac{1}{2}\times 17-\frac{3}{4}\times 55-\frac{1}{2}\times 57+\frac{1}{2}\times 59+\frac{1}{2}\times 61\Big]$	
	$-\frac{1}{2}\times 138-\frac{1}{2}\times 139-\frac{1}{2}\times 144-\frac{1}{4}\times 145-\frac{1}{4}\times 146+\frac{1}{2}\times 168$	*138*
CH8	$\Big[-54+58-2\times 64-2\times 65+66+67+2\times 68\Big]+145+2\times 164$	*145*
CH9	$-2\times 120-123$	*123*
CH10	$-\frac{1}{2}\times\Big(119+120+124\Big)$	*119*
CH11	$119+\frac{1}{2}\times 122+124-130$	*130*
CH12	$-\frac{1}{2}\times 119-120+\frac{1}{4}\times 122-\frac{1}{2}\times 124+128$	*128*
CH13	$-119-\frac{1}{2}\times 120+\frac{1}{4}\times 122-\frac{1}{2}\times 129+133$	*133*
CH14	$\frac{1}{2}\times\Big(119+120-122-124+2\times 126+129\Big)$	*126*
CH15	$119+\frac{3}{2}\times 120-\frac{3}{4}\times 122-125+2\times 126+129$	*125*
CH16	$\frac{1}{4}\times\Big(179-180+4\times 181+182+184\Big)$	*179*
CH17	$\frac{1}{2}\times\Big(180+184\Big)$	*180*
CH18	$209+\frac{1}{2}\times 210$	*209*
CH19	$211-212+213$	*212*
CH20	$214+\frac{1}{2}\times 215$	*214*
CH21	$\Big[2\times 1+2\times 3-4-2\times 5-\frac{1}{2}\times 8-\frac{1}{4}\times 9-11-\frac{1}{2}\times 12+\frac{1}{2}\times 14$	
	$+\frac{1}{2}\times 15+\frac{1}{4}\times 16+\frac{1}{4}\times 17+\frac{1}{2}\times 18+\frac{1}{2}\times 19-\frac{1}{2}\times 20-\frac{1}{2}\times 21$	
	$+38+39-41+42\Big]-122-2\times 124+2\times 126+129+\frac{5}{4}\times 138$	
	$+\frac{5}{4}\times 139-\frac{1}{2}\times 140-\frac{1}{2}\times 141-\frac{1}{2}\times 142-\frac{1}{2}\times 143+\frac{1}{4}\times 144$	
	$+\frac{1}{8}\times 145+\frac{1}{2}\times 148+\frac{1}{4}\times 149-2\times 180-184-214-\frac{1}{2}\times 215$	*141*

C.H.	linear combination of relations	corr. to
CH22	149	149
CH23	$154-155+157$	154
CH24	157	157
CH25	$153-158$	153
CH26	$-153-155+157+158$	155
CH27	$-153-155+157+158+160$	160
CH28	172	172
CH29	198	198
CH30	196	196
CH31	-219	219
CH32	$\Big[2\times 8+9-16-17-\frac{1}{2}\times 55+57+59-61\Big]$	
	$-138-139-144-\frac{1}{2}\times 145-\frac{3}{2}\times 146+168$	139
CH33	$\Big[\frac{1}{2}\times 55+57-61\Big]-\frac{1}{2}\times 146+168$	168
CH34	$\Big[-2\times 56-58+2\times 60+64+66-67\Big]+2\times 144-145-2\times 164$	144
CH35	$\Big[\frac{1}{2}\times 64+65-68\Big]-164$	164
CH36	$\Big[-2\times 69+70+2\times 77-78-79\Big]+171$	171
CH37	$\Big[\frac{1}{2}\times 64+65-68-71+72\Big]-164+177$	177
CH38	122	122
CH39	$\frac{1}{2}\times\Big(119-120+122-124\Big)$	120
CH40	$\frac{1}{2}\times\Big(-119-120+122+124\Big)$	124
CH41	129	129
CH42	$119-120-121+122+124+125-2\times 126$	121
CH43	$179-180+2\times 181+182$	182
CH44	$\frac{1}{2}\times\Big(179-180+2\times 181+182-184\Big)$	184
CH45	$\frac{1}{2}\times\Big(-180+181\Big)$	181
CH46	210	210
CH47	211	211
CH48	213	213
CH49	221	221
CH50	-204	204
CH51	$-204+205$	205
CH52	$\frac{1}{2}\times\Big[8+\frac{1}{2}\times 9-12-14-15-\frac{1}{2}\times 16-\frac{1}{2}\times 17-18+19+20+21\Big]$	
	$-\frac{1}{4}\times 138-\frac{1}{4}\times 139+\frac{1}{2}\times 140-\frac{1}{2}\times 141+\frac{1}{2}\times 142+\frac{1}{2}\times 143$	
	$-\frac{1}{4}\times 144-\frac{1}{8}\times 145+\frac{1}{2}\times 148-\frac{1}{4}\times 149+150+\frac{1}{2}\times 151$	143

C.H.	linear combination of relations	corr. to
CH53	*215*	*215*
CH54	$\Big[1 + 2 - 4 - 2\times 6 - 3\times 8 - 9 + 11 + 2\times 12 + \frac{3}{2}\times 14 + 15 + 2\times 16$	
	$+17 + 18 - 2\times 19 - 20 - 2\times 21 + \frac{1}{2}\times 35 + \frac{1}{2}\times 39 + 40 + 42\Big] + 119$	
	$-120 - 121 - 124 + 125 + 129 + 138 + 3\times 139 - 2\times 140 - 2\times 142$	
	$+144 + 145 + 146 - 2\times 148 + \frac{1}{2}\times 149 - 3\times 150 - 2\times 151 - 182$	
	$-184 - \frac{1}{2}\times 215$	*142*
CH55	$\Big[1 - 4 - \frac{1}{2}\times 8 - \frac{1}{4}\times 9 + \frac{1}{2}\times 12 + \frac{1}{2}\times 14 + \frac{1}{2}\times 15 + \frac{1}{4}\times 16 + \frac{1}{4}\times 17$	
	$+\frac{1}{2}\times 18 - \frac{1}{2}\times 19 - \frac{1}{2}\times 20 - \frac{1}{2}\times 21 + \frac{1}{2}\times 39 + 42\Big] + \frac{1}{2}\times 119$	
	$-\frac{1}{2}\times 120 - \frac{1}{2}\times 122 - \frac{1}{2}\times 124 + 129 + \frac{1}{4}\times 138 + \frac{5}{4}\times 139 - \frac{1}{2}\times 140$	
	$-\frac{1}{2}\times 141 - \frac{1}{2}\times 142 - \frac{1}{2}\times 143 + \frac{1}{4}\times 144 + \frac{1}{8}\times 145 - \frac{1}{2}\times 148$	
	$+\frac{1}{4}\times 149 - 150 - \frac{1}{2}\times 151 - 2\times 184 - \frac{1}{2}\times 215$	*150*
CH56	$2\times\Big[23 - 24 + 45 - 46\Big] - 2\times 154 + 2\times 160 - 171 + 172 - 2\times 177$	
	$+2\times 178 + 2\times 204 - 4\times 205 - 2\times 208$	*178*
CH57	$\Big[-8 + 16 + \frac{1}{2}\times 74 + 75\Big] + 139 + \frac{1}{2}\times 145 + \frac{1}{2}\times 146$	*146*
CH58	$\Big[8 - 16 - \frac{1}{2}\times 74 - 75\Big] - 139 - \frac{1}{2}\times 145 - \frac{1}{2}\times 146 - 208$	*208*

The group of relations *140*, *141*, *142*, *143*, *150* and *151* belongs to the group of Cayley-Hamilton relations CH2, CH21, CH52, CH54 and CH55, as explicit decomposition verifies. Therefore, one of these six relations has been missed. If

$$\mathit{151} \equiv 2\times(31) + (35) - 2\times(45) = 0 \tag{3.67}$$

is selected as the missed relation, then it is obvious that the monomial $Y_3^{(2)} \equiv (35)$ is not independent. Its explicit shape in accord to the applied elimination paradigm is

$$\begin{gathered}(35) = \\ -(40) + (44) + 4\times(45) + 2\times(79) + 2\times(85) - 8\times(107) \\ -4\times(111) + (120) - (122) + \tfrac{1}{3}\times(123) - 2\times(131) \\ +\tfrac{2}{3}\times(133) - 2\times(140) - 10\times(145) - (164) - 2\times(165) \\ +4\times(169) + 12\times(171) - 12\times(173) + 24\times(174) \\ -2\times(175) + 4\times(176) - 18\times(177) + 5\times(189) - 2\times(190).\end{gathered} \tag{3.68}$$

The group of relations *179* – *188* connects the monomials (86) – (92).

Among these, six relations are independent after application of the elimination paradigm. The relation

$$\mathit{186} \equiv (87) - (88) + (89) - (92) = 0 \tag{3.69}$$

is constructed by the choice $A_1 = u_\alpha$, $A_2 = u_\beta$ and $A_3 = [u^\alpha, u_\gamma] f_+^{\beta\gamma}$ in part one of Eq. (3.37), but does not appear in the list of Cayley-Hamilton relations in [BCE 00]. Therefore, another monomial of [BCE 00] is not independent. The monomial $Y_{28}^{(2)} \equiv (89)$ has the explicit shape

$$\begin{aligned}(89) =\\ \tfrac{1}{2}\Big((101) + (110) + (140) - (151) + (154) + (171) - 4 \times (172)\\ -4 \times (174) + 2 \times (176) - 4 \times (177) - (190) + 4 \times (191)\Big)\end{aligned} \tag{3.70}$$

with respect to the above elimination paradigm. Thus, the number of independent two-flavour-LECs is reduced by two. As the Cayley-Hamilton relations require a decomposition, it is not clear, if both relations identified as missing are indeed the relations, which were missed in [BCE 00].

Comparison of LECs of both Lagrangians

As a means of translation of LECs from one Lagrangian to the LECs from another, which has been derived via a different elimination paradigm, all independent relations have to be taken into account. The expressions for the eliminated monomials are substituted into one Lagrangian. Afterwards, the terms are sorted and reorganized. The Eqs. (3.68) and (3.70) demonstrate, that the resulting linear combinations are very lengthy and messy. Giving the explicit expressions for the dependent monomials would take many pages without giving further insight, so they do not appear here. The mathematica source code used for this purpose in the anomalous sector is given in the appendix A.4.

3.6 Anomalous next-to-next-to-leading-order Lagrangian $\mathcal{L}_6^\epsilon$

The anomalous sector is necessary for the renormalization of loops containing anomalous interaction vertices from the Wess-Zumino-Witten action [WZ 71, Wit 83]. That is why it is called anomalous sector. The specific properties of any monomial in the anomalous-sector are

1. contraction of the tensor structure with an ϵ tensor (due to this ϵ tensor, the anomalous sector is often called ϵ sector),

2. odd number of Goldstone bosons in every monomial at every step of expansion for purely strong or electromagnetic processes: odd intrinsic parity,

3. even parity.

The number of monomials is significantly smaller than in the even sector, therefore the chance of finding (in principle) measurable processes, which allow determination of single LECs is very good [Hac 08]. In the two-flavour case, it is inevitable to treat the $U(1)_V$ part of the chiral group and the related external fields as well. This construction process has been done earlier in [BGT 02, EFS 02]. The Lagrangian is compared to both of these previous works.

This section is organized similarly to section 3.5:

1. the complete list of a priori independent monomials is presented,

2. the complete list of all derived relations is summarized for general N_f and for three- and two-flavour cases,

3. the list is reduced to a maximal set of independent relations,

4. the list of monomials is reduced to a list of independent monomials,

5. the $U(1)_V$ part is discussed,

6. a comparison with the results of [BGT 02, EFS 02] is performed.

Additional segments of the source code used in comparing the Lagrangians are presented in the appendix A.4.

3.6.1 Monomials in the anomalous sector at chiral order six

The monomials of the anomalous sector are listed in table 3.23.

Table 3.23: Monomials at chiral order six in the anomalous sector

monomial	shape		contributes to	# of flavours
(1)	$\langle u^\alpha u^\beta u^\gamma u^\delta h^{\mu\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	5ϕ	
(2)	$\langle [u^\mu, u^\alpha u^\beta u^\gamma] h^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	5ϕ	3
(3)	$\langle u^\alpha [u^\mu, u^\beta] u^\gamma h^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	5ϕ	
(4)	$i\langle u^\alpha u^\beta u^\gamma u^\delta \chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	5ϕ	3
(5)	$\langle [u^\mu, u^\alpha u^\beta u^\gamma] f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$5\phi+\gamma$	3
(6)	$\langle u^\alpha [u^\mu, u^\beta] u^\gamma f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$5\phi+\gamma$	
(7)	$\langle [u^\mu u^\nu, u^\alpha u^\beta] f_-^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$5\phi+\gamma$	3
(8)	$\langle \{u^\mu u^\alpha u^\nu, u^\beta\} f_-^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$5\phi+\gamma$	
(9)	$\langle u^\mu u^\alpha u^\nu\rangle\langle u^\beta f_-^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$5\phi+\gamma$	3

monomial	shape		contributes to	# of flavours
(10)	$\langle u^\mu u^\alpha\rangle\langle\{u^\beta, u^\nu\} f_-^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$5\phi+\gamma$	N_f
(11)	$i\langle\{u^\alpha u^\beta, h^{\mu\nu}\} f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(12)	$i\langle[u^\mu, u^\alpha, h^{\beta\nu}] f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	3
(13)	$i\langle[u^\alpha, u^\mu, h^{\beta\nu}] f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(14)	$i\langle\{u^\alpha u^\beta, h^{\mu\gamma}\} f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	3
(15)	$i\langle u^\alpha h^{\mu\nu} u^\beta f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(16)	$i\langle[u^\mu, h^{\alpha\nu}, u^\beta] f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	3
(17)	$i\langle u^\alpha h^{\mu\beta} u^\gamma f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(18)	$i\langle u^\alpha u^\beta u^\gamma \nabla^\mu f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(19)	$i\langle\{u^\mu, u^\alpha u^\beta\}\nabla^\gamma f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(20)	$i\langle u^\alpha u^\mu u^\beta \nabla^\gamma f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(21)	$i\langle\{u^\mu, u^\alpha u^\beta\}\nabla^\nu f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(22)	$i\langle u^\alpha u^\mu u^\beta \nabla^\nu f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(23)	$i\langle[u^\mu u^\nu, u^\alpha]\nabla^\beta f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(24)	$\langle\{u^\alpha, h^{\mu\nu}\}\nabla^\beta f_-^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(25)	$\langle\{u^\mu, h^{\alpha\nu}\}\nabla^\beta f_-^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(26)	$\langle\{u^\alpha, h^{\mu\beta}\}\nabla^\nu f_-^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(27)	$\langle\{u^\alpha, h^{\mu\beta}\}\nabla^\gamma f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(28)	$\langle\{u^\alpha u^\beta, f_+^{\gamma\delta}\}\chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	3
(29)	$\langle u^\alpha f_+^{\beta\gamma} u^\delta \chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	3
(30)	$\langle u^\alpha u^\beta f_+^{\gamma\delta}\rangle\langle\chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	2
(31)	$\langle[u^\alpha u^\beta, f_-^{\gamma\delta}]\chi_+\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	3
(32)	$\langle u^\alpha f_-^{\beta\gamma}\rangle\langle u^\delta \chi_+\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	2
(33)	$i\langle[u^\alpha, \nabla^\beta f_+^{\gamma\delta}]\chi_+\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(34)	$i\langle[u^\alpha, f_+^{\beta\gamma}]\nabla^\delta \chi_+\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(35)	$i\langle\{u^\alpha, \nabla^\beta f_-^{\gamma\delta}\}\chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(36)	$i\langle\{u^\alpha, f_-^{\beta\gamma}\}\nabla^\delta \chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(37)	$i\langle u^\alpha \nabla^\beta f_-^{\gamma\delta}\rangle\langle\chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(38)	$i\langle u^\alpha f_-^{\beta\gamma}\rangle\langle\nabla^\delta \chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(39)	$\langle\nabla^\alpha f_-^{\beta\gamma}\nabla^\delta \chi_+\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+\gamma$	
(40)	$i\langle[u^\mu, u^\alpha, f_+^{\beta\gamma}] f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(41)	$i\langle[u^\alpha, u^\mu, f_+^{\beta\gamma}] f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(42)	$i\langle u^\alpha u^\beta\{f_+^{\mu\gamma}, f_-^{\delta\nu}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	3
(43)	$i\langle[u^\mu, u^\alpha, f_-^{\beta\gamma}] f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(44)	$i\langle[u^\alpha, u^\mu, f_-^{\beta\gamma}] f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	3
(45)	$i\langle u^\mu u^\nu [f_-^{\alpha\beta}, f_+^{\gamma\delta}]\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(46)	$i\langle[u^\mu, f_+^{\alpha\beta}, u^\gamma] f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(47)	$i\langle u^\alpha f_+^{\mu\beta} u^\gamma f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(48)	$i\langle[u^\mu, f_-^{\alpha\beta}, u^\gamma] f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	3

monomial	shape		contributes to	# of flavours
(49)	$\langle u^{\mu}\{\nabla^{\nu} f_{+}^{\alpha\beta}, f_{+}^{\gamma\delta}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$\phi+2\gamma$	
(50)	$\langle u^{\mu}\{\nabla^{\alpha} f_{+}^{\beta\nu}, f_{+}^{\gamma\delta}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$\phi+2\gamma$	
(51)	$\langle u^{\mu}\{\nabla^{\alpha} f_{+}^{\beta\gamma}, f_{+}^{\delta\nu}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$\phi+2\gamma$	
(52)	$\langle u^{\alpha}\{\nabla^{\mu} f_{+}^{\beta\nu}, f_{+}^{\gamma\delta}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$\phi+2\gamma$	3
(53)	$\langle u^{\alpha}\{\nabla^{\mu} f_{+}^{\beta\gamma}, f_{+}^{\delta\nu}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$\phi+2\gamma$	
(54)	$\langle u^{\alpha}\{\nabla^{\beta} f_{+}^{\mu\gamma}, f_{+}^{\delta\nu}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$\phi+2\gamma$	
(55)	$\langle u^{\mu}\{\nabla^{\nu} f_{-}^{\alpha\beta}, f_{-}^{\gamma\delta}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(56)	$\langle u^{\mu}\{\nabla^{\alpha} f_{-}^{\beta\nu}, f_{-}^{\gamma\delta}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(57)	$\langle u^{\mu}\{\nabla^{\alpha} f_{-}^{\beta\gamma}, f_{-}^{\delta\nu}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(58)	$\langle u^{\alpha}\{\nabla^{\mu} f_{-}^{\beta\nu}, f_{-}^{\gamma\delta}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	3
(59)	$\langle u^{\alpha}\{\nabla^{\mu} f_{-}^{\beta\gamma}, f_{-}^{\delta\nu}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(60)	$\langle u^{\alpha}\{\nabla^{\beta} f_{-}^{\mu\gamma}, f_{-}^{\delta\nu}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	
(61)	$i\langle[f_{+}^{\alpha\beta}, f_{-}^{\gamma\delta}]\chi_{+}\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$\phi+2\gamma$	2
(62)	$i\langle f_{+}^{\alpha\beta} f_{+}^{\gamma\delta}\chi_{-}\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$\phi+2\gamma$	3
(63)	$i\langle f_{+}^{\alpha\beta} f_{+}^{\gamma\delta}\rangle\langle\chi_{-}\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$\phi+2\gamma$	2
(64)	$i\langle f_{-}^{\alpha\beta} f_{-}^{\gamma\delta}\chi_{-}\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	3
(65)	$i\langle f_{-}^{\alpha\beta} f_{-}^{\gamma\delta}\rangle\langle\chi_{-}\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$	$3\phi+2\gamma$	2
(66)	$\langle f_{-}^{\mu\alpha}\{f_{+}^{\beta\gamma}, f_{+}^{\delta\nu}\}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$	$\phi+3\gamma$	

3.6.2 Relations in the anomalous sector at chiral order six

The list of relations is organzized in accord with their derivation technique.

Table 3.24: ϵ relations

relation	shape
1	$(1)+(2)-(3)=0$
2	$(5)-(6)=0$
3	$2\times(6)+(7)-(8)=0$
4	$(11)+(12)-(13)+2\times(14)=0$
5	$(15)-(16)-2\times(17)=0$
6	$(18)-(19)+(20)=0$
7	$2\times(18)+(21)-(22)=0$
8	$2\times(20)+(22)+(23)=0$
9	$(24)-(25)-(26)-2\times(27)=0$
10	$(40)-(41)+2\times(42)=0$
11	$-(40)-(41)+(43)+(44)+(45)=0$
12	$-2\times(42)+(43)-(44)=0$
13	$(46)-2\times(47)=0$

relation	shape
14	$-2 \times (47) + (48) = 0$
15	$(49) + 2 \times (52) + 2 \times (53) = 0$
16	$(50) - (52) - 2 \times (54) = 0$
17	$(51) - (53) + 2 \times (54) = 0$
18	$(55) + 2 \times (58) + 2 \times (59) = 0$
19	$(56) - (58) - 2 \times (60) = 0$
20	$(57) - (59) + 2 \times (60) = 0$
21	$(66) = 0$

Table 3.25: Bianchi identities in the anomalous sector

relation	shape
22	$-(5) + (6) + \frac{1}{2} \times (7) + 2 \times (19) + (21) = 0$
23	$-(6) + \frac{1}{2} \times (8) + 2 \times (20) + (22) = 0$
24	$\frac{1}{2} \times (7) - (23) = 0$
25	$\frac{1}{2} \times (11) + (15) + (24) = 0$
26	$\frac{1}{2} \times (13) + \frac{1}{2} \times (16) + (25) = 0$
27	$-\frac{1}{2} \times (12) - (14) + \frac{1}{2} \times (16) + 2 \times (17) + (26) + 2 \times (27) = 0$
28	$-\frac{1}{2} \times (28) + (29) + (35) = 0$
29	$-(30) + (37) = 0$
30	$-\frac{1}{2} \times (31) + (33) = 0$
31	$\frac{1}{2} \times (34) + (39) = 0$
32	$\frac{1}{2} \times (40) - \frac{1}{2} \times (46) + (57) = 0$
33	$\frac{1}{2} \times (41) - (42) + \frac{1}{2} \times (46) - 2 \times (47) + (59) - 2 \times (60) = 0$
34	$-(41) + \frac{1}{2} \times (45) + (46) + (49) + 2 \times (50) = 0$
35	$(42) + \frac{1}{2} \times (44) - 2 \times (47) + \frac{1}{2} \times (48) + (53) - 2 \times (54) = 0$
36	$\frac{1}{2} \times (43) - \frac{1}{2} \times (48) + (51) = 0$
37	$-(44) - \frac{1}{2} \times (45) + (48) + (55) + 2 \times (56) = 0$

Table 3.26: Partial-integration-induced relations in the anomalous sector

relation	shape
38	$(1) + (2) - (3) + (5) - (6) = 0$
39	$(5) + (6) + 2 \times (27) - (41) - (46) + 2 \times (56) + 2 \times (57) + 2 \times (60) = 0$
40	$\frac{1}{2} \times (5) + \frac{1}{2} \times (6) + (27) - (42) - 2 \times (47) + (56) + (57) + (59) - (60) = 0$
41	$\frac{1}{2} \times (7) - (8) - (24) + (26) + (43) + (48) + 2 \times (55) + (56) + (57) + (58) = 0$
42	$\frac{1}{2} \times (7) + (25) + \frac{1}{2} \times (45) + (57) = 0$

relation	shape
43	$(8) + (24) + (25) - (26) - (43) - (44) - (55) - (57) + (59) = 0$
44	$(11) + (12) + (16) + 2 \times (21) - (41) + (46) = 0$
45	$(13) + (15) + 2 \times (22) - (40) = 0$
46	$(13) + (16) - 2 \times (23) + (40) - (45) - (46) = 0$
47	$(14) + (17) + 2 \times (18) + (42) - (47) = 0$
48	$(14) + 2 \times (19) - (42) + (43) - (48) = 0$
49	$(17) - 2 \times (20) - (44) + (47) = 0$
50	$(31) - 4 \times (39) - (61) = 0$
51	$(33) + (34) - \frac{1}{2} \times (61) = 0$
52	$(35) + (36) + (64) = 0$
53	$(37) + (38) + \frac{1}{2} \times (65) = 0$
54	$(50) + (51) - (52) - (53) + (66) = 0$
55	$(56) + (57) - (58) - (59) = 0$

Table 3.27: Leading-order-equation-of-motion-induced relations in the anomalous sector

relation	shape
56	$(1) - (4) = 0$
57	$(11) + (28) - \frac{2}{N_f} \times (30) = 0$
58	$(15) - (29) - \frac{1}{N_f} \times (30) = 0$
59	$-(24) + (35) - \frac{2}{N_f} \times (37) = 0$
60	$2 \times (49) + (62) - \frac{1}{N_f} \times (63) = 0$
61	$2 \times (55) + (64) - \frac{1}{N_f} \times (65) = 0$

Table 3.28: $SU(3)$ trace relation in the anomalous sector

relation	shape
62	$(7) + (8) - 2 \times (9) - (10) = 0$

Table 3.29: $SU(2)$ trace relations in the anomalous sector

relation	shape
63	$(1) = 0$
64	$(2) = 0$

relation	shape
65	$(2) - (3) = 0$
66	$(4) = 0$
67	$(5) = 0$
68	$(5) - (6) = 0$
69	$(7) - 2 \times (8) = 0$
70	$(7) + (8) - 2 \times (9) - (10) = 0$
71	$(7) - 2 \times (9) = 0$
72	$(7) - 2 \times (10) = 0$
73	$(8) - (9) = 0$
74	$(11) = 0$
75	$(11) - 2 \times (15) = 0$
76	$(12) - (13) = 0$
77	$(12) + (16) = 0$
78	$(13) - (16) = 0$
79	$(14) = 0$
80	$(14) + 2 \times (17) = 0$
81	$(18) = 0$
82	$(19) = 0$
83	$(19) - 2 \times (20) = 0$
84	$(21) = 0$
85	$(21) - 2 \times (22) = 0$
86	$(23) = 0$
87	$(24) = 0$
88	$(25) = 0$
89	$(26) = 0$
90	$(27) = 0$
91	$(28) + 2 \times (29) = 0$
92	$(28) - (30) = 0$
93	$(28) - 2 \times (29) - 2 \times (30) = 0$
94	$(31) + 2 \times (32) = 0$
95	$(35) - (37) = 0$
96	$(36) - (38) = 0$
97	$(40) - (41) = 0$
98	$(40) - (46) = 0$
99	$(41) + (46) = 0$
100	$(42) = 0$
101	$(42) + 2 \times (47) = 0$
102	$(43) - (44) = 0$
103	$(43) - (484) = 0$

relation	shape
104	$(44)+(48)=0$
105	$(45)=0$
106	$(49)=0$
107	$(50)=0$
108	$(51)=0$
109	$(52)=0$
110	$(53)=0$
111	$(54)=0$
112	$(55)=0$
113	$(56)=0$
114	$(57)=0$
115	$(58)=0$
116	$(59)=0$
117	$(60)=0$
118	$2\times(62)-(63)=0$
119	$2\times(64)-(65)=0$
120	$(66)=0$

3.6.3 Reduction to maximal set of independent relations

The set of 66 monomials is not mutually independent, but connected by the set of 120 relations. The reduction scheme is the same as in section 3.5.3. According to the applied scheme, the following relations are dropped:

Table 3.30: Linearly dependent relations in the anomalous sector

# of flavours	eliminated relations
N_f	*23, 24, 27, 33, 35, 36, 37, 38, 40, 42, 43, 46, 47, 48, 49, 51, 54, 55, 59*
2	..., *62, 65, 66, 68, 72, 73, 79, 80, 83, 84, 85, 86, 87, 88, 90, 91, 92, 93, 95, 98, 99, 100, 101, 102, 103, 104, 105, 107, 108, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120*

The number of independent monomials is therefore 24 for an arbitrary number of flavours, 23 for three flavours[22] and 5 for two flavours.

[22]The one new relation in the three-flavour case is independent of the relations for general N_f. Therefore, $N_f = 3$ does not appear in table 3.30.

3.6.4 Reduction to minimal set of independent monomials

Since the number of Goldstone bosons is odd, no contact terms exist in the anomalous sector. The set of kept monomials of [BGT 02, EFS 02] are very similar. This work does not adapt to their choices in the two-flavour case, because four monomials are preferable as multiple trace monomials[23]. The differences are discussed in section 3.6.6. The chosen independent monomials are marked in table 3.23.

3.6.5 Extension of the chiral group in the anomalous sector

The extension of the chiral group is very important in the two-flavour case of the anomalous sector. Here, the entire set of counterterms absorbing infinities from loops due to the WZW action must be taken from the extension of the chiral group. Due to the contraction with an antisymmetric ϵ tensor, the variety of non-vanishing monomials containing $U(1)_V$ field strength tensors is much greater than in the even sector. The set of relations is richer than in the even sector, too.

The set of monomials, the list of relations and its reduction to a maximal set are presented. Afterwards, replacement rules for the electromagnetic case are discussed.

Monomials and symmetry relations

Table 3.31: Monomials of the extended chiral group in the anomalous sector

monomial	shape	
(1x)	$if^{\alpha\beta}\langle u^\gamma u^\delta h^{\mu\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(2x)	$if^{\alpha\beta}\langle [u^\mu, u^\gamma] h^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(3x)	$if^{\mu\alpha}\langle u^\beta u^\gamma h^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(4x)	$\nabla^\mu f^{\alpha\nu}\langle u^\beta u^\gamma u^\delta\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(5x)	$\nabla^\mu f^{\alpha\beta}\langle u^\gamma u^\delta u^\nu\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(6x)	$\nabla^\alpha f^{\mu\beta}\langle u^\gamma u^\delta u^\nu\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(7x)	$f^{\alpha\beta}\langle u^\gamma u^\delta \chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$
(8x)	$f^{\alpha\beta}\langle h^{\mu\nu} f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(9x)	$f^{\alpha\beta}\langle h^{\mu\gamma} f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(10x)	$f^{\mu\alpha}\langle h^{\beta\nu} f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(11x)	$\nabla^\mu f^{\alpha\beta}\langle u^\nu f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(12x)	$\nabla^\mu f^{\alpha\beta}\langle u^\gamma f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(13x)	$\nabla^\mu f^{\alpha\nu}\langle u^\beta f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(14x)	$\nabla^\alpha f^{\beta\gamma}\langle u^\mu f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$

[23] Two of them are connected to two monomials with flavour singlet field strength tensors in the electromagnetic case.

monomial	shape	
(15x)	$\nabla^\alpha f^{\mu\beta}\langle u^\nu f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(16x)	$\nabla^\alpha f^{\mu\beta}\langle u^\gamma f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(17x)	$f^{\alpha\beta}\langle u^\mu \nabla^\nu f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(18x)	$f^{\alpha\beta}\langle u^\mu \nabla^\gamma f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(19x)	$f^{\alpha\beta}\langle u^\gamma \nabla^\mu f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(20x)	$f^{\mu\alpha}\langle u^\nu \nabla^\beta f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(21x)	$f^{\mu\alpha}\langle u^\beta \nabla^\nu f_+^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(22x)	$f^{\mu\alpha}\langle u^\beta \nabla^\gamma f_+^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(23x)	$if^{\alpha\beta}\langle [u^\mu, u^\gamma] f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(24x)	$if^{\mu\alpha}\langle u^\beta u^\gamma f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(25x)	$if^{\mu\alpha}\langle [u^\beta, u^\nu] f_-^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(26x)	$if^{\alpha\beta}\langle f_+^{\gamma\delta}\chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$
(27x)	$f^{\mu\alpha}\langle f_+^{\beta\nu} f_-^{\gamma\delta}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(28x)	$f^{\mu\alpha}\langle f_+^{\beta\gamma} f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(29x)	$f^{\alpha\beta}\langle f_+^{\mu\gamma} f_-^{\delta\nu}\rangle$	$g_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}$
(30x)	$if^{\alpha\beta} f^{\gamma\delta}\langle \chi_-\rangle$	$\epsilon_{\alpha\beta\gamma\delta}$

The relations are collected in table 3.32.

Table 3.32: Relations of the extended chiral group in the anomalous sector

relation	shape
	ϵ relations
1x	$(1x) + (2x) + 2 \times (3x) = 0$
2x	$(4x) - 3 \times (6x) = 0$
3x	$(5x) - 2 \times (6x) = 0$
4x	$(8x) + 2 \times (9x) + 2 \times (10x) = 0$
5x	$(11x) + 2 \times (14x) - 2 \times (15x) = 0$
6x	$(12x) - (14x) - 2 \times (16x) = 0$
7x	$(13x) - (15x) - 2 \times (16x) = 0$
8x	$(17x) + 2 \times (18x) + 2 \times (20x) = 0$
9x	$(17x) + 2 \times (19x) + 2 \times (21x) = 0$
10x	$(18x) - (19x) + 2 \times (22x) = 0$
11x	$(23x) + 2 \times (24x) = 0$
12x	$(23x) - (25x) = 0$
13x	$(27x) - (29x) = 0$
14x	$(28x) + (29x) = 0$

relation	shape
	Bianchi identities
15x	$(5x) - 2 \times (6x) = 0$
16x	$(11x) - 2 \times (15x) = 0$
17x	$(12x) - 2 \times (16x) = 0$
18x	$(14x) = 0$
19x	$(17x) + 2 \times (18x) + (23x) = 0$
20x	$(20x) - \frac{1}{2} \times (25x) = 0$
21x	$(21x) + 2 \times (22x) + 2 \times (24x) + \frac{1}{2} \times (25x) = 0$
	Partial-integration-induced relations
22x	$(1x) + (2x) + 2 \times (5x) + (23x) = 0$
23x	$3 \times (3x) - 2 \times (4x) + 3 \times (24x) = 0$
24x	$-(3x) + 2 \times (6x) + (24x) + (25x) = 0$
25x	$(8x) + 2 \times (11x) + 2 \times (17x) = 0$
26x	$(9x) + 2 \times (12x) + 2 \times (19x) + (29x) = 0$
27x	$(9x) + 2 \times (14x) + 2 \times (18x) - (29x) = 0$
28x	$(10x) - 2 \times (13x) + 2 \times (21x) + (28x) = 0$
29x	$(10x) - 2 \times (15x) + 2 \times (20x) - (28x) = 0$
30x	$(16x) + (22x) + \frac{1}{2} \times (27x) = 0$
	Leading-order-equation-of-motion-induced relations
31x	$(1x) + (7x) = 0$
32x	$(8x) - (26x) = 0$

The following non-independent relations are eliminated:

1x, *4x*, *5x*, *6x*, *15x*, *19x*, *21x*, *24x*, *27* and *29x*.

Thus, a valid minimal set of independent monomials consists of

$(4x)$, $(7x)$, $(13x)$, $(19x)$, $(24x)$, $(26x)$, $(29x)$ and $(30x)$.

Replacement rules in the electromagnetic case

Under the same conditions like in the even sector (section 3.5.5), the replacement rules for the electromagnetic case can be formulated:

$$\begin{aligned} (30x) &= \frac{1}{8} \times (63), \\ (7x) &= \frac{1}{2\sqrt{2N_f}} \frac{\chi_3}{\chi_0} \times (30), \\ (26x) &= \frac{1}{2\sqrt{2N_f}} \frac{\chi_3}{\chi_0} \times (63), \end{aligned} \tag{3.71}$$

Thus, two isospin-breaking corrections arise and one LEC is fixed in the electromagnetic case.

3.6.6 Comparison to the anomalous Lagrangians of [BGT 02, EFS 02]

The motivation of this thesis was that no consistent rule for translation between the anomalous Lagrangians of [BGT 02, EFS 02] existed. They contain different sets of LECs due to use of different bases and elimination schemes. A comparison of both results with this work follows hereafter. The Lagrangian of [BGT 02] is investigated in table 3.33 Specialization to $N_f = 3$ relates O_{24}^W to O_{17}^W and O_{18}^W (relation *62*).

Table 3.33: operators of [BGT 02]

$SU(N_f)$case	$SU(2)$case	structure
O_1^W		(4)
O_2^W	o_1^W	$-(31)$
O_3^W		$-(32)$
O_4^W	o_2^W	(28)
O_5^W		(29)
O_6^W		(30)
O_7^W	o_3^W	(62)
O_8^W		(63)
O_9^W	o_4^W	(64)
O_{10}^W		(65)
O_{11}^W	o_5^W	(61)
O_{12}^W		(2)
O_{13}^W		$-(14)$
O_{14}^W		$-(12)$
O_{15}^W		$-(16)$
O_{16}^W		$-(5)$
O_{17}^W		(7)
O_{18}^W		$-(9)$
O_{19}^W		$-(42)$
O_{20}^W		$-(44)$
O_{21}^W		(48)
O_{22}^W		$-(52)$
O_{23}^W		$-(58)$
O_{24}^W		$-(10)$
extension of the chiral group		
	o_6^W	$(7x)$
	o_7^W	$(26x)$
	o_8^W	$(30x)$
	o_9^W	$-(3x) = -\frac{2}{3} \times (4x) + (24x)$
	o_{10}^W	$-(24x)$

$SU(N_f)$case	$SU(2)$case	structure
	o^W_{11}	$(9x) = -(13x) - (19x) - (24x) - \frac{1}{4} \times (26x)$
	o^W_{12}	$-(29x)$
	o^W_{13}	$(13x)$

A comparison to [EFS 02] is more tedious, because the bases are different. The main effect is multiplication with powers of 2 in accordance with table 2.4. The monomials of [EFS 02] are indicated here by small characters l with the indices of the corresponding LECs, which are denoted with large characters L.

Table 3.34: monomials of [EFS 02]

monomial of [EFS 02]	structure
$l_1^{6,\epsilon}$	$-8 \times (31)$
$l_2^{6,\epsilon}$	$-8 \times (32)$
$l_3^{6,\epsilon}$	$-4 \times (61)$
$l_4^{6,\epsilon}$	$16 \times (4)$
$l_5^{6,\epsilon}$	$-8 \times (28)$
$l_6^{6,\epsilon}$	$-8 \times (29)$
$l_7^{6,\epsilon}$	$-8 \times (30)$
$l_8^{6,\epsilon}$	$4 \times (62)$
$l_9^{6,\epsilon}$	$4 \times (63)$
$l_{10}^{6,\epsilon}$	$4 \times (64)$
$l_{11}^{6,\epsilon}$	$4 \times (65)$
$l_{12}^{6,\epsilon}$	$-16 \times (2)$
$l_{13}^{6,\epsilon}$	$-8 \times (12)$
$l_{14}^{6,\epsilon}$	$-8 \times (13) = -8 \times \Big((12) + 2 \times (14) - (28) + \frac{2}{3} \times (30)\Big)$
$l_{15}^{6,\epsilon}$	$-32 \times (8) = -32 \times \Big(2 \times (5) + (7)\Big)$
$l_{16}^{6,\epsilon}$	$-32 \times (7)$
$l_{17}^{6,\epsilon}$	$32 \times (10) = 64 \times \Big((5) + (7) - (9)\Big)$
$l_{18}^{6,\epsilon}$	$32 \times (9)$
$l_{19}^{6,\epsilon}$	$8 \times (52)$
$l_{20}^{6,\epsilon}$	$8 \times (58)$
$l_{21}^{6,\epsilon}$	$-16 \times (42)$
$l_{22}^{6,\epsilon}$	$-16 \times (43) = -16 \times \Big(2 \times (42) + (44)\Big)$
$l_{23}^{6,\epsilon}$	$-16 \times (45) =$
	$32 \times \Big((7) - (12) - 2 \times (14) - (16) + (28) - \frac{2}{3} \times (30) - (44) + (48)\Big)$
$l_{24}^{6,\epsilon}$	$16 \times (47) = 8 \times (48)$

monomial of [EFS 02]	structure
extension of the chiral group	
$l^{6,\epsilon}_{1'}$	$-4 \times (7x)$
$l^{6,\epsilon}_{2'}$	$2 \times (26x)$
$l^{6,\epsilon}_{3'}$	$(30x)$
$l^{6,\epsilon}_{4'}$	$-8 \times (4x)$
$l^{6,\epsilon}_{5'}$	$4 \times (19x)$
$l^{6,\epsilon}_{6'}$	$-4 \times (13x)$
$l^{6,\epsilon}_{7'}$	$-8 \times (24x)$
$l^{6,\epsilon}_{8'}$	$4 \times (29x)$

A comparison to the results of [BGT 02, EFS 02] shows that they are equivalent to each other and to the results of this work. The use of different bases generates powers of 2, whereas different ordering conventions generate signs.

In the two-flavour case, multiple traces were preferred in four cases:

$$\begin{aligned} (28) &\to (30), \\ (31) &\to (32), \\ (62) &\to (63), \\ (64) &\to (65). \end{aligned} \tag{3.72}$$

The correspondence of the two-flavour LECs is listed in table 3.35. The LECs due to flavour singlet interactions and the LECs of the three-flavour case are presented in the appendix A.3.

Table 3.35: Correspondence of LECs in the two-flavour case

this work	[BGT 02]	[EFS 02]
m_{30}	k^W_2	$4\ L^{6,\epsilon}_6$
m_{32}	$2\ k^W_1$	$16\ L^{6,\epsilon}_1$
m_{61}	$-k^W_5$	$-4\ L^{6,\epsilon}_3$
m_{63}	$\frac{1}{2}\ k^W_3$	$2\ L^{6,\epsilon}_8$
m_{65}	$\frac{1}{2}\ k^W_4$	$2\ L^{6,\epsilon}_{10}$

It is important to note that a transition from one minimal set of monomials to another set is non-trivial. Reformulation of one LEC from one basis to another may have impact on other LECs, too. The monomial $l_{14}^{6,\epsilon}$ illustrates this. The corresponding LEC $L_{14}^{6,\epsilon}$ is connected to the LEC m_{14} by

$$m_{14} = \; -16 \; L_{14}, \tag{3.73}$$

only if other LECs are also corrected[24]:

$$\begin{cases} m_{12} = & -8 \left(L_{13}^{6,\epsilon} + L_{14}^{6,\epsilon} \right), \\ m_{28} = & -8 \left(L_{5}^{6,\epsilon} - L_{14}^{6,\epsilon} \right), \\ m_{30} = & -8 \left(L_{7}^{6,\epsilon} + \frac{2}{3} \; L_{14}^{6,\epsilon} \right). \end{cases} \tag{3.74}$$

[24]Further complex dependencies of LECs between these sets exist. Eq. (3.74) is not complete; it is just an illustration!

Chapter 4

Axial anomaly, θ term and singlet η in ChPT

4.1 The $U(1)_A$ anomaly

4.1.1 The axial anomaly in QED

The following treatment of the axial anomaly is guided by the original work of Adler [Adl 69]. In a quantum field theory of fermionic fields minimally coupled to a gauge field (like QED[1]),

$$\mathcal{L}(x) = \overline{\psi}(x)\Big(i\partial\!\!\!/ - eA\!\!\!/(x) - m\Big)\psi(x) - \frac{1}{4}F_{\alpha\beta}(x)F^{\alpha\beta}(x), \tag{4.1}$$

the Ward identity for the axial-vector current

$$j^5_\mu(x) = \overline{\psi}(x)\gamma_\mu\gamma_5\psi(x) \tag{4.2}$$

is not satisfied. By use of the pseudoscalar density,

$$j^5(x) = \overline{\psi}(x)\gamma_5\psi(x), \tag{4.3}$$

the symmetry current of axial transformations

$$\psi(x) \mapsto e^{-i\kappa\gamma_5}\psi(x) \tag{4.4}$$

and the conservation law

$$\partial^\mu j^5_\mu(x) = 2imj^5(x) + \frac{\alpha_0}{4\pi}F^{\alpha\beta}(x)F^{\gamma\delta}(x)\epsilon_{\alpha\beta\gamma\delta} \tag{4.5}$$

are derived. This is a contradiction to the prediction of Noether's theorem,

$$\partial^\mu j^5_\mu(x) = 2imj^5(x). \tag{4.6}$$

[1] It is important to note that, since QED does not have any operator for a local axial-vector current, it is, in fact, anomaly free. But the mechanism of the anomaly is actually the same as in QCD.

The additional term is due to a fermionic triangle loop coupled to two vector currents and one axial-vector current and its crossing-symmetric counterpart (Fig. 4.1).

Figure 4.1: Triangle loop diagrams are responsible for the axial anomaly.

4.1.2 Transition to QCD – axial anomaly of the chiral group and θ term

The mechanism of section 4.1.1 works in almost exactly the same manner in QCD. The main differences are:

1. N_f quark flavours run around in the loop, thus the contribution to the axial-vector current is N_f times the QED contribution,
2. the photon current is replaced by the 8 possible gluon currents:
$$F^{\alpha\beta}(x)F^{\gamma\delta}(x) \mapsto 2\big\langle G^{\alpha\beta}(x)G^{\gamma\delta}(x)\big\rangle,$$
3. the coupling constant is replaced: $\alpha_0 \mapsto \frac{1}{N_c}\frac{g_3^2}{4\pi}$ and
4. the pseudoscalar mesons of ChPT provide local axial-vector current operators.

Therefore, the axial-vector current conservation law in QCD is modified due to the anomaly:

$$\partial^\mu j_\mu^5(x) = 2imj^5(x) + \frac{N_f}{N_c}\frac{g_3^2}{8\pi^2}\big\langle G^{\alpha\beta}(x)G^{\gamma\delta}(x)\big\rangle_c \epsilon_{\alpha\beta\gamma\delta}. \tag{4.7}$$

The so-called θ term is another structure, which can be included in the QCD Lagrangian. It contains the instanton[2] solutions of QCD, it is suppressed by $\frac{1}{N_c}$ and it is proportional to the vacuum angle θ_0:

$$\mathcal{L}_{\text{QCD}}^{\theta\text{-term}}(x) = \theta_0 \frac{g_3^2}{8\pi^2 N_c}\big\langle G^{\alpha\beta}(x)G^{\gamma\delta}(x)\big\rangle_c \epsilon_{\alpha\beta\gamma\delta}. \tag{4.8}$$

The θ term can be included in the external field formalism by introducing an external field $\theta(x)$ in place of θ_0. An expression for the transformation

[2]The instanton solutions are discussed in [BPST 75].

of the generating functional can be given explicitly, if the external field $\theta(x)$ transforms as (see [GL 85])

$$\theta(x) \mapsto \theta(x) - 2\theta_A. \tag{4.9}$$

4.2 The singlet η in ChPT

The inclusion of the singlet η (proportional to ϕ_0) at leading order in ChPT requires a revision of section 3.2. It has first been performed by several authors using different approaches (e.g. [VV 80, Wit 80, GL 85]). The elimination of (2) of table 3.9 is not possible anymore. In fact, only the ϕ_0 contribution survives:

$$\langle u_\alpha \rangle = -\frac{N_f}{N_c}\frac{1}{F_0}\partial_\alpha \phi_0. \tag{4.10}$$

Thus, the contribution of monomial (2) is

$$(2) = \frac{N_f^2}{N_c^2}\frac{1}{F_0^2}\partial_\alpha \phi_0 \partial^\alpha \phi_0, \tag{4.11}$$

with a LEC of order $\mathcal{O}(1)$ in the $\frac{1}{N_c}$-expansion. The incoming and outgoing meson fields couple to separate quark loops. Their colour and flavour content would be independent without an interaction. The interaction is due to quark-antiquark annihilation into at least two gluons, which are reabsorbed in the second loop. The corresponding coupling constants $\frac{g_3}{N_c}$ are responsible for the overall suppression by $\frac{1}{N_c}$.

On the other hand, the monomial (1) breaks up into a sum of separate octet- and singlet-contributions

$$(1) = \langle u_\alpha u^\alpha \rangle_{\text{octet}} + \frac{N_f}{N_c^2}\frac{1}{F_0^2}\partial_\alpha \phi_0 \partial^\alpha \phi_0. \tag{4.12}$$

The LEC of (1) is used in the usual form $m_1 = \frac{F_0^2}{4}$ and the LEC of (2) is taken proportional to it:

$$m_2 = \frac{F_0^2}{4}\frac{C}{N_c}. \tag{4.13}$$

That is why C is of order $\mathcal{O}(1)$, due to the usual way of large-N_c counting (see Eq. (2.91)). Since the monomial (1) splits up in the aforementioned way, the monomial (2) can be eliminated entirely, if a renormalization of the singlet η is performed:

$$\eta_1 = \sqrt{1 + \frac{N_f}{N_c}C}\sqrt{\frac{N_f}{2}}\frac{1}{N_c}\phi_0. \tag{4.14}$$

The new LEC C can be determined from the weak neutral current decay of the octet η and singlet η respectively. At leading chiral order (and at next-to-leading order in large-N_c counting), the Lagrangians for both decays are

$$\mathcal{L}^{\eta_1 Z^0} = +\tfrac{F_0}{2}(1+\tfrac{N_f}{N_c}\tfrac{C}{2})\tfrac{1}{\sqrt{2N_f}}\partial^\alpha\eta_1\tfrac{g}{\cos\vartheta_W}Z_\alpha, \quad (4.15)$$

$$\mathcal{L}^{\eta_8 Z^0} = +\tfrac{F_0}{2}\partial^\alpha\eta_8\tfrac{g}{\cos\vartheta_W}Z_\alpha. \quad (4.16)$$

The external field configuration modeling the weak neutral current in the three-flavour case is derived in the appendix B.1.

The decay constants of the nonet are identical at leading order in *both* expansions [GL 85, Sch 03]. Decay constants are investigated in terms of weak decays. Only the monomial (1) describes weak interactions at leading order. The explicit symmetry breaking due to quark masses does not affect these decay constants at chiral order two. Because the singlet η has a correction at next-to-leading order (Eq. (4.14)) in the $\frac{1}{N_c}$ expansion, the singlet decay constant is shifted.

Using the physical values for F_1 and F_8 generates an error which is of higher chiral order. Thus, at chiral order two, the identifications

$$\begin{aligned} F_0(1+\frac{N_f}{N_c}\frac{C}{2})\frac{1}{\sqrt{2N_f}} &= F_1, \\ F_0 &= F_8, \end{aligned}$$

are correct up to next-to-leading order in large-N_c counting. The LEC C is determined as

$$C = \frac{N_c}{N_f}2(\sqrt{2N_f}\frac{F_1}{F_8}-1). \quad (4.17)$$

The decay constants[3] are taken from [KSD 96]:

$$F_\pi = 93\,\mathrm{MeV}, \quad F_8 = 1.25\,F_\pi, \quad F_1 = 1.06\,F_\pi, \quad (4.18)$$

and the numerical value of the new LEC is determined:

$$C = 0.718\,N_c = 2.154. \quad (4.19)$$

4.3 Implementation of the anomaly and of the θ term in chiral perturbation theory

Up to this point, the leading-order chiral Lagrangian is

$$\mathcal{L}_2 = \frac{F_0^2}{4}(\langle u_\alpha u^\alpha\rangle + \langle\chi_+\rangle). \quad (4.20)$$

To incorporate the anomaly and the θ term into the leading-order chiral Lagrangian requires the introduction of a structure which

[3]Since these are experimental values, they are based on the physical value $N_c = 3$.

1. transforms linearly[4] in the parameter θ_A of an axial transformation,
2. preserves all other symmetries of the Lagrangian,
3. is suppressed by $\frac{1}{N_c}$,
4. generates an additional mass term for the singlet η with the correct N_c dependence (see [Wit 79b]),
5. but does not contain arbitrary higher powers of the singlet η.

All of these criteria are satisfied by the Lagrangian[5]

$$\mathcal{L}_{U(1)_A} = F_0^2 \frac{a}{N_c} (\log \det (u))^2 = -\frac{a}{N_c} \frac{3}{2} \eta_1 \eta_1 . \tag{4.21}$$

The quantity a has dimension of mass^2 and is of order $\mathcal{O}(1)$ in large-N_c counting. The derivation of this Lagrangian by different methods is demonstrated in [VV 80, Wit 80].

Before the discussion of the axial symmetry breaking Lagrangian $\mathcal{L}_{U(1)_A}$ can be continued, another aspect of QCD must be discussed. The external field χ representing the quark mass matrix (times a constant) is, in general, a complex (3×3) matrix[6].

Performing suitable $SU(3)_L \times SU(3)_R \times U(1)_V$-transformations, it can be brought to diagonal form, though a phase may remain:

$$\chi^{\text{diagonal}} = e^{i \frac{\theta_m}{N_f}} \chi^{\text{real}} . \tag{4.22}$$

This phase can be removed from the mass term by an axial transformation of the quark fields. In chiral perturbation theory, the corresponding transformation of the Goldstone boson matrix

$$u(x) \mapsto \exp\left(-i \frac{\theta_m}{2N_f}\right) u(x) \tag{4.23}$$

eliminates the phase in the building block $\langle \chi_+ \rangle$. The same axial transformation generates two additional θ_m-dependent structures in the axial symmetry

[4] The transformation law produces a term quadratic in the parameter θ_A, too. However, in the case of global transformations, θ_A is only a constant and the quadratic term does not contribute to the dynamics.

[5] The instanton solutions can be included into the effective Lagrangian directly, but they can be eliminated in favour of the singlet η. In this case the vacuum angle θ_0 appears in the effective Lagrangian, too. This approach is demonstrated in [VV 80]. The angle θ_0 is incorporated in the corresponding way in Eq. (4.24).

[6] Because the quark masses do have their origin in the parity-violating electroweak sector, there is no a priori reason, why the quark mass matrix should be diagonal, real and positive in flavour space.

breaking Lagrangian $\mathcal{L}_{U(1)_A}$:

$$\mathcal{L}_{U(1)_A} \overset{\text{axial transf.}}{\mapsto} \mathcal{L}_{U(1)_A} + iF_0^2 \frac{a}{N_c}(\theta_0 + \theta_m) \log\det(u) - \frac{F_0^2}{4}\frac{a}{N_c}(\theta_0 + \theta_m)^2. \tag{4.24}$$

First, the Lagrangian obviously depends only on the sum of $\theta_0 + \theta_m$. Therefore, the vacuum angle is redefined as $\theta = \theta_0 + \theta_m$. The second new structure is an irrelevant constant[7]. The presence of the first new structure has profound effects. Due to the fact that it is linear in η_1, the vacuum expectation value of $u(x)$ cannot be the unit matrix anymore. It must be changed to absorb the linear term. Since χ^{real} is diagonal, the ansatz for the vacuum expectation value of u can be made as a diagonal matrix as well:

$$u_{\text{VEV}} = \text{diag}(e^{-\frac{i}{2}\varphi_u}, e^{-\frac{i}{2}\varphi_d}, e^{-\frac{i}{2}\varphi_s}). \tag{4.25}$$

Stationarity of the potential $\big(V(u) \equiv -\frac{F^2}{4}\ (3) - \mathcal{L}_{U(1)_A}\big)$ in u_{VEV} leads to the three Dashen-Nuyts equations [Das 71, Nuy 71, Wit 80]

$$\chi_f^{\text{real}} \sin\varphi_f = \frac{a}{N_c}(\theta - \sum_{j=u,d,s} \varphi_j), \quad f = u, d, s. \tag{4.26}$$

The Dashen-Nuyts equations are then inserted into the monomial (3), and u_{VEV} is eliminated:

$$\begin{aligned} & \tfrac{F^2}{4}\langle \chi_+^{\text{real}}\big(u_{\text{VEV}}\, u(x)\big)\rangle + \mathcal{L}_{U(1)_A}\big(u_{\text{VEV}}\, u(x)\big) \\ & +iF_0^2 \tfrac{a}{N_c}\theta\ \log\det(u_{\text{VEV}}\ u(x)) - \tfrac{F_0^2}{4}\tfrac{a}{N_c}\theta^2 \\ = \ & \tfrac{F^2}{4}\langle \chi_+^{\text{new}}\big(u(x)\big)\rangle + \mathcal{L}_{U(1)_A}\big(u(x)\big) \\ & -i\tfrac{F^2}{4}\tfrac{a}{N_c}\theta_{\text{new}}\Big(\langle u^2 - u^{\dagger 2}\rangle - 4\log\det\big(u(x)\big)\Big) \\ & -\tfrac{F^2}{4}\tfrac{a}{N_c}\theta_{\text{new}}^2. \end{aligned} \tag{4.27}$$

The θ parameter as well as the field χ[8] have been redefined[9]:

$$\theta_{\text{new}} \equiv \theta - \sum_{f=u,d,s} \varphi_f, \tag{4.28}$$

$$\chi^{\text{new}} \equiv \text{diag}(\chi_f^{\text{real}} \cos\varphi_f). \tag{4.29}$$

The structure linear in θ has been fine-tuned to cancel the contribution, which is linear in the fields:

$$\begin{aligned} \mathcal{L}_{PV} &= -i\tfrac{F^2}{4}\tfrac{a}{N_c}\theta\ \Big(\langle u^2 - u^{\dagger 2}\rangle - 4\log\det\big(u(x)\big)\Big) \\ &= -i\tfrac{F^2}{4}\tfrac{a}{N_c}\theta\ 2\sum_{j=1}^{\infty}\tfrac{1}{(2j+1)!}(\tfrac{i}{F})^{2j+1}\sum_{k=0}^{2j+1}(\tfrac{1}{N_c}\phi_0)^k\langle\phi_{\text{octet}}^{2j+1-k}\rangle. \end{aligned} \tag{4.30}$$

[7]Later, it is linearly shifted, but still constant.

[8]The relation to the scalar quark condensate of section 3.2.2 now has to be performed with the new field χ^{new} due to the necessary change of the vacuum.

[9]Of course, the indication *new* is omitted hereafter

This new anomalous interaction term contains only odd numbers of pseudoscalar meson fields and it violates parity[10]. It is of leading order $\mathcal{O}(N_c^{-3/2})$ in large-N_c counting. If parity violating processes are described in ChPT, then they must contain an odd number of vertices of this interaction. The final form of the Lagrangian at leading chiral order and at order $\mathcal{O}(N_c^{-3/2})$ in large-N_c counting is

$$\mathcal{L}(x) = \mathcal{L}_2[u(x)] + \mathcal{L}_{U(1)_A}[u(x)] + \mathcal{L}_{PV}[u(x)]. \tag{4.31}$$

4.4 η-η' mixing at next-to-leading order in large-N_c counting and anomalous mass

Two different approaches deal with the mixing of η_8 and η_1 to η and η'. The topic is far from trivial for multiple reasons:

- Since $SU(3)_V$ symmetry is broken explicitly by the difference between the strange quark and the up and down quark masses, the second-order chiral Lagrangian includes a term mixing octet and singlet.

- The main contribution to the mass of the singlet does not originate in the quark masses, but it is due to the anomaly. The anomalous mass term is of zeroth chiral order. Then the kinetic term of the singlet must be of the same order. Since there is no contribution of the octet at zeroth chiral order, there can be no mixing, before quark masses are inlcuded.

The standard approach is not interested in the underlying dynamics of the singlet and of the singlet-octet interaction. It does not care about power counting. The approach consistent with power counting is considered thereafter. The mixing angle Θ is expanded in both chiral orders as well as in $\frac{1}{N_c}$ up to next-to-leading order. The result is far off from the result of the naive mixing, but inclusion of the next chiral order generates a serious (and necessary) increase in consistency[11]. Furthermore, another order in large-N_c counting can be included.

Structure of the Goldstone boson mass matrix

A review of section 3.2.2 reveals that the field χ has components proportional to λ_0 and λ_8 in the isospin-symmetric limit[12] ($\hat{m} = m_u = m_d$):

[10] There are special values of θ which ensure parity conservation. This is discussed in chapter 5.

[11] The main reason for the deviation of the η mass from the value, which is predicted by the Gell-Mann–Okubo mass relation, is due to chiral logarithms [GL 85].

[12] If the isospin symmetric limit had not been taken, (due to a component proportional to λ_3) the π^0 would contribute to the mixing, too. This is not considered here.

$$\begin{aligned}\chi &= 2B_0\ \mathrm{diag}(\hat{m},\hat{m},m_s) \\ &= B_0\sqrt{\tfrac{2}{3}}(2\hat{m}+m_s)\lambda_0 + \tfrac{2B_0}{\sqrt{3}}(\hat{m}-m_s)\lambda_8 \\ &= \chi_0\ \lambda_0 + \chi_8\ \lambda_8. \end{aligned} \tag{4.32}$$

The component χ_8 proportional to λ_8 forces mixing of η_8 and η_1[13]. Because measurable particles require orthogonal states, the conclusion is inevitable that η_8 and η_1 are not the physical particles η and η'. The leading order quark mass contribution to the octet-singlet mass matrix is

$$\begin{pmatrix} M_{88}^2 & M_{18}^2 \\ M_{18}^2 & M_{11}^2 \end{pmatrix} = \begin{pmatrix} \left(\sqrt{\tfrac{2}{3}}\chi_0 - \tfrac{1}{\sqrt{3}}\chi_8\right) & \sqrt{\tfrac{2}{3}}\chi_8 \\ \sqrt{\tfrac{2}{3}}\chi_8 & \sqrt{\tfrac{2}{3}}\chi_0 \end{pmatrix}. \tag{4.33}$$

When the anomalous mass term (Eq. (4.21)) and the proper renormalisation of the singlet (Eq. (4.14)) are considered, the mass matrix is transformed into

$$\begin{pmatrix} M_{88}^2 & M_{18}^2 \\ M_{18}^2 & M_{11}^2 \end{pmatrix} = \begin{pmatrix} \left(\sqrt{\tfrac{2}{3}}\chi_0 - \tfrac{1}{\sqrt{3}}\chi_8\right) & \sqrt{\tfrac{2}{3}}\chi_8\left(1-\tfrac{3}{N_c}\tfrac{C}{2}\right) \\ \sqrt{\tfrac{2}{3}}\chi_8\left(1-\tfrac{3}{N_c}\tfrac{C}{2}\right) & \left(\sqrt{\tfrac{2}{3}}\chi_0 + \tfrac{3}{N_c}a\right)\left(1-\tfrac{3}{N_c}C\right) \end{pmatrix}. \tag{4.34}$$

An ansatz with a mixing angle Θ ensures orthogonality of the physical particles:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_1 \end{pmatrix}. \tag{4.35}$$

The Lagrangian is expanded in the fields, until it contains two powers in total of the fields η_8 and η_1[14]. Any dependence on Minkowski space-time coordinates is suppressed. The external fields r_α and l_α are set to zero, the field χ is identified with the Goldstone boson mass matrix (see section 3.2).

$$\begin{aligned}\mathcal{L}|^{\eta_8^2,\eta_1^2,\eta_8\eta_1} &= \frac{1}{2}\Big(\partial_\alpha\eta_8\partial^\alpha\eta_8 + \partial_\alpha\eta_1\partial^\alpha\eta_1\Big) \\ &- \frac{1}{2}\Big(M_{88}^2\,\eta_8^2 + M_{11}^2\,\eta_1^2 + 2M_{18}^2\,\eta_8\eta_1\Big). \end{aligned} \tag{4.36}$$

[13] The naive ansatz does not bother with the fact that the singlet has to be renormalized: $\eta_1^{\mathrm{naive}} = \sqrt{\frac{N_f}{2}\frac{1}{N_c}}$. This renormalization of the singlet is used in Eq. 4.33.

[14] Since fields appear in a total power of two, any decay constants are eliminated.

The mixing ansatz is inserted into the leading order Lagrangian:

$$\begin{aligned}
\mathcal{L}|^{\eta^2,\eta'^2,\eta\eta'} &= \frac{1}{2}\Big(\partial_\alpha\eta\partial^\alpha\eta + \partial_\alpha\eta'\partial^\alpha\eta'\Big) && (4.37)\\
&- \frac{1}{2}\Big(M_{88}^2\cos^2(\Theta) + M_{11}^2\sin^2(\Theta) - 2M_{18}^2\sin(2\Theta)\Big)\,\eta^2 && (4.38)\\
&- \frac{1}{2}\Big(M_{88}^2\sin^2(\Theta) + M_{11}^2\cos^2(\Theta) + 2M_{18}^2\sin(2\Theta)\Big)\,\eta'^2 && (4.39)\\
&- \frac{1}{2}\Big(2M_{18}^2\sin(2\Theta)\big[\frac{M_{88}^2 - M_{11}^2}{2M_{18}^2} + \frac{1}{\tan(2\Theta)}\big]\Big)\,\eta\eta'. && (4.40)
\end{aligned}$$

Even though the kinetic term is free from mixing for any value of Θ, mixing disappears from the mass term only for a fixed value of the angle Θ. At this spot, the methods differ.

Naive approach

The octet η is subject to the Gell-Mann–Okubo mass relations

$$\begin{aligned}
M_\pi^2 &= B_0\;2\hat{m} &&= \chi_0 + \tfrac{\chi_8}{\sqrt{3}},\\
M_K^2 &= B_0\;(\hat{m} + m_s) &&= \chi_0 - \tfrac{1}{2}\tfrac{\chi_8}{\sqrt{3}},\\
M_{\eta_8}^2 &= B_0\;\tfrac{2}{3}(\hat{m} + 2m_s) &&= \chi_0 - \tfrac{\chi_8}{\sqrt{3}}
\end{aligned} \qquad (4.41)$$

in the naive standard approach. The masses of the physical mesons are fixed (here taken from [PDG 2008]):

$$\begin{aligned}
M_{\pi^\pm} &= 139.6\,\text{MeV}, & M_{\pi^0} &= 135.0\,\text{MeV},\\
M_{K^\pm} &= 493.7\,\text{MeV}, & M_{K^0} &= 497.6\,\text{MeV},\\
M_\eta &= 547.5\,\text{MeV}, & M_{\eta'} &= 957.8\,\text{MeV},
\end{aligned} \qquad (4.42)$$

while pion and kaon masses are chosen as weighted averages,

$$\begin{aligned}
M_\pi &\equiv \tfrac{1}{3}\Big(M_{\pi^0} + 2M_{\pi^\pm}\Big) = 138.1\,\text{MeV},\\
M_K &\equiv \tfrac{1}{2}\Big(M_{K^\pm} + M_{K^0}\Big) = 495.7\,\text{MeV}.
\end{aligned} \qquad (4.43)$$

The three equations

$$\begin{aligned}
M_\eta^2 &= \Big(M_{88}^2\cos^2(\Theta) + M_{11}^2\sin^2(\Theta) - 2M_{18}^2\sin(2\Theta)\Big),\\
M_{\eta'}^2 &= \Big(M_{88}^2\sin^2(\Theta) + M_{11}^2\cos^2(\Theta) + 2M_{18}^2\sin(2\Theta)\Big),\\
0 &= \tfrac{M_{88}^2 - M_{11}^2}{2M_{18}^2} + \tfrac{1}{\tan(2\Theta)},
\end{aligned} \qquad (4.44)$$

are transformed to

$$M_{88}^2 = M_\eta^2 \cos^2(\Theta) + M_{\eta'}^2 \sin^2(\Theta), \tag{4.45}$$

and solved for Θ:

$$\tan^2\Theta = \frac{M_{88}^2 - M_\eta^2}{M_{\eta'}^2 - M_{88}^2}. \tag{4.46}$$

Insertion of the masses determines the numerical value for Θ:

$$\Theta = -10.76^\circ. \tag{4.47}$$

The underlying assumption is that η-η' mixing is the dominant cause for the difference between the η mass and the expected mass due to the Gell-Mann–Okubo mass relations. If the dominant cause are chiral logarithms, results are different [GL 85].

Mixing with consistent power counting

The zeroth chiral order Lagrangian does contain only dynamics of the singlet. This requires that the leading chiral and $\frac{1}{N_c}$-order contribution to the angle vanishes. Insertion of the full mass matrix into Eq. (4.40) produces the condition for vanishing mixing:

$$\tan(2\Theta) = \frac{2M_{18}^2}{M_{11}^2 - M_{88}^2}, \tag{4.48}$$

which must be expanded in $\frac{1}{N_c}$:

$$\tan(2\Theta) = 2\sqrt{2}\Big(1 + \frac{3}{N_c}\big(\frac{C\chi_0 - a}{\frac{\chi_8}{\sqrt{3}}} - \frac{C}{2}\big)\Big). \tag{4.49}$$

Identification with the physical masses determines the anomalous mass:

$$a = \frac{N_c}{3}\Big(M_\eta^2 + M_{\eta'}^2 - 2(\chi_0 - \frac{1}{2}\frac{\chi_8}{\sqrt{3}})\Big). \tag{4.50}$$

Insertion of experimental values for the measured quantities and $N_c = 3$ determines the numerical values of the constants:

$$\begin{aligned} a = 7.257\times 10^5\,\mathrm{MeV}^2, \quad \sqrt{a} = 851.9\,\mathrm{MeV}, \\ \Theta|^{N_c\to\infty} = 35.26^\circ, \quad \Theta = 11.15^\circ. \end{aligned} \tag{4.51}$$

Due to the fact that the result of the naive mixing is closer to experimental values $-10^\circ \gtrsim \Theta \gtrsim -20^\circ$, the naive angle[15] is used in chapter 5. The price of using the naive angle is that large-N_c counting cannot be treated correctly.

On the other hand, the anomalous mass does not depend on the angle. Due to this fact, even with an unphysical angle, the parameter a from this calculation is probably close to its physical value and can be used in further applications.

[15]Because of the assumption that η-η' mixing is the main cause which drives off the η mass in contrast to the results in [GL 85], bad convergence is not unexpected.

Chapter 5

Parity-violating decays of the η

Two parity-violating decays of the η are investigated here:

1. $\eta \to 2\pi^0$,
2. $\eta \to 4\pi^0$.

Due to the fact that both decay rates are proportional to the square of the vacuum angle θ, their measured rates (or rather, the upper limit for the rates) fix an upper limit for the vacuum angle θ. A comparison to previous results was not possible.

In accord with the Dashen-Nuyts equations $\big($Eq. (4.26)$\big)$, physics is invariant under a shift of θ by integer multiples of 2π, if one angle φ_f is rotated through the same amount. Even though the vacuum expectation value u_{VEV} is not necessarily invariant under such a shift, it is always possible to choose the rotations of the angles φ_f in such a manner that u_{VEV} does nothing but change its sign[1]. Since u_{VEV} is just the square root of U from [Wit 80], the results are equivalent. Beyond that, the Lagrangian is invariant under this change of sign, because only even total powers of u and its adjoint appear in the Lagrangian (see section 2.4.3), except

$$\log \det u = \frac{1}{2} \log \det u^2,$$

which can be expressed in terms of an even power.

Since a parity transformation turns the Goldstone boson matrix into its inverse, it does the same to its vacuum expectation value. Thus, the angles φ_f change signs and the vacuum is only invariant, if all angles φ_f satisfy (modulo integer multiples of 2π)

$$\varphi_f = 0, \pm\,\pi, \quad f = u, d, s. \tag{5.1}$$

[1] If φ_s is rotated through 6π, but φ_u and φ_d through -2π each, the sum is rotated through 2π and $\text{sign}(u_{\text{VEV}})\, u_{\text{VEV}}$ is invariant.

Due to this restriction of the angles φ_f, the left-hand sides of the Dashen-Nuyts equations (4.26) vanish so that the vacuum angle θ must satisfy

$$\theta_{\text{new}} \equiv \theta - \sum_{f=u,d,s} \varphi_f = 0. \tag{5.2}$$

Parity conservation demands that the original vacuum angle θ must be an integer multiple of π. If parity violation exists in QCD, it is small and therefore the vacuum angle satisfies either $\theta \approx 0$ or $\theta \approx \pi$.

The second decay channel is strongly suppressed in phase space. But since it has a very clear signature for detection, it could be hardly mistaken for another kind of event and would give an exceptionally clear signal.

5.1 Decay into two pions: $\eta \to 2\pi^0$

5.1.1 Invariant matrix element

The interaction is given by the leading term in the sum of Eq. (4.30):

$$\mathcal{L}_{\text{int}} = -\frac{F_0^2}{4} 2a \frac{\theta}{N_c} \frac{1}{3! F_0^3} \Big(\frac{1}{N_c} \phi_0 \phi_3^2 \langle \lambda_3^2 \rangle + \phi_8 \phi_3^2 \langle \lambda_8 \lambda_3^2 \rangle \Big). \tag{5.3}$$

Two fields are neutral pions, the third is the η_8 or the η_1:

$$\phi_8 \phi_3^2 \langle \lambda_8 \lambda_3^2 \rangle = 2 d_{338} \eta_8 (\pi^0)^2 = \frac{2}{\sqrt{3}} \eta_8 (\pi^0)^2, \tag{5.4}$$

$$\begin{aligned} \frac{1}{N_c} \phi_0 \phi_3^2 \langle \lambda_3^2 \rangle &= \sqrt{\frac{2}{N_f}} \frac{1}{\sqrt{1 + \frac{N_f}{N_c} C}} \eta_1 (\pi^0)^2 \langle \lambda_3^2 \rangle \\ &= 2 \sqrt{\frac{2}{N_f}} (1 - \frac{N_f}{N_c} \frac{C}{2} + \mathcal{O}(N_c^{-2})) \eta_1 (\pi^0)^2. \end{aligned} \tag{5.5}$$

Thus, the interaction with the physical η is given by (from here: $N_f = 3$)

$$\mathcal{L}^{\eta(\pi^0)^2} = \frac{1}{6\sqrt{3}} \frac{a}{F_0} \frac{\theta}{N_c} \Big(\big[\cos(\Theta) + \sqrt{2} \sin(\Theta) \big] - \frac{3}{N_c} \frac{C}{\sqrt{2}} \sin(\Theta) \Big) \eta (\pi^0)^2. \tag{5.6}$$

The invariant matrix element is symbolically given by

$$\begin{aligned} i\mathcal{M} &= i \langle \pi^0(\vec{p}_1), \pi^0(\vec{p}_2) | \mathcal{L}^{\eta(\pi^0)^2} | \eta(\vec{q}) \rangle \\ &= i \frac{1}{3\sqrt{3}} \frac{\theta}{N_c} \frac{a}{F_0} \Big(\big[\cos(\Theta) + \sqrt{2} \sin(\Theta) \big] - \frac{3}{N_c} \frac{C}{\sqrt{2}} \sin(\Theta) \Big) + \mathcal{O}(N_c^{-\frac{7}{2}}), \end{aligned} \tag{5.7}$$

where a combinatorial factor 2 shows up. It is due to two possible contractions of the pion fields. Squaring the matrix element is straightforward:

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{1}{27}\frac{a^2}{F_\pi^2}\Big(\left[1+\sin(2\Theta)+\sin^2(\Theta)\right]-\frac{3}{N_c}2C\sin^2(\Theta)\Big)\frac{3}{N_c}(\frac{\theta}{N_c})^2 \\
&\quad +\mathcal{O}(N_c^{-5}) \\
&= (2.475-\frac{1}{N_c}1.667)\times 10^{-2}\frac{a^2}{F_\pi^2}\frac{3}{N_c}(\frac{\theta}{N_c})^2+\mathcal{O}(N_c^{-5}) \\
&= (1.507-\frac{1}{N_c}1.015)\times 10^{6}\,\mathrm{MeV}^2\,\frac{3}{N_c}(\frac{\theta}{N_c})^2+\mathcal{O}(N_c^{-5}). \qquad (5.8)
\end{aligned}$$

5.1.2 Phase space

The phase space of this decay into two final particles[2] is especially simple, since the matrix element is constant and no intermediate states appear in the recursion formula from [BK 73]:

$$\begin{aligned}
R_2(M_\eta^2) &= \frac{1}{M_\eta}2^{-2}\int d\Omega\frac{\lambda^{\frac{1}{2}}(M_\eta^2;M_\pi^2;M_\pi^2)}{2M_\eta} \\
&= \frac{\pi}{2}\sqrt{1-(\frac{2M_\pi}{M_\eta})^2}, \qquad (5.9)
\end{aligned}$$

where the triangle function was used in the form

$$\lambda(x,y,z)=x^2-2x(y+z)+(y-z)^2. \qquad (5.10)$$

5.1.3 Partial decay width

The partial decay width for the given channel is

$$\Gamma(\eta\to 2\pi^0)=\frac{1}{2M_\eta(2\pi)^2}|\mathcal{M}|^2R_2(M_\eta^2). \qquad (5.11)$$

Insertion of Eqs. (4.42), (5.8) and (5.9) ends in the final result

$$\Gamma(\eta\to 2\pi^0;\frac{\theta}{N_c})=(47.64-\frac{1}{N_c}32.09)\times 10^3\,\mathrm{keV}\,\frac{3}{N_c}(\frac{\theta}{N_c})^2+\mathcal{O}(N_c^{-5}). \qquad (5.12)$$

The experimental upper limit (from [PDG 2008]) of the partial decay width is

$$\begin{aligned}
\Gamma_{\exp}(\eta\to 2\pi^0) &< 4.3\times 10^{-4}\underbrace{\Gamma(\eta)}_{=1.3\,\mathrm{keV}} \\
&= 5.6\times 10^{-4}\,\mathrm{keV}. \qquad (5.13)
\end{aligned}$$

Consistency demands for $N_c=3$ an upper limit of

$$|\theta|<3.694\times 10^{-4}. \qquad (5.14)$$

[2] Since the final particles are neutral pions, M_π stands for the mass of the neutral pion (from Eq. (4.42)) only, instead of some averaged mass (see Eq. (4.43)).

5.2 Decay into four pions: $\eta \to 4\pi^0$

5.2.1 Invariant matrix element

The interaction is given by the next-to-leading term in the sum of Eq. (4.30):

$$\mathcal{L}_{\text{int}} = \frac{F_0^2}{4}\, 2a\frac{\theta}{N_c}\frac{1}{5!F_0^5}(\frac{1}{N_c}\phi_0\phi_3^4\langle\lambda_3^4\rangle + \phi_8\phi_3^4\langle\lambda_8\lambda_3^4\rangle). \tag{5.15}$$

Four of the fields are neutral pions, the fifth is the η_8 or the η_1:

$$\begin{aligned}\phi_8\phi_3^4\langle\lambda_8\lambda_3^4\rangle &= (\frac{8}{3}d_{338} + 2d_{338}^2 d_{888})\eta_8(\pi^0)^4\\ &= \frac{2}{\sqrt{3}}\eta_8(\pi^0)^4,\end{aligned} \tag{5.16}$$

$$\begin{aligned}\frac{1}{N_c}\phi_0\phi_3^4\langle\lambda_3^4\rangle &= \sqrt{\frac{2}{N_f}}\frac{1}{\sqrt{1+\frac{N_f}{N_c}C}}\eta_1(\pi^0)^4\langle(\lambda_3^2)^2\rangle\\ &= 2\sqrt{\frac{2}{N_f}}(1-\frac{N_f}{N_c}\frac{C}{2}+\mathcal{O}(N_c^{-2}))\eta_1(\pi^0)^4.\end{aligned} \tag{5.17}$$

Thus, the interaction with the physical η is given by (from here: $N_f = 3$)

$$\mathcal{L}^{\eta(\pi^0)^4} = \frac{1}{5!\sqrt{3}}\frac{a}{F_0^3}\frac{\theta}{N_c}\Big(\big[\cos(\Theta)+\sqrt{2}\sin(\Theta)\big] - \frac{3}{N_c}\frac{C}{\sqrt{2}}\sin(\Theta)\Big)\eta(\pi^0)^4. \tag{5.18}$$

The invariant matrix element squared is determined in the same manner as in the three-body case, except that the combinatorical factor is 4! instead of 2:

$$i\mathcal{M} = i\frac{a}{5\sqrt{3}F_0^2}\frac{\theta}{N_c}\Big(\big[\cos(\Theta)+\sqrt{2}\sin(\Theta)\big] - \frac{3}{N_c}\frac{C}{\sqrt{2}}\sin(\Theta)\Big) + \mathcal{O}(N_c^{-\frac{9}{2}}) \tag{5.19}$$

and

$$\begin{aligned}|\mathcal{M}|^2 &= \frac{1}{75}(\frac{a}{F_\pi^3})^2\Big(\big[1+\sin(2\Theta)+\sin^2(\Theta)\big] - \frac{3}{N_c}2C\sin^2(\Theta)\Big)(\frac{3}{N_c})^3(\frac{\theta}{N_c})^2\\ &\quad +\mathcal{O}(N_c^{-7})\\ &= (8.909 - \frac{1}{N_c}6.000)\times 10^{-3}\frac{a^2}{F_\pi^6}(\frac{3}{N_c})^3(\frac{\theta}{N_c})^2 + \mathcal{O}(N_c^{-7})\\ &= (7.252 - \frac{1}{N_c}4.884)\times 10^{-3}\,\text{MeV}^{-2}(\frac{3}{N_c})^3(\frac{\theta}{N_c})^2 + \mathcal{O}(N_c^{-7}).\end{aligned} \tag{5.20}$$

5.2.2 Phase space

Phase space is more complicated here. Even though the matrix element is constant, the recursion formula requires two intermediate states, hence three recursions. The recursion formula (see [BK 73]) for an initial state with mass M_n, ending in a final state with mass M_{n-1} under emission of a particle with mass m_n yields a phase space factor

$$R_n(M_n^2) = \int dM_{n-1}^2 \int d\Omega_{n-1} \frac{\lambda^{\frac{1}{2}}(M_n^2; M_{n-1}^2; m_n^2)}{8M_n^2} R_{n-1}(M_{n-1}^2). \quad (5.21)$$

Successive application of Eq. (5.21) yields

$$R_n(M_n^2) \frac{2^{-n}}{M_n} \int \prod_{i=1}^{n-1} d\Omega_i \int \prod_{j=2}^{n-1} dM_j \prod_{k=2}^{n} Q_k, \quad (5.22)$$

where

$$Q_k = \frac{\lambda^{\frac{1}{2}}(M_k^2; M_{k-1}^2; m_k^2)}{2M_k}. \quad (5.23)$$

The interdependencies of the intermediate states are taken into account by the boundaries of the mass integrations. The angular integrations are completely independent and can be done easily. The intermediate mass integrations cannot be done analytically in the general case. But introduction of a dimensionless total kinetic energy normalized to the pion mass

$$e = \frac{m_\eta - 4m_\pi}{m_\pi} = 5.6 \times 10^{-2} \quad (5.24)$$

and of dimensionless kinetic energies of the intermediate states

$$t_j = \frac{M_j - jm_\pi}{m_\pi}, j = 2, 3 \quad (5.25)$$

simplifies the next-to-leading-o22rder calculation in the non-relativistic limit a lot. The expansion parameter is e, since both $t_j < e$. The total phase space factor is

$$R_4(M_\eta^2) = \frac{\pi^3}{2} \frac{M_\pi^2}{M_\eta^2} I_3, \quad (5.26)$$

where

$$I_3 = \int_0^e dt_3 \frac{\lambda^{\frac{1}{2}}(M_\eta^2(e); M_3^2(t_3); M_\pi^2)}{M_3(t_3)} I_2(t_3), \quad (5.27)$$

$$I_2(t_3) = \int_0^{t_3} dt_2 \frac{\lambda^{\frac{1}{2}}(M_3^2(t_3); M_2^2(t_2); M_\pi^2) \lambda^{\frac{1}{2}}(M_2^2(t_2); M_\pi^2; M_\pi^2)}{M_2(t_2)}. \quad (5.28)$$

Evaluating I_2 requires making use of

$$\frac{\lambda^{\frac{1}{2}}(x; y; y)}{\sqrt{x}} = \frac{\sqrt{x(x - 4y)}}{\sqrt{x}} = \sqrt{x - 4y} \quad (5.29)$$

for simplification of the t_2 integration:

$$\frac{\lambda^{\frac{1}{2}}(M_2^2(t_2);M_\pi^2;M_\pi^2)}{M_2(t_2)} \quad = \quad 2M_\pi\sqrt{t_2}[1+\frac{t_2}{8}]+\mathcal{O}(e^{\frac{5}{2}}).+ \tag{5.30}$$

Further simplification requires introduction of the difference of kinetic energies $\delta_2 = t_3 - t_2$ which is the energy carried away by the second pion and which is of order $\mathcal{O}(e)$, too:

$$\lambda^{\frac{1}{2}}(M_3^2(t_3);M_2^2(t_2);M_\pi^2) = 4\sqrt{3}M_\pi^2\sqrt{\delta_2}[1+\frac{11}{24}\delta_2+\frac{5}{12}t_2]+\mathcal{O}(e^{\frac{5}{2}}). \tag{5.31}$$

Now the t_2 integration can be done by partial integration or use of mathematical tables (like [Bro 03]):

$$\begin{aligned} I_2(t_3) &= \int_0^{t_3} dt_2 \frac{\lambda^{\frac{1}{2}}(M_3^2(t_3);M_2^2(t_2);M_\pi^2)\lambda^{\frac{1}{2}}(M_2^2(t_2);M_\pi^2;M_\pi^2)}{M_2(t_2)} \\ &= 8\sqrt{3}M_\pi^3\int_0^{t_3} dt_2\sqrt{-t_2^2+t_2t_3}[1+\frac{11}{24}t_3+\frac{1}{12}t_2]+\mathcal{O}(e^4) \\ &= \sqrt{3}\pi M_\pi^3 t_3^2[1+\frac{t_3}{2}]+\mathcal{O}(e^4). \end{aligned} \tag{5.32}$$

The t_3 integration also requires the introduction of a difference of kinetic energies $\delta_3 = e - t_3$, which is of order $\mathcal{O}(e)$:

$$\lambda^{\frac{1}{2}}(M_\eta^2(e);M_3^2(t_3);M_\pi^2) = 4\sqrt{6}M_\pi^2\sqrt{\delta_3}[1+\frac{19}{48}\delta+\frac{7}{24}t_3]+\mathcal{O}(e^{\frac{5}{2}}). \tag{5.33}$$

The denominator must be expanded, too:

$$\frac{1}{M_3(t_3)} = \frac{M_\pi^{-1}}{3}\frac{1}{1+\frac{t_3}{3}} = \frac{M_\pi^{-1}}{3}[1-\frac{t_3}{3}]+\mathcal{O}(e^2). \tag{5.34}$$

The t_3 integration is done by multiple partial integrations

$$\begin{aligned} I_3 &= \int_0^{e} dt_3 \frac{\lambda^{\frac{1}{2}}(M_\eta^2(e);M_3^2(t_3);M_\pi^2)}{M_3(t_3)}\; I_2(t_3) \\ &= 4\sqrt{2}\pi M_\pi^4\int_0^{e} dt_3\sqrt{e-t_3}t_3^2[1+\frac{19}{48}e+\frac{1}{16}t_3]+\mathcal{O}(e^{\frac{11}{2}}) \\ &= \frac{2^6\sqrt{2}\pi}{105}M_\pi^4 e^{\frac{7}{2}}[1+\frac{7}{24}e]+\mathcal{O}(e^{\frac{11}{2}}). \end{aligned} \tag{5.35}$$

Therefore, the phase space factor in the non-relativistic limit with next-to-leading-order corrections is

$$R_4(M_\eta^2) = \frac{\pi^4 2^5\sqrt{2}}{105}\frac{M_\pi^6}{M_\eta^2}e^{\frac{7}{2}}[1+\frac{7}{24}e]+\mathcal{O}(e^{\frac{11}{2}}). \tag{5.36}$$

Now, it is possible to verify that the leading correction is indeed small,

$$\frac{7}{24}e = 1.6 \times 10^{-2} \ll 1, \tag{5.37}$$

and the nonrelativistic approximation can be justified. The exact numerical evaluation (see [Mor 94]) of Eq. (5.26) provides

$$R_4^{\text{exact}} = 3.626 \times 10^4 \,\text{MeV}^2, \tag{5.38}$$

the analytic evaluation yields

$$R_4^{\text{analytic}} = [3.538 + 0.058] \times 10^4 \,\text{MeV}^2, \tag{5.39}$$

where the next-to-leading order has been included. The non-relativistic result (without correction) is not exactly consistent with the non-relativistic result in [Mor 94] ($R_4^{\text{Morris}} = 3.638 \times 10^4 \,\text{MeV}^2$). The formula from [BK 73] applied by Morris replaces in the fraction M_π^6/M_η^2 in the final result the mass of the initial particle by the sum of the masses of the final particles:

$$\frac{M_\eta^2}{16M_\pi^2} = 1.0280. \tag{5.40}$$

5.2.3 Partial decay width

The partial decay width for the four pion channel is

$$\Gamma(\eta \to 4\pi^0) = \frac{1}{2M_\eta(2\pi)^8} |\mathcal{M}|^2 R_4(M_\eta^2). \tag{5.41}$$

Insertion of Eqs. (4.42), (5.20) and (5.38) ends in the final result

$$\begin{aligned} \Gamma(\eta \to 4\pi^0; \frac{\theta}{N_c}) = & \quad (9.887 + \tfrac{1}{N_c}6.659) \times 10^{-5} \,\text{keV}\, (\tfrac{3}{N_c})^3 (\tfrac{\theta}{N_c})^2 \\ & + \mathcal{O}(N_c^{-7}). \end{aligned} \tag{5.42}$$

The partial decay width has an upper limit [PDG 2008] of

$$\begin{aligned} \Gamma_{\text{exp}}(\eta \to 4\pi^0) & < 6.9 \times 10^{-7} \underbrace{\Gamma(\eta)}_{=1.3\,\text{keV}} \\ & = 9.0 \times 10^{-7} \,\text{keV}. \end{aligned} \tag{5.43}$$

Consistency demands for $N_c = 3$ an upper limit of

$$|\theta| < 0.3250. \tag{5.44}$$

Chapter 6

Summary and concluding remarks

After a short introduction on the basics of QCD and dynamically broken chiral symmetry, a non-linear realization of the chiral group was introduced. Its connection to the different basis systems, which are applied in mesonic ChPT, was demonstrated. A translational scheme was given. Different methods for the construction of chirally invariant structures were demonstrated and their behaviour in a $\frac{1}{N_c}$ expansion clarified.

Next-to-next-to-leading-order Lagrangian

In the u basis, the construction principles and the nature of additional symmetry relations were demonstrated at leading and next-to-leading order. The Lagrangian of Gasser and Leutwyler from [GL 85] was translated into the u basis.

The next-to-next-to-leading-order Lagrangian was constructed. A (probably) complete set of symmetry relations was provided. It was demonstrated that two trace relations in the even sector had been missed in the previous construction in [BCE 00] and in one case, it could be clearly demonstrated, which relation had been missed. The extension of the chiral group in the even sector was discussed.

The composite Cayley-Hamilton relations in [BCE 00] were decomposed in terms of the symmetry relations. Two different maximal sets of independent monomials are provided, one with a preference of single traces for a more transparent view on large-N_c counting, the other with a preference of multiple traces, which possess an easier flavour structure.

In the anomalous sector, the LECs in [BGT 02, EFS 02] were related to each other. The equivalence of both Lagrangians was proved.

Even though the results of this work and the small deviations (only for the two flavour case) from previous results seem to indicate that the minimal set of independent monomials has been found, there is no way to be absolutely sure.

Parity violation

The appearance of the anomaly in QCD was discussed. It was demonstrated, how the inclusion of the singlet η necessitates the introduction of the anomaly into ChPT and a different vacuum state. Moreover, it requires a mixing of η_8 and η_1 to η and η'. Properties of the mixing angle in the chiral and $\frac{1}{N_c}$ expansions were discussed. It was pointed out that a parity-violating interaction arises, which is proportional to the vacuum angle.

Two parity violating decays of the η due to this interaction were discussed. Both decay rates are constrained by experiment. The decay rates were calculated and upper constraints for the vacuum angle were determined. A comparison to previous results was not possible. The newly determined constraints are less tight than those derived from other processes or quantities, such as the electric dipole moment of the neutron.

Conclusions and outlook

The results of the investigation of effective Lagrangians in this thesis demonstrate that there is much to be done:

- The inclusion of the singlet η must be extended to higher chiral orders.
- η-η' mixing with consistent large-N_c counting cannot be understood in terms of a perturbative expansion, yet.
- The improved Lagrangians allow a better understanding of the pseudoscalar octet.
- The systematic study of symmetry relations may provide further insight for baryonic Lagrangians, too.

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Appendix A

Lagrangian at next-to-next-to-leading order

A.1 Single-trace solution to the even sector

A solution preferring single-trace monomials is presented in table A.1. It does not match the monomials of [BCE 00], since monomials which require less fields were preferred in some cases. Eliminated monomials are denoted by [...]. Further, the exact expressions for the monomials here in terms of the monomials of [BCE 00] were not worked out completely. The correspondence is always clear.

Table A.1: Single trace soution

monomial	# of flavours	two-flavour case	three-flavour case
(1)	2	Y_{24}	Y_{40}
(2)	2	Y_{25}	Y_{44}
(5)	2	Y_{26}	Y_{46}
(6)	3		Y_{41}
(10)	3		Y_{43}
(30)	2	Y_1	Y_1
(31)	3	$[\to Y_3]$	$\to Y_{47}$
(32)	3		$\to Y_4$
(36)	2	Y_2	Y_3
(39)	3		Y_2
(40)	3		$\to Y_{45}$
(51)	2	Y_4	Y_5
(52)	2	Y_5	Y_{10}
(53)	3		Y_8
(54)	3		Y_6
(55)	3		Y_{11}
(56)	3		Y_9

monomial	# of flavours	two-flavour case	three-flavour case
(57)	3		Y_7
(62)	2	Y_{12}	Y_{22}
(64)	2	Y_{13}	Y_{25}
(66)	3		Y_{24}
(68)	3		Y_{23}
(72)	2	Y_6	Y_{12}
(74)	3		Y_{13}
(88)	2	Y_{27}	Y_{50}
(89)	2	$[Y_{28}]$	Y_{51}
(91)	3		Y_{49}
(95)	2	Y_{36}	Y_{66}
(96)	2	$-Y_{38}$	$-Y_{69}$
(97)	2	Y_{37}	Y_{67}
(99)	3		$-Y_{68}$
(101)	3		$\to Y_{48}$
(109)	2	$-Y_{53}$	$-Y_{88}$
(110)	2	$-Y_{51}$	$-Y_{90}$
(111)	3		$\to Y_{42}$
(112)	2	Y_7	Y_{14}
(113)	2	Y_9	Y_{17}
(114)	3		Y_{18}
(115)	2	Y_8	Y_{15}
(116)	3		Y_{16}
(118)	2	Y_{14}	Y_{26}
(119)	2	Y_{16}	Y_{29}
(120)	3		Y_{30}
(121)	2	Y_{15}	Y_{27}
(122)	3		Y_{28}
(125)	2	$\to Y_{20}$	$\to Y_{34}$
(127)	2	$\to Y_{21}$	$\to Y_{35}$
(129)	3		$\to Y_{36}$
(130)	2	$\to Y_{22}$	$\to Y_{38}$
(131)	2	$\to Y_{54}$	$\to Y_{91}$
(132)	2	$\to Y_{23}$	$\to Y_{39}$
(133)	2	$\to Y_{57}$	$\to Y_{37}$
(134)	3		Y_{63}
(135)	2	$-Y_{35}$	$-Y_{65}$
(136)	3		Y_{64}
(137)	3	Y_{48}	$-Y_{84}$

monomial	# of flavours	two-flavour case	three-flavour case
(139)	3		$\to Y_{52}$
(142)	2	$-Y_{49}$	$\to Y_{85}$
(144)	3		$\to Y_{86}$
(145)	3		$\to Y_{83}$
(146)	2	Y_{29}	Y_{53}
(147)	2	Y_{31}	Y_{56}
(148)	2	Y_{32}	Y_{57}
(149)	2	$-Y_{33}$	Y_{59}
(150)	2	Y_{30}	Y_{55}
(151)	3		Y_{54}
(152)	3		Y_{58}
(153)	3		$-Y_{60}$
(156)	2	Y_{39}	Y_{70}
(157)	2	Y_{41}	Y_{73}
(158)	2	Y_{42}	Y_{74}
(159)	2	$-Y_{33}$	Y_{76}
(160)	2	Y_{40}	Y_{72}
(161)	3		Y_{71}
(162)	3		Y_{75}
(163)	3		$-Y_{77}$
(169)	2	$-Y_{52}$	$-Y_{89}$
(171)	2	$\to Y_{44}$	$\to Y_{78}$
(173)	2	$\to Y_{56}$	$\to Y_{93}$
(176)	2	Y_{50}	Y_{87}
(178)	2	Y_{10}	Y_{19}
(179)	2	Y_{11}	Y_{20}
(180)	3		Y_{21}
(181)	2	Y_{17}	Y_{31}
(182)	2	Y_{19}	Y_{33}
(183)	2	Y_{18}	Y_{32}
(184)	3		$\to Y_{94}$
(185)	2	Y_{34}	Y_{61}
(186)	3		Y_{62}
(187)	2	Y_{46}	Y_{80}
(188)	3		Y_{81}
(189)	2	Y_{47}	Y_{82}
(190)	2	$\frac{1}{2}Y_{45}$	$\frac{1}{2}Y_{79}$
(191)	2	$\to Y_{55}$	$\to Y_{92}$

A.2 Three-flavour Cayley-Hamilton relations of [BCE 00]

In the same manner as in section 3.5.6, a comparison of Cayley-Hamilton relations with the corresponding $SU(3)$ trace relations is done. The results are summarized in table A.2.

Table A.2: Comparison to Cayley-Hamilton relations

C.H.	linear combination of relations	corr. to
CH1	$\mathit{99}$	*99*
CH2	$\Big[-\frac{2}{3}\times\mathit{1}+\frac{2}{3}\times\mathit{2}+\frac{4}{3}\times\mathit{3}-\frac{2}{3}\times\mathit{6}-\frac{1}{6}\times\mathit{8}+\frac{11}{12}\times\mathit{9}+\frac{5}{3}\times\mathit{11}-\frac{1}{6}\times\mathit{12}$	
	$+\frac{1}{6}\times\mathit{14}-\frac{5}{6}\times\mathit{15}-\frac{1}{4}\times\mathit{16}-\frac{1}{4}\times\mathit{17}-\frac{1}{2}\times\mathit{18}-\frac{1}{2}\times\mathit{19}+\frac{1}{2}\times\mathit{20}$	
	$-\frac{1}{2}\times\mathit{21}+\frac{1}{3}\times\mathit{35}+\frac{2}{3}\times\mathit{38}-\frac{1}{3}\times\mathit{39}+\frac{1}{3}\times\mathit{40}+\frac{1}{3}\times\mathit{54}-\mathit{55}-\frac{2}{3}\times\mathit{56}$	
	$+\mathit{57}+\frac{1}{2}\times\mathit{59}+\frac{1}{3}\times\mathit{64}-\frac{2}{3}\times\mathit{65}+\frac{2}{3}\times\mathit{74}-\frac{1}{3}\times\mathit{75}\Big]-\frac{2}{3}\times\mathit{86}$	
	$+\frac{2}{3}\times\mathit{91}-\frac{1}{2}\times\mathit{94}-\mathit{95}+\frac{1}{3}\times\mathit{96}-\frac{1}{4}\times\mathit{97}-\frac{1}{2}\times\mathit{99}-\frac{2}{3}\times\mathit{108}$	*95*
CH3	$-\mathit{103}$	*103*
CH4	$\mathit{102}$	*102*
CH5	$-\mathit{101}+3\times\mathit{102}$	*101*
CH6	$-\mathit{112}$	*112*
CH7	$\Big[\times\mathit{8}+\frac{1}{2}\times\mathit{9}-\frac{1}{2}\times\mathit{16}-\frac{1}{2}\times\mathit{17}-\mathit{55}-\mathit{57}+\frac{1}{2}\times\mathit{59}+\mathit{61}\Big]-\mathit{94}-\frac{1}{2}\times\mathit{97}$	*94*
CH8	$2\times\Big[-\mathit{54}-\mathit{56}+\frac{1}{2}\times\mathit{58}+\mathit{60}-\mathit{64}+\mathit{66}+\frac{1}{2}\times\mathit{67}+\mathit{68}\Big]+2\times\mathit{97}$	*97*
CH9	$-\mathit{88}$	*88*
CH10	$-\mathit{86}$	*86*
CH11	$-\mathit{85}+3\times\mathit{86}$	*85*
CH12	$\frac{1}{2}\times\Big(-\mathit{86}+\mathit{86}+\mathit{89}-\mathit{90}\Big)$	*87*
CH13	$\frac{1}{2}\times\Big(-\mathit{86}-\mathit{86}+\mathit{89}+\mathit{90}\Big)-\mathit{91}$	*91*
CH14	$\mathit{91}$	*91*
CH15	$\frac{1}{2}\times\Big(3\times\mathit{86}-\mathit{87}+\mathit{89}+\mathit{90}\Big)+2\times\mathit{91}$	*90*
CH16	$\mathit{107}+\frac{1}{2}\times\mathit{108}$	*107*
CH17	$\mathit{108}$	*108*
CH18	$\mathit{114}$	*114*
CH19	$\mathit{115}$	*115*
CH20	$\mathit{116}$	*116*
CH21	$\Big[2\times\mathit{1}+2\times\mathit{3}-\mathit{4}-2\times\mathit{5}-\frac{1}{2}\times\mathit{8}-\frac{1}{4}\times\mathit{9}-\mathit{11}-\frac{1}{2}\times\mathit{12}+\frac{1}{2}\times\mathit{14}$	
	$+\frac{1}{2}\times\mathit{15}+\frac{1}{4}\times\mathit{16}+\frac{1}{4}\times\mathit{17}+\frac{1}{2}\times\mathit{18}+\frac{1}{2}\times\mathit{19}-\frac{1}{2}\times\mathit{20}-\frac{1}{2}\times\mathit{21}+\mathit{38}$	
	$+\mathit{39}-\mathit{41}+\mathit{42}\Big]-2\times\mathit{86}+2\times\mathit{91}+\frac{5}{2}\times\mathit{94}-\mathit{95}-\mathit{96}+\frac{1}{4}\times\mathit{97}+\frac{1}{2}\times\mathit{99}$	
	$-2\times\mathit{108}-\mathit{116}$	*96*

A.3 Translation prescription for the Lagrangians of [BGT 02, EFS 02]

Table A.3 offers a translation prescription of LECs in the anomalous sector for the three-flavour case and for the extension of the chiral group.

Table A.3: Translation prescription for LECs

this work	[BGT 02]	[EFS 02]
m_2	K^W_{12}	$-16\ L^{6,\epsilon}_{12}$
m_4	K^W_1	$16\ L^{6,\epsilon}_4$
m_5	$-K^W_{16}$	$-64\ \left(L^{6,\epsilon}_{15} - L^{6,\epsilon}_{17}\right)$
m_7	K^W_{17}	$-32\ \left(L^{6,\epsilon}_{15} + L^{6,\epsilon}_{16} - 2\ L^{6,\epsilon}_{17} - L^{6,\epsilon}_{23}\right)$
m_9	$-K^W_{18}$	$-32\ \left(2\ L^{6,\epsilon}_{17} - L^{6,\epsilon}_{18}\right)$
m_{10}	$-K^W_{24}$	$16\ L^{6,\epsilon}$
m_{12}	$-K^W_{14}$	$-8\ \left(L^{6,\epsilon}_{13} + L^{6,\epsilon}_{14} - 4\ L^{6,\epsilon}_{23} + 4\ L^{6,\epsilon}_{23}\right)$
m_{14}	$-K^W_{13}$	$-16\ \left(L^{6,\epsilon}_{14} + 4\ L^{6,\epsilon}_{23}\right)$
m_{16}	$-K^W_{15}$	$-32\ L^{6,\epsilon}_{23}$
m_{28}	K^W_4	$-8\ \left(L^{6,\epsilon}_5 - L^{6,\epsilon}_{14} - 4\ L^{6,\epsilon}_{23}\right)$
m_{29}	K^W_5	$-8\ L^{6,\epsilon}_6$
m_{30}	K^W_6	$-8\ \left(L^{6,\epsilon}_7 + \frac{2}{3}\ L^{6,\epsilon}_{14} + \frac{8}{3}\ L^{6,\epsilon}_{23}\right)$
m_{31}	$-K^W_2$	$-8\ L^{6,\epsilon}_1$
m_{32}	$-K^W_3$	$-8\ L^{6,\epsilon}_2$
m_{42}	$-K^W_{19}$	$-16\ \left(L^{6,\epsilon}_{21} + 2\ L^{6,\epsilon}_{22}\right)$
m_{44}	$-K^W_{20}$	$-16\ \left(L^{6,\epsilon}_{22} + 2\ L^{6,\epsilon}_{23}\right)$
m_{48}	K^W_{21}	$8\ \left(4\ L^{6,\epsilon}_{23} + L^{6,\epsilon}_{24}\right)$
m_{52}	$-K^W_{22}$	$8\ L^{6,\epsilon}_{19}$
m_{58}	$-K^W_{23}$	$8\ L^{6,\epsilon}_{20}$
m_{61}	K^W_{11}	$-4\ L^{6,\epsilon}_3$
m_{62}	K^W_7	$4\ L^{6,\epsilon}_7$
m_{63}	K^W_8	$4\ L^{6,\epsilon}_8$
m_{64}	K^W_9	$4\ L^{6,\epsilon}_9$
m_{65}	K^W_{10}	$4\ L^{6,\epsilon}_{10}$
	extension of the chiral group	
m_{4x}	$-\frac{2}{3}\ c^W_9$	$-8\ L^{6,\epsilon}_{4'}$
m_{7x}	$-c^W_6$	$-4\ L^{6,\epsilon}_{1'}$

this work	[BGT 02]	[EFS 02]
m_{13x}	$-c_{11}^W + c_{13}^W$	$-4\ L_{13'}^{6,\epsilon}$
m_{19x}	$-c_{11}^W$	$4\ L_{5'}^{6,\epsilon}$
m_{24x}	$c_9^W - c_{10}^W - c_{11}^W$	$-8\ L_{7'}^{6,\epsilon}$
m_{26x}	$c_7^W - \frac{1}{4}\ c_{11}^W$	$2\ L_{2'}^{6,\epsilon}$
m_{29x}	$-c_{12}^W$	$4\ L_{8'}^{6,\epsilon}$
m_{30x}	c_8^W	$L_{3'}^{6,\epsilon}$

A.4 Mathematica source code

Most of the source code is taken from the work with two flavours in the even sector. Differences concerning the anomalous sector are denoted in the corresponding spots. Commentary is written in italics, mathematica keywords are in bold print and variables defined by the author are slanted. Programme output is in medium weight. Different mathematica cells are in different paragraphs. The same cells are split by an intermediate commentary in some occasions.

Input of relations and construction of a maximal set

A table is generated in the beginning, which makes the symbolic notation more pleasing to the eye. The list 'blanklist' is a dummy-list with as many entries as a priori independent monomials exist, which are replaced by the coefficients of the derived symmetry relations. The symmetry relations, which are typed in by hand are later appended to the initially empty list 'relations'. This procedure allows later addition of relations or reordering of relations with only little effort. Furthermore, it is easier to implement this approach in mathematica than generating list entries before the entire list is specified.

```
eventerms=Table["("<>ToString[cnt]<>")",{cnt,1,191}];
blanklist=Table[dummy[cnt],{cnt,1,191}];
relations={};
relcnt=1;
appendix[relat_]:=Module[{},Return[Append[relations,relat]]];
```

The next paragraph demonstrates, how symmetry relations are inserted. The typing order is the same as in the tables of section 3.5.2 or 3.6.2. A few examples are sufficient.

```
rel[relcnt] =
  blanklist/.{dummy[1] → 1, dummy[2] → −2, dummy[3] → 1, dummy[16] → −1,
  dummy[18] →, dummy[86] → 1, dummy[89] → 2, dummy[102] → −1};
relcnt++;
rel[relcnt] =
  blanklist/.{dummy[2] → −1, dummy[3] → 1, dummy[4] → −1, dummy[5] → 1,
  dummy[20] → −1, dummy[21] → 2, dummy[86] → 1, dummy[88] → −2,
  dummy[104] → 1};
relcnt++;
⋮
```

The aforementioned collection into a (number of relations × number of monomials)-matrix named 'relations' follows.

```
cnt = 1;
While[cnt < relcnt, relations = appendix[rel[cnt]]; cnt++; ];
```

The next command replaces all remaining dummy variables by zero. Then a (number of monomials)-dimensional vector equation is set up (see Eq. (3.53). In every **Solve**-*command, the number of light flavours is specified.*

```
evensector = Table[relations[[cnt]], {cnt, 1, relcnt − 1}]
        /.Table[dummy[cnt] → 0, {cnt, 1, Length[eventerms]}];
Print[Solve[ Sum_{cnt=1}^{Length[evensector]} evensector[[cnt]]coeff[cnt] == 0
        /.{N → 2}, Table[coeff[cnt], {cnt, 1, Length[evensector]}]] [[1]]];
```

The first **Print** *command yields a list of dependencies:*

$$\text{coeff}[1] \to \text{coeff}[111] - 2\text{coeff}[117] + \frac{\text{coeff}[193]}{2} + \text{coeff}[194] - \text{coeff}[216] - \text{coeff}[218]$$

$$\text{coeff}[2] \to -\text{coeff}[195] - \text{coeff}[216] - \text{coeff}[217]$$

$$\text{coeff}[3] \to \text{coeff}[111] - 2\text{coeff}[117] + \text{coeff}[193] + 2\text{coeff}[217]$$

$$\text{coeff}[4] \to -\frac{\text{coeff}[111]}{2} + \text{coeff}[117] - \text{coeff}[194] + \text{coeff}[216] + \text{coeff}[217] + \text{coeff}[218]$$

$$\text{coeff}[5] \to -\text{coeff}[111] + 2\text{coeff}[117] - \text{coeff}[193] - 2\text{coeff}[217]$$

$$\text{coeff}[6] \to 2\text{coeff}[195] + 2\text{coeff}[216] + 2\text{coeff}[217]$$

$$\text{coeff}[7] \to \frac{2\text{coeff}[80]}{3}$$

$$\begin{aligned}\text{coeff}[8] \to\ & -\text{coeff}[57] - \text{coeff}[98] + \frac{\text{coeff}[100]}{2} - \frac{\text{coeff}[110]}{2} + \frac{\text{coeff}[117]}{2} - 2\text{coeff}[147] \\ & +\text{coeff}[152] + \text{coeff}[167] + \text{coeff}[168] - 2\text{coeff}[169] - 2\text{coeff}[170] - 2\text{coeff}[189] \\ & +2\text{coeff}[190] + 2\text{coeff}[191] - \text{coeff}[192] + 2\text{coeff}[206] + 3\text{coeff}[216] + 3\text{coeff}[217] \\ & +\frac{\text{coeff}[218]}{2}\end{aligned}$$

$$\begin{aligned}\text{coeff}[9] \to\ & -\frac{\text{coeff}[57]}{2} + \text{coeff}[74] - \frac{\text{coeff}[80]}{3} - \frac{\text{coeff}[98]}{2} + \frac{\text{coeff}[100]}{4} - \frac{\text{coeff}[110]}{4} + \frac{\text{coeff}[117]}{4} \\ & -\text{coeff}[147] + \frac{\text{coeff}[152]}{2} + \frac{\text{coeff}[167]}{2} + \frac{\text{coeff}[168]}{2} - \text{coeff}[169] - \text{coeff}[170] \\ & +\text{coeff}[190] + \text{coeff}[191] - \frac{\text{coeff}[192]}{2} + \text{coeff}[206] + \text{coeff}[216] + \text{coeff}[217] \\ & +\frac{\text{coeff}[218]}{4}\end{aligned}$$

$$\text{coeff}[10] \to 0$$

$$\vdots$$

Each of the above variables on the left-hand side is inserted into the list named 'redundant' and replaced one-by-one, until the reduction scheme is fulfilled. After some renaming procedures,

```
redundant={{7},{25},...,{222}};
  esector=Delete[evensector,redundant];
  evensector=esector;
  Clear[esector];
  esector=relcnt-Length[redundant];
  relcnt=esector;
  Clear[esector];
  Clear[redundant];
```

a second **Print** *command verifies that the remaining relations are linearly independent. The price of this procedure is that the enumeration of the remaining independent relations is not longer consistent with the original enumeration. The original enumeration must be restored by means of taking the eliminated relations into account manually. This is tedious, but not problematic.*

$$\textbf{Print}\Big[\textbf{Solve}\Big[\sum_{cnt=1}^{\textbf{Length}[evensector]} evensector[[cnt]]coeff[cnt] == 0 \;/.\{N \to 2\}, \textbf{Table}\big[coeff[cnt], \{cnt, 1, \textbf{Length}[evensector]\}\big]\Big][[1]]\Big];$$

$$\textbf{Print}[\textbf{Length}[evensector]];$$

The remaining set of independent relations can be solved as a system of linear equations and is saved into a variable 'lsg'.

$$lsg = \textbf{Solve}\Big[\textbf{Table}\Big[\sum_{cnt2=1}^{\textbf{Length}[eventerms]} evensector[[cnt1, cnt2]]blanklist[[cnt2]] == 0/.\{N \to 2\}, \{cnt1, 1, \textbf{Length}[evensector]\}\Big], \textbf{Table}[dummy[cnt], \{cnt, 1, \textbf{Length}[eventerms]\}]\Big];$$

$$lsg[[1]]/.\textbf{Table}\big[dummy[cnt] \to eventerms[[cnt]], \{cnt, 1, \textbf{Length}[eventerms]\}\big]$$

$$\textbf{Length}[lsg[[1]]]$$

Comparison to Cayley-Hamilton relations

The next code segment is used for comparison of relations. After the monomials of [BCE 00] are reformulated in terms of the monomials of section 3.5.2, they are inserted into the Cayley-Hamilton relations, which are typed into a function BCE. Two examples are given.

$BCErel = 1;$
$BCE[BCErel] =$
$blanklist/.\,\{dummy[30] \to 2, dummy[36] \to 1, dummy[39] \to -\frac{1}{2}, dummy[44] \to -1\};$
$BCErel{+}{+};$
$BCE[BCErel] =$
$\frac{blanklist}{3}/.\{dummy[1] \to -4, dummy[2] \to 8, dummy[3] \to -2, dummy[4] \to -4,$
$dummy[5] \to 2, dummy[6] \to 1, dummy[7] \to -1, dummy[30] \to 2, dummy[35] \to 3,$
$dummy[36] \to -2, dummy[45] \to -3, dummy[62] \to 3, dummy[64] \to -3,$
$dummy[66] \to -1, dummy[68] \to -\frac{3}{2}, dummy[86] \to -2, dummy[87] \to 2,$
$dummy[88] \to -4, dummy[89] \to -8, dummy[91] \to 2, dummy[95] \to 4,$
$dummy[96] \to -2, dummy[97] \to -2, dummy[99] \to 4, dummy[118] \to 2,$
$dummy[119] \to -2, dummy[137] \to 2, dummy[138] \to -1, dummy[156] \to -2,$
$dummy[157] \to 6, dummy[158] \to -2, dummy[159] \to 2, dummy[160] \to 2,$
$dummy[162] \to -1, dummy[163] \to -1\};$
$BCErel{+}{+};$

The funtions 'BCE' are collected into a **Table** *'BCEch'. Each entry is decomposed as a linear combination of the maximal set of independent relations.*

$$
\begin{aligned}
&\mathit{BCEch} = \textbf{Table}\\
&\quad\Big[\mathit{BCE}[\mathit{BCEcnt}]/.\textbf{Table}\big[\mathit{dummy}[\mathit{cnt}] \to 0, \{\mathit{cnt}, 1, \textbf{Length}[\mathit{eventerms}]\}\big],\\
&\quad\{\mathit{BCEcnt}, 1, \mathit{BCErel} - 1\}\Big];\\
&\textbf{For}\Big[\mathit{BCEcnt} = 1, \mathit{BCEcnt} \le \mathit{BCErel} - 1, \mathit{BCEcnt}{+}{+},\\
&\quad\mathit{BCEdecomp}[\mathit{BCEcnt}] = \textbf{Solve}\\
&\quad\Bigg[\sum_{\mathit{cnt}=1}^{\textbf{Length}[\mathit{evensector}]} \mathit{coeff}[\mathit{cnt}]\mathit{evensector}[[\mathit{cnt}]] == \mathit{BCEch}[[\mathit{BCEcnt}]]\\
&\quad/.\{N \to 2\}, \textbf{Table}\big[\mathit{coeff}[\mathit{cnt}], \{\mathit{cnt}, 1, \textbf{Length}[\mathit{evensector}]\}\big]\Bigg]\Big];
\end{aligned}
$$

The commands in the ensuing **For** *loop generate output. Each of the coefficients of the linear combinations is appended to a list, which is printed in* **TableForm**. *The total number of relations required in the linear combination is also printed.*

$$
\begin{aligned}
&\textbf{For}\Big[\mathit{BCEcnt} = 1, \mathit{BCEcnt} \le \mathit{BCErel} - 1, \mathit{BCEcnt}{+}{+}, \textbf{Print}[\mathit{BCEcnt}];\\
&\quad\mathit{testcnt} = 0; \mathit{testlist} = \{\};\\
&\quad\textbf{For}\Big[\mathit{cnt} = 1, \mathit{cnt} \le \textbf{Length}[\mathit{evensector}], \mathit{cnt}{+}{+},\\
&\quad\quad\textbf{If}\Big[\mathit{BCEdecomp}[\mathit{BCEcnt}][[1, \mathit{cnt}, 2]] \ne 0,\\
&\quad\quad\mathit{testnewlist} = \textbf{Append}\big[\mathit{testlist}, \mathit{BCEdecomp}[\mathit{BCEcnt}][[1, \mathit{cnt}]]\big];\\
&\quad\quad\mathit{testlist} = \mathit{testnewlist}; \mathit{testcnt}{+}{+}\Big]\Big];\\
&\quad\textbf{Print}[\mathit{TableForm}[\mathit{testlist}]];\\
&\quad\textbf{Clear}[\mathit{testlist}]; \textbf{Clear}[\mathit{testnewlist}];\\
&\quad\textbf{Print}[\mathit{testcnt}, \text{" Relations are necessary."}]\Big];\\
&\textbf{Clear}[\mathit{BCErel}];
\end{aligned}
$$

1
coeff[98] $\to 1$
coeff[99] $\to \frac{1}{2}$
2 Relations are necessary.

2
coeff[1] $\to -\frac{2}{3}$
coeff[2] $\to \frac{2}{3}$
⋮
coeff[117] $\to -\frac{1}{3}$
coeff[120] $\to -\frac{1}{3}$
48 Relations are necessary.
⋮

At this point, the original enumeration has to be restored by hand. Which trace relation corresponds to a given Cayley-Hamilton relation is clear in most cases and in some cases up to a decision of the investigator.

Application of the elimination paradigm

The subsequent code seqment is concerned with the construction of a solution in accord with an elimination paradigm. The following function 'selecting' deletes the entries at the positions denoted by its arguments from 'blanklist'. The deleted monomials are those, which are not expressed in terms of others.

```
selecting[col1_, col2_, ..., col55_]:=
Module[{elims = {}, vars = blanklist, elimstemp = elims, varstemp = {}, cnt},
    For[cnt = 1, cnt ≤ Length[eventerms] − Length[evensector], cnt++,
        elimstemp = Append
        [elims, {vars[[col1]], vars[[col2]], ..., vars[[col55]]}[[cnt]]];
        elims = elimstemp];
    varstemp = Delete[vars, {{col1}, {col2}, ..., {col55}}];
    vars = varstemp; Return[vars]; ];
```

Applications of 'selecting' generate solutions. Starting at a list of arguments from 'lsg' they are exchanged one-by-one, until the desired form of the solution is generated. The **Print[Length[...]]** *command helps ensure that none of the selected monomials are linearly dependent on the rest of them. This scheme allows a good flexibility concerning the shape of the solutions.*

```
multipletracesolution = Solve[Table
        [ Sum_{cnt2=1}^{Length[eventerms]} evensector[[cnt1, cnt2]] blanklist[[cnt2]] == 0
        /.N → 2, {cnt1, 1, Length[evensector]}], selecting[15, 39, ..., 191]][[1]];
  Print[multipletracesolution
        /.Table[dummy[cnt] → eventerms[[cnt]], {cnt, 1, Length[eventerms]}]];
  Print[Length[multipletracesolution]];
```

The generated output[1] *starts like this:*

$$\begin{aligned}(1) \to \tfrac{1}{48}\big(&12(15) + 6(39) + 6(40) + 18(44) + 42(45) + 48(79) + 36(85) - 36(101) - 96(107)\\ &-36(110) - 48(111) + 12(120) - 12(122) + 4(123) - 36(131) + 12(133) - 72(140)\\ &-132(145) + 36(151) - 36(154) - 9(161) - 15(164) - 48(165) + 48(169)\\ &+108(171) + 144(172) - 138(173) + 420(174) - 60(175) - 24(176) - 108(177)\\ &+60(189) + 48(190) - 144(191)\big)\end{aligned}$$

[1] This output is where the enhancement of LECs can be read off directly. Any monomial appearing on the right hand side takes on the worst N_c-dependency of the monomials on the left hand sides, which are expressed in terms of it.

$$(2) \rightarrow \frac{1}{48}\big(12(15) - 6(39) + 6(40) - 6(44) - 30(45) - 24(79) - 36(85) + 12(101) + 96(107) + 12(110) + 48(111) - 12(120) + 12(122) - 4(123) + 36(131) - 12(133) + 48(140) + 132(145) - 12(151) + 12(154) + 3(161) + 9(164) + 24(165) - 48(169) - 132(171) - 48(172) + 150(173) - 348(174) + 36(175) - 24(176) + 180(177) - 60(189) + 48(191)\big)$$

Construction of new relations

The following list 'BCEterms' contains the correspondences of table 3.21.

$$\boldsymbol{BCEterms = \{dummy[30], dummy[36], dummy[35], dummy[51],}$$

$\vdots$

$$\boldsymbol{\tfrac{1}{8}(dummy[173] + dummy[176]) - \tfrac{1}{32}(dummy[167] + dummy[170]),}$$
$$\boldsymbol{\tfrac{1}{2}\Big(\tfrac{1}{2}(dummy[112] + dummy[113] + dummy[118] + dummy[119]) + (dummy[124] + dummy[125]) - 2(dummy[128] + dummy[129]) - (dummy[130] + dummy[131]) + (dummy[132] + dummy[133])\Big)\};}$$

A solution (here: 'singletracesolution', because it is more similar to the solution of [BCE 00]) is inserted into the monomials of [BCE 00]. The Y_i *are decomposed in terms of the solution.*

```
BCEmonomials = BCEterms/.singletracesolution;
Print[ColumnForm[Table
    [{cnt, BCEmonomials[[cnt]]}, {cnt, 1, Length[BCEmonomials]}]]];
```

Some excerpts of output follow. The monomials Y_{27} *and* Y_{28} *are obviously equivalent, the other dependence is more subtle.*

$\{1, \text{dummy}[30]\}$
$\{2, \text{dummy}[36]\}$
$\{3, \frac{1}{6}(20\text{dummy}[1] - 8\text{dummy}[2] - 12\text{dummy}[5] - 8\text{dummy}[30] + 2\text{dummy}[36] + 12\text{dummy}[64] - 16\text{dummy}[88] - 8\text{dummy}[95] + 8\text{dummy}[97] - 8\text{dummy}[118] + 2\text{dummy}[119] + 3\text{dummy}[121] - 4\text{dummy}[137] + 8\text{dummy}[156] - 8\text{dummy}[157] + 8\text{dummy}[158] - 6\text{dummy}[159] - 2\text{dummy}[160])\}$
$\{4, \text{dummy}[51]\}$
$\vdots$
$\{27, \text{dummy}[88]\}$
$\{28, -\text{dummy}[88]\}$
$\vdots$
$\{48, \text{dummy}[137]\}$
$\{49, -\frac{\text{dummy}[137]}{2} - \text{dummy}[142]\}$
$\vdots$

The next step uses the independence of the monomials to demand relations between the Y_i.

```
ColumnForm[DeleteCases[Table
  [{comcnt,
    ∂_dummy[comcnt] ( Sum[coeff[cnt] BCEmonomials[[cnt]], {cnt, 1, Length[BCEmonomials]}] ) == 0}
   /.Table[dummy[cnt] → 0, {cnt, 1, Length[eventerms]}],
   {comcnt, 1, Length[eventerms]}], {_Integer, True}]]
```

Here, the Y_i are represented by 'coeff[i]'. The first column denotes, from which independent monomial the relation is derived.

$$\{1, \tfrac{10\mathrm{coeff}[3]}{3} + \mathrm{coeff}[24] = 0\}$$
$$\vdots$$
$$\{36, \mathrm{coeff}[2] + \tfrac{\mathrm{coeff}[3]}{3} = 0\}$$
$$\{51, \mathrm{coeff}[4] = 0\}$$
$$\vdots$$

The new system of equations is solved in the usual way. The **Length** *command allows quick verification that the number of independent relations in both this thesis and [BCE 00] coincide.*

```
newrel = ColumnForm[Solve[Table
   [∂_dummy[comcnt] ( Sum[coeff[cnt] BCEmonomials[[cnt]], {cnt, 1, Length[BCEmonomials]}] ) == 0,
   {comcnt, 1, Length[eventerms]}],
   Table[coeff[cnt], {cnt, 1, Length[BCEmonomials]}]][[1]]]
Length[newrel[[1]]]
```

Excerpts of the output follow:

$$\mathrm{coeff}[1] \to 2\mathrm{coeff}[48]$$
$$\mathrm{coeff}[2] \to -\tfrac{\mathrm{coeff}[48]}{2}$$
$$\mathrm{coeff}[3] \to \tfrac{3\mathrm{coeff}[48]}{2}$$
$$\mathrm{coeff}[4] \to 0$$
$$\vdots$$
$$\mathrm{coeff}[27] \to \mathrm{coeff}[28] + 4\mathrm{coeff}[48]$$
$$\vdots$$

The coefficients of the new relations can be read off directly, here $Y_{27} + Y_{28} = 0$ and $2Y_1 - \frac{1}{2}Y_2 + \ldots + Y_{48} + \ldots = 0$.

Matching of LECs

The rest of the source code originates in the three-flavour case of the anomalous sector. Nomenclature of variables is somewhat different. In the variable names 'eventerms' and 'evensector' "even" is replaced by "ϵ". Furthermore, the solution is denoted by 'lösung'. The Lagrangian $\mathcal{L}$ is produced as a sum of all monomials times some arbitrary constants 'C[cnt]', the original LECs.

$$\mathcal{L} = \sum_{cnt=1}^{\mathbf{Length}[\epsilon terms]} dummy[cnt]C[cnt];$$

The **Module** *'manualsort' inserts the solution 'lösung' in the Lagrangian. Due to the linearity of the Lagrangian in the monomials, the best way of reorganizing the Lagrangian sorted as*

$$\sum_{\text{LECs i}} C[i] \times (\sum_{\text{monomials j}} (\text{j})_i)$$

in a form

$$\sum_{\text{monomials i}} (\text{i}) \times (\sum_{\text{LECs j}} C_i[j])$$

is achieved by doing a first order Taylor expansion in the monomials. This reorganisation is done by 'manualsort', too.

$$\begin{aligned}
&\textbf{manualsort}[\mathcal{L}_]:= \\
&\mathbf{Module}\Big[\{lagrangian = 0, lag = 0, termcnt, reducedlist, reduceddummy\}, \\
&\quad reducedlist[termcnt_]:=\mathbf{Join} \\
&\qquad \Big[\mathbf{Table}\big[dummy[cnt], \{cnt, 1, termcnt - 1\}\big], \\
&\qquad\quad \mathbf{Table}\Big[dummy[cnt], \{cnt, termcnt + 1, \mathbf{Length}[\epsilon terms]\}\Big]\Big]; \\
&\quad reduceddummy[termcnt_, cnt_]:=reducedlist[termcnt][[cnt]]; \\
&\quad lagrangian = 0; \\
&\quad \mathbf{For}\Big[termcnt = 1, termcnt \leq \mathbf{Length}[\epsilon terms], termcnt{+}{+}, \\
&\qquad lag = lagrangian \\
&\qquad\quad +\Big(\mathcal{L}/.\text{lösung}/.\mathbf{Table}\Big[reduceddummy[termcnt, cnt] \to 0, \\
&\qquad\qquad\qquad \{cnt, 1, \mathbf{Length}[\epsilon terms] - 1\}\Big]\Big); \\
&\qquad lagrangian = lag; \Big]; \\
&\quad lag = \sum_{cnt=1}^{\mathbf{Length}[\epsilon terms]} \partial_{dummy[cnt]} lagrangiandummy[cnt]; \mathbf{Return}[lag]; \Big]; \\
&\qquad\qquad \mathcal{L}angrangian = manualsort[\mathcal{L}];
\end{aligned}$$

The linear combinations of LECs, which have been derived before, are given new names.

$$m_i \equiv \sum_{\text{LECs j}} C_i[j]$$

In the ensuing paragraph, the sets of LECs of this work and of [BGT 02, EFS 02] are typed in manually.

mycouplings=

0	*m2*	0	*m4*	*m5*	0	*m7*	0	*m9*
m10	0	*m12*	0	*m14*	0	*m16*	0	0
0	0	0	0	0	0	0	0	0
m28	*m29*	*m30*	*m31*	*m32*	0	0	0	0
0	0	0	0	0	*m42*	0	*m44*	0
0	0	*m48*	0	0	0	*m52*	0	0
0	0	0	*m58*	0	0	*m61*	*m62*	*m63*
m64	*m65*	0						

tecouplings=

0	16*L12*	0	−16*L4*	0	0	−32*L16*	−32*L15*	32*L18*
32*L17*	0	−8*L13*	−8*L14*	0	0	0	0	0
0	0	0	0	0	0	0	0	0
−8*L5*	−8*L6*	−8*L7*	−8*L1*	−8*L2*	0	0	0	0
0	0	0	0	0	−16*L21*	−16*L22*	0	−16*L23*
0	−16*L24*	0	0	0	0	−8*L19*	0	0
0	0	0	−8*L20*	0	0	−4*L3*	4*L8*	4*L9*
4*L10*	4*L11*	0						

bicouplings=

0	*k12*	0	*k1*	−*k16*	0	*k17*	0	−*k18*
k24	0	−*k14*	0	−*k13*	0	−*k15*	0	0
0	0	0	0	0	0	0	0	0
k4	*k5*	*k6*	−*k2*	−*k3*	0	0	0	0
	0	0	0	0	−*k19*	0	−*k20*	0
0	0	*k21*	0	0	0	−*k22*	0	0
0	0	0	−*k23*	0	0	*k11*	*k7*	*k8*
k9	*k10*	0						

When a minimal set of LECs is inserted into the Lagrangian, which has been reduced to the minimal set of independent monomials, a result equivalent to the sorting algorithm results. The LECs originating in another minimal set can be inserted, too.

$$\begin{aligned}
&\textit{LECcompare} = \textbf{Table}\\
&\quad\Big[\Big\{\partial_{\textit{dummy}[\textit{cnt1}]}\mathcal{L}\textit{angrangian}\\
&\qquad /.\textbf{Table}\Big[C[\textit{cnt}] \to \textit{mycouplings}[[\textit{cnt}]], \{\textit{cnt}, 1, \textbf{Length}[\epsilon\textit{terms}]\}\Big],\\
&\quad\ \partial_{\textit{dummy}[\textit{cnt1}]}\mathcal{L}\textit{angrangian}\\
&\qquad /.\textbf{Table}\Big[C[\textit{cnt}] \to \textit{tecouplings}[[\textit{cnt}]], \{\textit{cnt}, 1, \textbf{Length}[\epsilon\textit{terms}]\}\Big],\\
&\quad\ \partial_{\textit{dummy}[\textit{cnt1}]}\mathcal{L}\textit{angrangian}\\
&\qquad /.\textbf{Table}\Big[C[\textit{cnt}] \to \textit{bicouplings}[[\textit{cnt}]], \{\textit{cnt}, 1, \textbf{Length}[\epsilon\textit{terms}]\}\Big]\Big\},\\
&\quad\{\textit{cnt1}, 1, \textbf{Length}[\epsilon\textit{terms}]\}\Big];
\end{aligned}$$

The following step eliminates the cases of eliminated monomials.

```
LEClist = LECcompare;
LECcompare = DeleteCases[LEClist, {0, 0, 0}]; Clear[LEClist];
```

The last command provides a formatted list, where the corresponding linear combinations of LECs can be read off. An excerpt of the output ensues.

```
Print[TableForm[LECcompare, TableAlignments → Center,
  TableHeadings →
    {None, {"my LECs", "T.E.'s LEC's", "Bij.'s LECs"}}]];
```

my LECs	T.E.'s LEC's	Bij.'s LECs
m2	16L12	k12
m4	−16L4	k1
2m10 + m5	−64L15 + 64L17	−k16 − 2k24
2m10 + m7	−32L15 − 32L16 + 64L17 + 32L23	k17 − 2k24
⋮	⋮	⋮

Appendix B

Singlet η and anomalous processes

B.1 Weak neutral currents for three light flavours

The representation of weak neutral currents in the external field formalism is constructed from the external field part of the QCD Lagrangian. In addition to Eq. (2.5), the axial-vector flavour singlet has to be included:

$$\mathcal{L}^{ext} = \overline{q}_R\Big(\not r + \frac{1}{N_c}(\not v^{(s)} + \not a^{(s)})\Big)q_R + \overline{q}_L\Big(\not l + \frac{1}{N_c}(\not v^{(s)} - \not a^{(s)})\Big)q_L. \tag{B.1}$$

The weak current interaction with quarks of the three light flavours is induced by minimal coupling:

$$\begin{aligned} \mathcal{L}^{q\overline{q}Z^0} = -\tfrac{g}{2\cos\vartheta_W} Z_\alpha \times \quad & \Big(\overline{u}\gamma^\alpha\{[\tfrac{1}{2} - \tfrac{4}{3}\sin^2\vartheta_W] - \tfrac{1}{2}\gamma_5\}u \\ & +\overline{d}\gamma^\alpha\{[-\tfrac{1}{2} + \tfrac{2}{3}\sin^2\vartheta_W] + \tfrac{1}{2}\gamma_5\}d \\ & +\overline{s}\gamma^\alpha\{[-\tfrac{1}{2} + \tfrac{2}{3}\sin^2\vartheta_W] + \tfrac{1}{2}\gamma_5\}s\Big). \end{aligned} \tag{B.2}$$

It is decomposed in terms of left- and right-handed quark fields. Contributions are due to the three diagonal generators only. Each flavour and handedness generates one equation, e.g.:

$$R,\, u:\quad \frac{1}{2}\Big(\frac{2}{N_c}\big(v_\alpha^{(s)} + a_\alpha^{(s)}\big) + r_\alpha^3 + \frac{r_\alpha^8}{\sqrt{3}}\Big) = \big(-\frac{4}{3}\sin^2\vartheta_W\big)\big(\frac{-g}{2\cos\vartheta_W}Z_\alpha\big). \tag{B.3}$$

As a consequence, the external field configuration

$$\begin{aligned} v_\alpha^{(s)} = -a_\alpha^{(s)} &= \quad -\tfrac{1}{2}\tfrac{N_c}{N_f}\,\tfrac{1}{2}\tfrac{g}{\cos\vartheta_W}Z_\alpha, \\ l_\alpha^3 = \sqrt{3}l_\alpha^8 &= -2\cos^2\vartheta_W\,\tfrac{1}{2}\tfrac{g}{\cos\vartheta_W}Z_\alpha, \\ r_\alpha^3 = \sqrt{3}r_\alpha^8 &= \quad 2\sin^2\vartheta_W\,\tfrac{1}{2}\tfrac{g}{\cos\vartheta_W}Z_\alpha. \end{aligned} \tag{B.4}$$

describes interactions with weak neutral currents.

Bibliography

[AD 68] S.L. Adler and R.F. Dashen, Current Algebras and Applications to Particle Physics, W.A. Benjamin (New York, 1968).

[Adl 69] S. L. Adler, Phys. Rev. 177, 2426 (1969).

[BCE 00] J. Bijnens, G. Colangelo, and G. Ecker, Annals Phys. 280, 100 (2000), hep- ph/9907333.

[BGT 02] J. Bijnens, L. Girlanda, and P. Talavera, Eur. Phys. J. C23, 539 (2002), hep-ph/0110400.

[Bij 91] J. Bijnens, Talk at the Workshop on Effective Field Theories of the Standard Model, Dobogoko, Hungary, Aug 22-26, 1991.

[BK 73] E. Byckling and K. Kajantie, Particle Kinematics (London: Wiley, 1973).

[BPST 75] A. A. Belavin, A. M. Polyakov, A. S. Shvarts, and Y. S. Tyupkin, Phys. Lett. B59, 85 (1975).

[Bro 03] I. Bro 03 et al., Teubner-Taschenbuch der Mathematik (Stuttgart: Teubner, 2003).

[CGL 01] G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Rev. Lett. 86, 5008 (2001), hep-ph/0103063.

[Col 66] S. Coleman, J. Math. Phys. 7, 787 (1966).

[Das 71] R. F. Dashen, Phys. Rev. D3, 1879 (1971).

[EFS 02] T. Ebertshauser, H. W. Fearing, and S. Scherer, Phys. Rev. D65, 054033 (2002), hep-ph/0110261.

[FS 96] H. W. Fearing and S. Scherer, Phys. Rev. D53, 315 (1996), hep-ph/9408346.

[GL 85] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985).

[Hac 08] Ch. Hacker, PhD Thesis, Mainz, 2008.

[Hoo 74] G. 't Hooft, Nucl. Phys. B72, 461 (1974).

[KSD 96] G. Knochlein, S. Scherer, and D. Drechsel, Prog. Part. Nucl. Phys. 36, 137 (1996), hep-ph/9510374.

[Mor 94] D. Morris, Estimating the Strong Decay Width $\Gamma(\eta \to 4\pi^0)$, 1994.

[Nuy 71] J. Nuyts, Phys. Rev. Lett. 26, 1604 (1971).

[PDG 2008] W.-M. Yao et al., Journal of Physics G 33, 1+ (2006).

[Sch 03] S. Scherer, Adv. Nucl. Phys. 27, 277 (2003), hep-ph/0210398.

[SF 95] S. Scherer and H. W. Fearing, Phys. Rev. D52, 6445 (1995), hep-ph/9408298.

[VV 80] P. Di Vecchia and G. Veneziano, Nucl. Phys. B171, 253 (1980).

[Wei 79] S. Weinberg, Phenomenological Lagrangians, 1979.

[Wit 79a] E. Witten, Nucl. Phys. B160, 57 (1979).

[Wit 79b] E. Witten, Nucl. Phys. B156, 269 (1979).

[Wit 80] E. Witten, Ann. Phys. 128, 363 (1980).

[Wit 83] E. Witten, Nucl. Phys. B223, 422 (1983).

[WZ 71] J. Wess and B. Zumino, Phys. Lett. B37, 95 (1971).