Mittwoch, 10. April 2024 12:02

FUNKTIONEN MEHRERE VARIABELN

Definition: Sei U eine Teilmenge des $\mathbb{R}^n$.

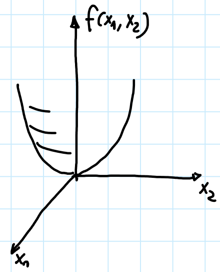
$$f: U \longrightarrow \mathbb{R}$$

$$(x_1, x_2, \dots, x_n) \longrightarrow f(x_1, x_2, \dots x_n) = y \qquad \text{SKALAR}$$

BEISPIEL $U = \mathbb{R}^2$

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$(x_1, x_2) \longrightarrow f(x_1, x_2) = x_1^2 + x_2^2 \in \mathbb{R}$$



PARTIELLE ABLEITUNG

Definition Die Funktion f ist PARTIELL DIFFERENZIERBAR IN DER i-TEN KOORDINATE, falls der Grenzwert

$$\lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} = \frac{\partial}{\partial x_i} f(x_1, \dots, x_i, \dots x_n) \qquad \text{i-te Partielle Ableitung}$$

existiert.

Man hält alle anderen Variabeln $x_1, \dots x_{i-1}, x_{i+1}, \dots x_n$ fest und nimmt die gewöhnliche Ableitung nach der Variablen x_i.

PARTIELLE DIFFERENZIEBARKEIT

Definition: Wir nennen eine Funktion PARTIELL DIFFERENZIEBAR, falls sie in alle Variabeln partiell differenzierbar ist.

Ebenso nennen wir eine Funktion STETIG PARTIELL DIFFERENZIERBAR, falls sie partiell differenzierbar ist und alle Ableitungen stetig sind.

BEISPIEL

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$(x_1, x_2, x_3) \longrightarrow f(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\frac{\partial}{\partial x_1} f(x_1, x_2, x_3) = \frac{1 \cdot \cancel{2} x_1}{\cancel{2}\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\frac{\partial}{\partial x_2} f(x_1, x_2, x_3) = \frac{1 \cdot \cancel{2} x_2}{\cancel{2}\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\frac{\partial}{\partial x_3} f(x_1, x_2, x_3) = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$f: \mathbb{R}^3 \to \mathbb{R}$$

$$(x_1, x_2, x_3) \to f(x_1, x_2, x_3) = \sqrt{x_1^3 + x_2 + x_3^2}$$

$$\frac{\partial}{\partial x_1} f(x_1, x_2, x_3) = \frac{3 \cdot x_1^2}{2\sqrt{x_1^3 + x_2 + x_3^2}}$$

$$\frac{\partial}{\partial x_2} f(x_1, x_2, x_3) = \frac{1}{2\sqrt{x_1^3 + x_2 + x_3^2}}$$

$$\frac{\partial}{\partial x_3} f(x_1, x_2, x_3) = \frac{\cancel{2} x_3^2}{\cancel{2}\sqrt{x_1^3 + x_2 + x_3^2}}$$

QUIZ: $f(x, y) = x^2 e^{y^2}$

$$\frac{\partial}{\partial y} f(x,y) \stackrel{?}{=} x^2 \cdot e^{y^2} \cdot 2y$$

A) $x^2 e^{y^2}$ B) $2x^2 e^{y^2}$ C) $x e^{y^2}$ D) $2x^2 y e^{y^2}$

$$\frac{\partial}{\partial x}\left(x^2 e^{y^2}\right) = 2x e^{y^2}$$

QUIZ: $f(x,t) = A \sin(x - vt)$

$$\frac{\partial}{\partial t} f(x,t) \stackrel{?}{=} A \cos(x - vt)(-v) = -vA \cos(x - vt)$$

A) $A \cos(x - vt)$

B) $-A \cos(x - vt)$

C) $v A \cos(x - vt)$

D) $-vA\cos(x-vt)$

HÖHERE PARTIELLE ABLEITUNGEN

<u>Definition</u>: Wir können partielle Ableitungen auch hintereinander ausführen und erhalten höhere Ableitungen

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f(x_1, \dots, x_m) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f(x_1, \dots x_m)\right)$$

Zuerst Ableitung nach x_j und dann Ableitung nach x_i

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} f \overset{?}{=} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f$$

THEOREM Sei f zweimal stetig partielle differenzierbar. Dann giet für die partiellen Ableitungen.

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f(x_1, \dots x_m) = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} f(x_1, \dots x_m)$$

<u>BEISPIEL</u>

$$f: \mathbb{R}^3 \to \mathbb{R}$$
$$(x_1, x_2, x_3) \longrightarrow f(x_1, x_2, x_3) = x_1^3 + 3x_1x_2^2 + x_1x_2x_3$$

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} f(x_1, x_2, x_3) = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\left(x_1^3 + 3x_1x_2^2 + x_1x_2x_3\right) = \frac{\partial}{\partial x_1}\left(3\cdot x_1 \cdot 2x_2 + x_1x_3\right) = \frac{\partial}{\partial x_1}\left(6x_1x_2 + x_1x_3\right) = 6x_2 + x_3$$

$$\frac{\partial x_1^3}{\partial x_2} \overset{?}{=} 0$$

$$\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} f(x_1, x_2, x_3) = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\left(x_1^3 + 3x_1x_2^2 + x_1x_2x_3\right) = \frac{\partial}{\partial x_2}\left(3x_1^2 + 3x_2^2 + x_2x_3\right) = 6x_2 + x_3$$

$$\underbrace{\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}}\, f(x_1, x_2, x_3) = \frac{\partial^2}{\partial x_1^2} f(x_1, x_2, x_3) = \frac{\partial}{\partial x_1}\left(3x_1^2 + 3x_2^2 + x_2x_3\right) = 6x_1$$

<u>QUIZ</u>: $f(x_1, x_2) = 3x_1^2x_2^2$ $\quad 3x_1^2x_2^3$

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} f(x_1, x_2) \overset{?}{=} \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\left(3x_1^2x_2^2\right) = \frac{\partial}{\partial x_1}\left(3x_1^2 \cdot 2x_2\right) = \frac{\partial}{\partial x_1}\left(6x_1^2 \cdot x_2\right) = 6\cdot x_2 \cdot 2x_1 \overset{!}{=} 12x_1x_2$$

A) 0

B) $9x_1^2$

C) $12x_1x_2$ $\longrightarrow$ $18x_1x_2^2$

D) $9x_1^2x_2^2 + 6x_1x_2^3$

10:30 Zurück.

DAS TOTALE DIFFERENTIAL

$$(f \circ x): \mathbb{R} \longrightarrow \mathbb{R}$$
$$x: \mathbb{R} \longrightarrow \mathbb{R}^m$$
$$t \longrightarrow (x_1, \dots x_m)$$
$$f: \mathbb{R}^m \longrightarrow \mathbb{R}$$

$$(f \circ x): \mathbb{R} \to \mathbb{R}$$
$$t \longrightarrow f(x(t)) = f(x_1(t), \dots x_m(t))$$

$(x_1, \dots x_m) \longrightarrow f(x_1, \dots x_m)$

$\partial \leftrightarrow d$

Kettenregeln $\quad \frac{df}{dt} = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} \cdot \frac{\partial x_i}{\partial t}$

DAS TOTALE DIFFERENTIAL $\quad df = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\cancel{\partial t}} \cancel{dt} = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} dx_i$

GRADIENT

$f: \mathbb{R}^m \to \mathbb{R}$

$(x_1, \dots x_m) \longrightarrow f(x_1, \dots x_m)$

$$\text{grad } f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{pmatrix} = \bar{\nabla} f \qquad \text{VEKTOR}$$

Nabla-Operator $\quad \bar{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_m} \end{pmatrix}$

DIVERGENZ UND ROTATION

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$(x_1, x_2, x_3) \longrightarrow (f_1, f_2, f_3) = \bar{f}$

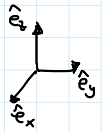
Mit Hilfe von $\bar{\nabla}$ können wir die Divergenz definieren

$$\text{div } \bar{f} = \bar{\nabla} \cdot \bar{f} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{\partial}{\partial x_1} f_1 + \frac{\partial}{\partial x_2} f_2 + \frac{\partial}{\partial x_3} f_3$$

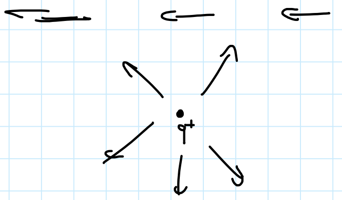
$$\text{div } \bar{f} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z$$

und Rotation "curl"

$$\text{rot } \bar{f} = \bar{\nabla} \times \bar{f} = \begin{pmatrix} \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \\ \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \\ \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \end{pmatrix}$$



$$\bar{f} = -x^2 \hat{e}_y = \begin{pmatrix} 0 \\ -x^2 \\ 0 \end{pmatrix} \qquad \text{div } f = 0 + \frac{\partial}{\partial y}(-x^2) + 0 = 0 \qquad \text{rot } \bar{f} = \begin{pmatrix} 0 - \frac{\partial}{\partial z}(-x^2) \\ 0 \\ \frac{\partial}{\partial x}(-x^2) - 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2x \end{pmatrix}$$



LAPLACE OPERATOR

$f: \mathbb{R}^m \to \mathbb{R}$ (zweimal differenzierbar)

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x} = \frac{\partial^2}{\partial x^2}$$

LAPLACE-OPERATOR

$$\text{div. grad } f = \vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 f$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} f = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial x} f + \frac{\partial}{\partial y}\frac{\partial}{\partial y} f + \frac{\partial}{\partial z}\frac{\partial}{\partial z} f\right)$$

$$= \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f$$

Operator $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$