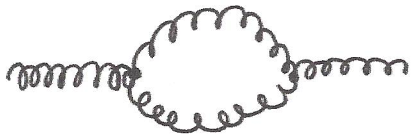
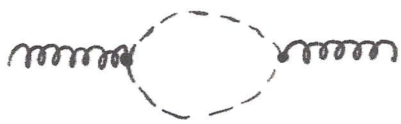
GLUON SELF-ENERGY

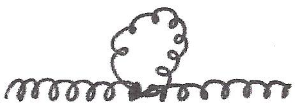
$i\,\Pi^{ab}_{\mu\nu}(k)$ $=$



$i\,\Pi^{ab}_{G,\mu\nu}(k)$

$+$ 

$i\,\Pi^{ab}_{GH,\mu\nu}(k)$

$+$ 

$i\,\Pi^{ab}_{T,\mu\nu}(k)$

$+$ 

$i\,\Pi^{ab}_{F,\mu\nu}(k)$

$+$ 

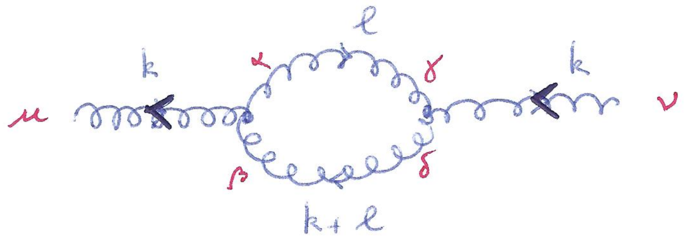
$i\,\Pi^{ab}_{CT,\mu\nu}(k)$

GAUGE INV.
$$k^\mu\,\Pi^{ab}_{\mu\nu}(k) = 0$$
$$k^\nu\,\Pi^{ab}_{\mu\nu}(k) = 0$$

$$\boxed{\Pi^{ab}_{\mu\nu}(k) = \delta_{ab}\left(k_\mu k_\nu - k^2 g_{\mu\nu}\right)\Pi(k^2)}$$

$$\hookrightarrow\ \Pi(k^2) = g^{\mu\nu}\left(\frac{-1}{(D-1)k^2}\right)\Pi^{aa}_{\mu\nu}(k)$$

$\Rightarrow$ GLUON LOOP FOR $\xi = 1$.



SUM OVER b & c BUT NOT OVER a !

$$i\,\Pi_{GL}(k^2) = -\frac{1}{3k^2}\cdot\frac{1}{2}\int\frac{d^4\ell}{(2\pi)^4}\left[-g\, f_{abc}\, V_\mu{}^{\alpha\beta}(-k,-\ell,k+\ell)\right]$$

SYMM. FACTOR

$$\frac{(-i\,g_{\alpha\gamma})}{\ell^2+i\varepsilon}\cdot\frac{(-i\,g_{\beta\delta})}{(k+\ell)^2+i\varepsilon}$$

$$\cdot\left[-g\, f_{abc}\, V^{\mu\gamma\delta}(k,\ell,-k-\ell)\right]$$

$$= \frac{g^2 f_{abc} f_{abc}}{3k^2}\cdot\frac{1}{2}\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{\left[\ell^2+i\varepsilon\right]\left[(k+\ell)^2+i\varepsilon\right]}$$

$$\cdot\underbrace{V_\mu{}^{\alpha\beta}(-k,-\ell,k+\ell)\,V^\mu{}_{\alpha\beta}(k,\ell,-k-\ell)}_{-18\,(k^2+\ell^2+k.\ell)}$$

$$= -\frac{6g^2 f_{abc} f_{abc}}{2k^2}\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{(k^2+\ell^2+k.\ell)}{\left[(\ell+kx)^2+k^2x(1-x)\right]^2}$$

$$= -\frac{6g^2 f_{abc} f_{abc}}{2k^2}\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{k^2+(\ell-kx)^2+k.(\ell-kx)}{\left[\ell^2+k^2x(1-x)\right]^2}$$

$$= -\frac{6g^2 f_{abc} f_{abc}}{2k^2}\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{k^2(1+x^2-x)+\ell^2}{\left[\ell^2+k^2x(1-x)\right]^2}$$

$\Rightarrow$ HELP

$$V_\mu{}^{\alpha\beta}(-k,-\ell,k+\ell)\;V^\mu{}_{\alpha\beta}(k,\ell,-k-\ell)$$

$$= \left\{ g_\mu^\alpha(-k+\ell)^\beta + g^{\alpha\beta}(-2\ell-k)_\mu + g_\mu^\beta(2k+\ell)^\alpha \right\}$$

$$\cdot \left\{ g_\alpha^\mu(k-\ell)_\beta + g_{\alpha\beta}(2\ell+k)^\mu + g_\beta^\mu(-2k-\ell)_\alpha \right\}$$

$$= -4(k-\ell)^2 + (-k+\ell).(2\ell+k) + (-k+\ell).(-2k-\ell)$$

$$-4(2\ell+k)^2 + (-2\ell-k).(k-\ell) + (-2\ell-k).(-2k-\ell)$$

$$-4(2k+\ell)^2 + (2k+\ell).(k-\ell) + (2k+\ell).(2\ell+k)$$

$$= -4(k-\ell)^2 + (-k+\ell)^2$$

$$-4(2\ell+k)^2 + (-2\ell-k)^2$$

$$-4(2k+\ell)^2 + (2k+\ell)^2$$

$$= -3\left\{(k-\ell)^2 + (2\ell+k)^2 + (2k+\ell)^2\right\}$$

$$= -3\left\{k^2 - 2k.\ell + \ell^2 + 4\ell^2 + 4k.\ell + k^2 + 4k^2 + 4k.\ell + \ell^2\right\}$$

$$= -18\left\{k^2 + \ell^2 + k.\ell\right\}$$

$\downarrow$ WE ARE ONLY INTERESTED IN MOST DIVERGENT TERM

$$= -\frac{6g^2 f_{abc} f_{abc}}{2k^2} \int_0^1 dx \frac{i}{(4\pi)^2} \Big\{ k^2 (1+x^2-x)\, \Gamma(\varepsilon)$$

$$+ 2\,\Gamma(-1+\varepsilon) \left[k^2 x(1-x)\right]^{1-\varepsilon} \Big\}$$

$$= -\frac{6g^2 f_{abc} f_{abc}}{2k^2} \frac{i}{(4\pi)^2} \cdot \frac{1}{\varepsilon} k^2 \int_0^1 dx\, \underbrace{\underbrace{\left\{ (1+x^2-x) - 2x + 2x^2 \right\}}_{1+1-\frac{3}{2}}}_{1/2}$$

$$= \frac{i(-3)g^2}{2(4\pi)^2} f_{abc} f_{abc} \cdot \frac{1}{\varepsilon}$$

$$\therefore \quad i\Pi_{GL}(k^2) = i\frac{1}{\varepsilon}\frac{g^2}{(4\pi)^2} \cdot \left(\frac{-3}{2}\right) \cdot \underbrace{f_{abc} f_{abc}}_{\equiv \underbrace{\sum_{b,c} f_{abc} f_{abc}}_{3\delta_{aa} = 3}}$$

$$\boxed{i\Pi_{GL}(k^2) = i\frac{1}{\varepsilon}\frac{g^2}{(4\pi)^2}\left(-\frac{9}{2}\right) + O(\varepsilon^0)}$$

FOR $\xi \neq 1$ $\quad \boxed{i\Pi_{GL}(k^2) = i\frac{1}{\varepsilon}\frac{g^2}{(4\pi)^2} \cdot \frac{1}{2}\left\{-9 - 3(1-\xi)\right\}}$

$\Longrightarrow$ GAUGE INV. OF GLUON SELF-ENERGY

- $k_\mu \, i\Pi^{\mu\nu}_{GL}(k)$

$$= - g^2 f_{abc} f_{abc} \frac{1}{2} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 + i\varepsilon][(k+\ell)^2 + i\varepsilon]} k_\mu V^{\mu\alpha\beta}(-k,-\ell,k+\ell) V^{\nu}{}_{\alpha\beta}(k,\ell,-k-\ell)$$

$$= - g^2 f_{abc} f_{abc} \frac{1}{2} \cdot \int \frac{d^4\ell}{(2\pi)^4} \frac{\{ k^\nu (-3k^2 - 7k.\ell - 2\ell^2) + \ell^\nu (-5k^2 - 10 k.\ell) \}}{[\ell^2 + i\varepsilon][(k+\ell)^2 + i\varepsilon]}$$

$$= - g^2 f_{abc} f_{abc} \frac{1}{2} \cdot \int_0^1 dx \cdot \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 + k^2 x(1-x)]^2} \{ k^\nu [-3k^2 - 7k.(\ell - kx) - 2(\ell - kx)^2 + 5k^2 x + 10\, x\, k.(\ell - kx)] + \ell^\nu [-5k^2 - 10\, k.(\ell - kx)] \}$$

$$= - g^2 f_{abc} f_{abc} \frac{1}{2} \cdot \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 + k^2 x(1-x)]^2} \{ k^\nu [-3k^2 (1-2x)^2 - 2\ell^2] - 10\, k.\ell\, \ell^\nu \}$$

$\Rightarrow$ HELP

$$V^{\mu\alpha\beta}(-k,-\ell,k+\ell)\,V^{\nu}{}_{\alpha\beta}(k,\ell,-k-\ell)$$

$$= \left\{ g^{\mu\alpha}(-k+\ell)^{\beta} + g^{\alpha\beta}(-2\ell-k)^{\mu} + g^{\beta\mu}(2k+\ell)^{\alpha} \right\}$$

$$\cdot \left\{ g^{\nu}_{\alpha}(k-\ell)_{\beta} + g_{\alpha\beta}(2\ell+k)^{\nu} + g^{\nu}_{\beta}(-2k-\ell)_{\alpha} \right\}$$

$$= -(k-\ell)^2 g^{\mu\nu} + (-k+\ell)^{\mu}(2\ell+k)^{\nu} + (-k+\ell)^{\nu}(-2k-\ell)^{\mu}$$

$$+ (k-\ell)^{\nu}(-2\ell-k)^{\mu} - 4(2\ell+k)^{\mu}(2\ell+k)^{\nu} + (-2\ell-k)^{\mu}(-2k-\ell)^{\nu}$$

$$+ (k-\ell)^{\mu}(2k+\ell)^{\nu} + (2k+\ell)^{\mu}(2\ell+k)^{\nu} - g^{\mu\nu}(2k+\ell)^2$$

$$= -g^{\mu\nu}\left((k-\ell)^2 + (2k+\ell)^2\right)$$

$$+ (k-\ell)^{\mu}(k-\ell)^{\nu} + (k-\ell)^{\nu}(k-\ell)^{\mu}$$

$$+ (2\ell+k)^{\nu}(-3\ell)^{\mu} + (2\ell+k)^{\mu}(-3\ell)^{\nu}$$

$$\therefore \quad k_{\mu} V^{\mu\alpha\beta}(-k,-\ell,k+\ell)\,V^{\nu}{}_{\alpha\beta}(k,\ell,-k-\ell)$$

$$= -k^{\nu}\left((k-\ell)^2 + (2k+\ell)^2\right)$$

$$+ 2(k^2 - k.\ell)(k-\ell)^{\nu} - 3k.\ell\,(2\ell+k)^{\nu} + (k^2 + 2\ell.k)(-3\ell)^{\nu}$$

$$= k^{\nu}\left\{-3k^2 - 7k.\ell - 2\ell^2\right\} + \ell^{\nu}\left\{-5k^2 - 10k.\ell\right\}$$

$\downarrow$ MOST DIV TERM

$$= - g^2 f_{abc} f_{abc} \frac{1}{2}$$

$$\cdot \frac{i}{(4\pi)^2} \frac{1}{\varepsilon} \int_0^1 dx \left\{ k^\nu \left[-3k^2(1-2x)^2 + 4k^2 x(1-x) \right] + \frac{10}{2} k^\nu \left[k^2 x (1-x) \right] \right\}$$

$$= -i \frac{g^2}{(4\pi)^2} f_{abc} f_{abc} \frac{1}{2} \cdot \frac{1}{\varepsilon}$$

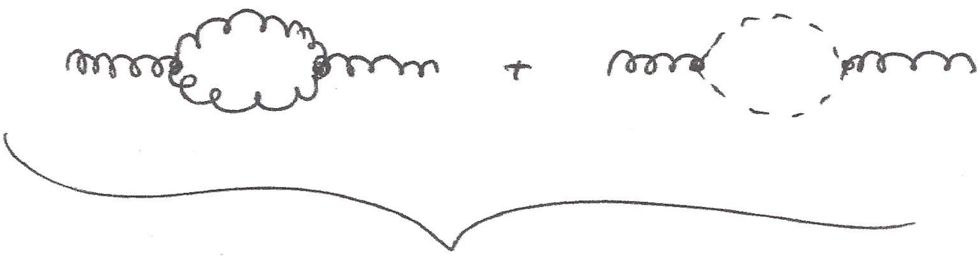
$$\cdot 3 k^\nu k^2 \int_0^1 dx \left(-1 + 7x - 7x^2 \right)$$

$$= -i \frac{g^2}{(4\pi)^2} f_{abc} f_{abc} \frac{1}{2} \cdot \frac{1}{\varepsilon} \frac{1}{2} k^\nu k^2$$

NOT ZERO !

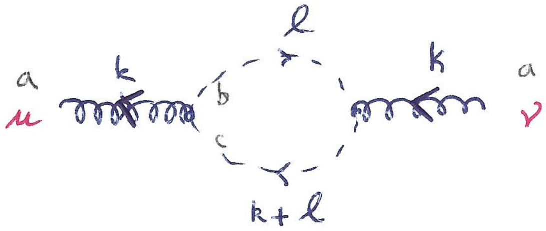
$\downarrow$

ADD GHOST LOOP

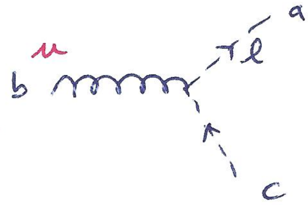


SUM IS GAUGE INV.

$\Rightarrow$ GHOST LOOP FOR $\xi = 1$



$-\dfrac{i}{l^2}\,\delta_{ab}$



$+\, g\, f_{abc}\, l^{\mu}$

- $$i\,\Pi^{\mu\nu}_{GH}(k) = (-1)\int \frac{d^4 l}{(2\pi)^4}\,\left[g\, f_{bac}\, l^{\mu}\right] \frac{i^2}{(k+l)^2\, l^2}\left[+\, g f_{cab}\,(k+l)^{\nu}\right]$$

GHOST FIELDS ARE ANTI-COMMUTING
(LIKE FERMIONS)

$$= (-1)\; g^2 f_{abc} f_{abc}$$

$$\cdot \int \frac{d^4 l}{(2\pi)^4}\; \frac{l^{\mu}\,(k+l)^{\nu}}{\left[(k+l)^2 + i\varepsilon\right]\left[l^2 + i\varepsilon\right]}$$

$$= (-1)\, g^2 f_{abc} f_{abc} \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4}\; \frac{(l - kx)^{\mu}\,(l + k(1-x))^{\nu}}{\left[l^2 + k^2 x\,(1-x)\right]^2}$$

$\downarrow$ MOST DIV. TERM

$$= (-1)\; i\,\frac{g^2}{(4\pi)^2}\; f_{abc} f_{abc} \cdot \frac{1}{\varepsilon}$$

$$\cdot \int_0^1 dx\, \left\{ -\,k^{\mu} k^{\nu}\, x\,(1-x) \;-\; \frac{1}{2}\, g^{\mu\nu}\, k^2\, x\,(1-x)\right\}$$

$$= (-1)\;\, i\,\frac{g^2}{(4\pi)^2}\; f_{abc} f_{abc}\; \frac{1}{\varepsilon}\left(-\frac{1}{6}\right)\left(k^{\mu} k^{\nu} + \frac{1}{2}\, g^{\mu\nu} k^2\right)$$

- $$i\, k_\mu\, \Pi^{\mu\nu}_{GH}(k)$$

$$= (-1)\; \frac{i g^2}{(4\pi)^2}\, f_{abc}\, f_{abc}\, \frac{1}{\varepsilon} \left(-\frac{1}{4} k^2\right) k^\nu$$

- $$i\, k_\mu \left(\Pi^{\mu\nu}_{GL}(k) + \Pi^{\mu\nu}_{GH}(k) \right)$$

$$= -\frac{i g^2}{(4\pi)^2}\, f_{abc}\, f_{abc}\, \frac{1}{\varepsilon}\, k^2 k^\nu \left(\frac{1}{4} - \frac{1}{4} \right)$$

$$\overset{!}{=} 0$$

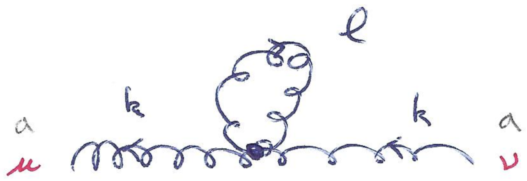
∴ SUM IS GAUGE INV.

$$\Pi_{GH}(k^2) = -\frac{1}{3k^2}\, \Pi^{\mu}_{GH\,\mu}(k)$$

$$= -\frac{g^2}{(4\pi)^2}\, \underbrace{f_{abc}\, f_{abc}}_{3}\, \frac{1}{\varepsilon} \cdot \frac{1}{6} = -\frac{g^2}{(4\pi)^2}\, \frac{1}{2\varepsilon}$$

- $$\boxed{\begin{aligned} &\Pi_{GL}(k^2) + \Pi_{GH}(k^2) \\ &= \frac{1}{\varepsilon} \cdot \frac{g^2}{(4\pi)^2} \cdot \frac{1}{2} \left\{ -10 - 3(1-\xi) \right\} \end{aligned}}$$

$\Longrightarrow$ TADPOLE CONTR.



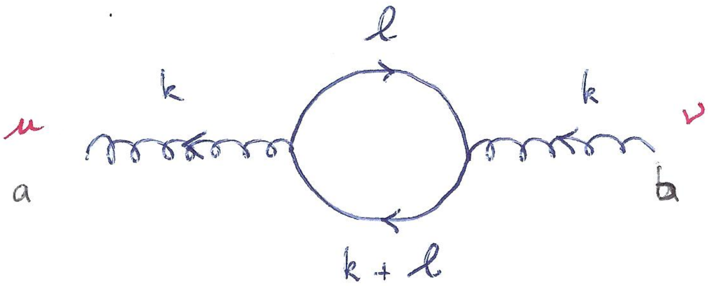
$$i\,\Pi_T(k^2) = -\frac{1}{3k^2}\int\frac{d^4\ell}{(2\pi)^4}\left[-ig^2\,W^{\mu\beta\alpha\mu}_{abba}\right]\cdot\frac{i}{\ell^2}\left[-g_{\alpha\beta}+\frac{\ell_\alpha\ell_\beta}{\ell^2}(1-\xi)\right]$$

- $$g_{\mu\nu}\,W^{\mu\beta\alpha\nu}_{abba} = -f^2_{abe}\left\{g^{\mu\alpha}g^{\beta}_{\mu} - g^{\mu}_{\mu}g^{\beta\alpha} + g^{\mu\beta}g^{\alpha}_{\mu} - g^{\mu}_{\mu}g^{\alpha\beta}\right\}$$
$$= 6\,f^2_{abe}\,g^{\alpha\beta}$$

$$i\,\Pi_T(k^2) = -\frac{1}{3k^2}\,g^2\,6f^2_{abe}\underbrace{\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2}}_{=0}\left[-4+(1-\xi)\right]$$

$$\therefore\quad \underline{\underline{i\,\Pi_T(k^2) = 0.}}$$

$\Rightarrow$ QUARK LOOP



$$i\,\Pi_F^{ab}(k^2) = -\frac{1}{3k^2}\,(-1)\cdot N_F$$

(-1): FERMION LOOP

N_F: SUM OVER ALL QUARK FLAVORS (u, d, s ...) IN LOOP

$$\cdot\, \mathrm{Tr}\left\{\int \frac{d^4 l}{(2\pi)^4}\cdot\left[-ig\frac{\lambda_a}{2}\gamma^\mu\right]\frac{i(\not k+\not l+m)}{(k+l)^2-m^2}\left[-ig\frac{\lambda_b}{2}\gamma_\mu\right]\cdot\frac{i(\not l+m)}{l^2-m^2}\right\}$$

Tr: SUM OVER COLORS IN LOOP & TRACE OVER DIRAC MATRICES

$$i\,\Pi_F^{ab}(k^2) = +\frac{1}{3k^2}\,N_f\,\underbrace{\mathrm{Tr}\left\{\frac{\lambda_a}{2}\frac{\lambda_b}{2}\right\}}_{\frac{1}{2}\delta_{ab}}\,g^2$$

$$\cdot\int\frac{d^4 l}{(2\pi)^4}\,\frac{\mathrm{Tr}\left\{\gamma^\mu(\not k+\not l+m)\gamma_\mu(\not l+m)\right\}}{\left[(k+l)^2-m^2\right]\left[l^2-m^2\right]}$$

$$\Rightarrow\ \mathrm{Tr}\left\{\gamma^\mu(\not k+\not l+m)\gamma_\mu(\not l+m)\right\}$$

$$= -2\,\mathrm{Tr}\left\{(\not k+\not l-2m)(\not l+m)\right\} = -8\left[l\cdot(k+l)-2m^2\right]$$

$$\Pi_F^{ab}(k^2) \equiv \delta_{ab} \; \Pi_F(k^2)$$

$$i\,\Pi_F(k^2) = \frac{1}{3k^2} \, N_f \, g^2 (-4)$$

$$\cdot \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \, \frac{\left[(\ell - kx)\cdot(\ell + k(1-x)) - 2m^2\right]}{\left[\ell^2 + k^2 x(1-x) - m^2\right]^2}$$

$$= i \, \frac{g^2}{(4\pi)^2} \, N_f \, \frac{(-4)}{3k^2} \, \frac{1}{\varepsilon}$$

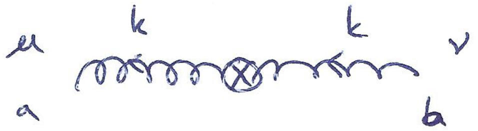
$$\cdot \int_0^1 dx \, \underbrace{\left\{ -2\left[k^2 x(1-x) - \cancel{m^2}\right] - k^2 x(1-x) - \cancel{2m^2} \right\}}_{-\frac{1}{2}k^2}$$

$$\boxed{\Pi_F(k^2) = \frac{g^2}{(4\pi)^2} \, N_f \, \frac{2}{3} \cdot \frac{1}{\varepsilon}}$$

$\Rightarrow$ SUM

$$\boxed{\begin{aligned} &\Pi_{GL} + \Pi_{GH} + \underset{\overset{\shortparallel}{0}}{\Pi_T} + \Pi_F \\ &= \frac{1}{\varepsilon} \frac{g^2}{(4\pi)^2} \frac{1}{2} \Big\{ \big(\underset{\overset{\uparrow}{GL+GH}}{-10} + \underset{\overset{\uparrow}{Q}}{\tfrac{4}{3} N_f} \big) - 3 \underset{\overset{\uparrow}{GL+GH}}{(1-\xi)} \Big\} \end{aligned}}$$

$\Rightarrow$ COUNTERTERM



$$+ i (Z_3 - 1) \delta_{ab} \left(k^\mu k^\nu - k^2 g^{\mu\nu} \right)$$

$$\Downarrow$$

$$\Pi_{CT}(k^2) = Z_3 - 1$$

$\downarrow$ Z_3 HAS TO CANCEL TERM IN $1/\varepsilon$

$$\boxed{Z_3 = 1 - \frac{g^2}{(4\pi)^2} \frac{1}{\varepsilon} \left\{ \left(-5 + \frac{2}{3} N_f \right) - \frac{3}{2} (1-\xi) \right\}}$$