

RENORMALIZATION IN QCD

⇒ LOOP DIAGRAMS

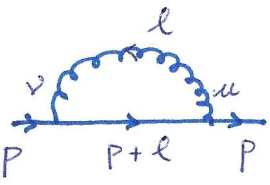
↳ LOOP DIAGRAMS CORRESPOND WITH QUANTUM CORRECTIONS

• QUARK SELF-ENERGY



FREE QUARK PROPAGATOR

$$\frac{i(\not{P} + m)}{P^2 - m^2}$$



GLUON 'DRESSING' OF THE QUARK.
(SELF ENERGY) $\Sigma(P)$ MODIFIES ITS PROPERTIES

$$-i\Sigma(P) \equiv \int \frac{d^4l}{(2\pi)^4} \frac{i}{l^2} \left[-g_{\mu\nu} + (1-\xi) \frac{l_\mu l_\nu}{l^2} \right]$$

$$\cdot \frac{1}{3} \text{Tr} \left\{ \frac{\lambda_a}{2} \frac{\lambda_a}{2} \right\} \cdot [-ig\gamma^\mu] \frac{i(\not{P} + \not{l} + m)}{[(P+l)^2 - m^2]} [-ig\gamma^\nu]$$

AVERAGE OVER 3 COLORS $\frac{1}{2} \cdot \underbrace{\delta_{aa}}_8$
 COLOR FACTOR : $\frac{4}{3}$

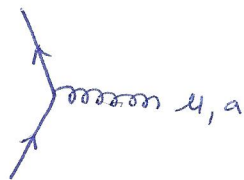
FOR $l \gg$ PART OF LOOP

$$-i\Sigma(P) \rightarrow \int d^4l \frac{1}{l^2} \frac{l}{l^2} \Rightarrow \text{LINEARLY DIVERGENT}$$

QUARK - GLUON VERTEX

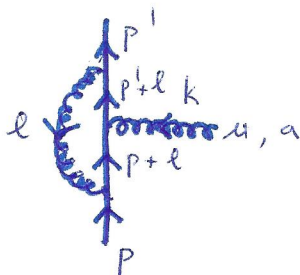
↳ LOWEST ORDER

$O(g)$

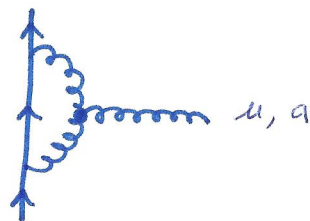


$$-ig \frac{\lambda_a}{2} \gamma^u$$

↳ $O(g^3)$



$$-ig \Lambda_a^{(1)u}$$



$$-ig \Lambda_a^{(2)u}$$

$$-ig \Lambda_a^{(1)u} = \int \frac{d^4 l}{(2\pi)^4} \left[-ig \frac{\lambda_b}{2} \gamma^\beta \right] \frac{i(p'+l+m)}{(p'+l)^2 - m^2} \left[-ig \frac{\lambda_a}{2} \gamma^u \right]$$

$$\frac{i(p+l+m)}{(p+l)^2 - m^2} \left[-ig \frac{\lambda_b}{2} \gamma^\alpha \right]$$

$$\cdot \frac{i}{l^2} \left[-g_{\alpha\beta} + (1-\xi) \frac{l_\alpha l_\beta}{l^2} \right]$$

$$= -ig (ig^2) \frac{\lambda_b}{2} \frac{\lambda_a}{2} \frac{\lambda_b}{2}$$

$$\cdot \int \frac{d^4 l}{(2\pi)^4} \frac{\gamma^\beta (p'+l+m) \gamma^u (p+l+m) \gamma^\alpha}{[(p'+l)^2 - m^2] [(p+l)^2 - m^2] l^2} \left\{ -g_{\alpha\beta} + (1-\xi) \frac{l_\alpha l_\beta}{l^2} \right\}$$

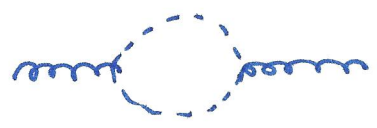
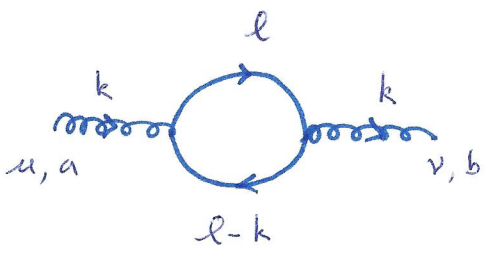
$$\xrightarrow{l \gg} \int d^4 l \frac{l^2}{l^6} \sim \int d^4 l \frac{1}{l^4}$$

LOGARITHMIC
DIVERGENCE

GLUON SELF-ENERGY



$$\delta_{ab} \frac{i}{k^2} \left[-g_{\mu\nu} + (1-\xi) \frac{k_\mu k_\nu}{k^2} \right]$$



$$-i \Pi_{ab}^{\mu\nu}(k) \stackrel{\text{QUARK LOOP}}{=} = \int \frac{d^4 l}{(2\pi)^4} \underset{\text{FERMION LOOP}}{(-1) \text{Tr}} \left\{ \left[-ig \frac{\lambda_b}{2} \gamma^\nu \right] \frac{i(\not{l} + m)}{l^2 - m^2} \left[-ig \frac{\lambda_a}{2} \gamma^\mu \right] \frac{i(\not{l} - k + m)}{(l-k)^2 - m^2} \right\}$$

$$= - \underbrace{\text{Tr} \left\{ \frac{\lambda_b}{2} \frac{\lambda_a}{2} \right\}}_{\frac{1}{2} \delta_{ab}} \cdot \int \frac{d^4 l}{(2\pi)^4} \frac{\text{Tr} \left\{ \gamma^\nu (\not{l} + m) \gamma^\mu (\not{l} - k + m) \right\}}{[l^2 - m^2] [(l-k)^2 - m^2]}$$

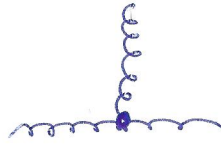
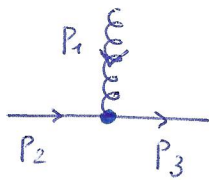
$$\xrightarrow{l \gg} \int d^4 l \frac{l^2}{l^4} \sim \int d^4 l \frac{1}{l^2}$$

QUADRATIC DIVERGENCE

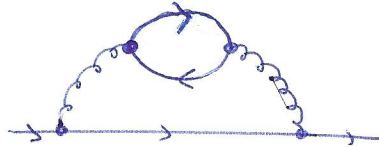


DEGREE OF DIVERGENCE OF ARBITRARY DIAGRAM

FOR EVERY VERTEX OF TYPE i



e.g. ENTERS



- EVERY MOMENTUM $\leadsto \int \frac{d^4 p_i}{(2\pi)^4}$

- EVERY FERMION PROPAGATOR CONNECTING 2 VERTICES LEADS TO A "MOMENTUM" POWER AT EACH VERTEX

$$\left(\frac{1}{2}\right) \int d^4 p_i \frac{p_i + m}{p_i^2 - m^2} \rightarrow \left(\frac{3}{2}\right)$$

BECAUSE
PROPAGATOR
CONNECTS 2 VERTICES

- EVERY BOSON PROPAGATOR LEADS TO A "MOMENTUM" POWER AT EACH VERTEX

$$\frac{1}{2} \int d^4 p_i \cdot \frac{1}{p_i^2 - m^2} \left(-g_{\mu\nu} + \frac{p_{i\mu} p_{i\nu}}{p_i^2} (1 - \xi) \right) \rightarrow \left(1\right)$$

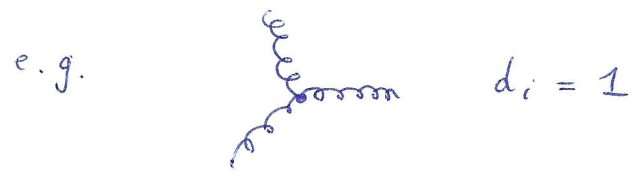
- FOR EVERY VERTEX : THERE IS 1 ENERGY - MOMENTUM CONSERVATION



$\delta^4 \left(\sum_i p_i \right) \rightarrow$ REDUCING MOMENTUM POWER COUNTING BY 4 AT EACH VERTEX.

\Downarrow
-4

- IF WE HAVE DERIVATIVE INTERACTIONS



$d_i = \#$ DERIVATIVES IN INTERACTION \mathcal{L} .

\Downarrow
LEADS TO MOMENTUM POWER d_i AT EACH VERTEX.

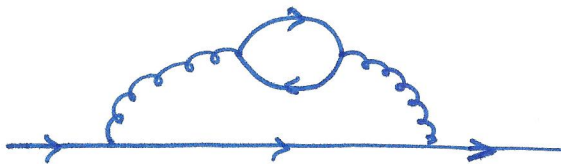
∴ DEGREE OF DIVERGENCE (DD) WHEN SUMMING OVER ALL VERTICES

$$DD = \frac{\# \text{ MOMENTUM POWERS IN NUMERATOR}}{\# \text{ MOMENTUM POWERS IN DENOMINATOR}}$$

$$DD = \sum_i n_i \cdot \left(d_i + \frac{3}{2} f_i + b_i - 4 \right)$$

\uparrow SUM OVER ALL VERTICES OF TYPE i
 \uparrow # VERTICES OF TYPE i
 \uparrow # DERIVATIVES IN \mathcal{L} AT VERTEX i
 \uparrow # FERMION LINES ENTERING VERTEX i
 \uparrow # BOSON LINES ENTERING VERTEX i
 \uparrow ENERGY - MOMENTUM CONSERV.

• CORRECT FOR EXTERNAL LINES



→ BECAUSE FOR EXTERNAL LINES: NO PROPAGATOR

→ WE SUBTRACTED ONE ENERGY-MOMENTUM CONSERVATION TOO MUCH, BECAUSE ONE CORRESPONDS WITH OVERALL ENERGY-MOMENTUM CONSERVATION



LEADS TO NO REDUCTION IN MOMENTUM POWER.

$$\bullet \bullet \bullet \quad \mathbb{D} = \sum_i m_i \cdot \delta_i + \left(4 - \frac{3}{2} N_f - N_b \right)$$

N_f : # EXTERNAL FERMION LINES

N_b : # " " BOSON "

δ_i : INDEX OF DIVERGENCE OF \mathcal{L}_i

$$\underline{\underline{\delta_i \equiv d_i + \frac{3}{2} f_i + b_i - 4}}$$

- IF $\delta_i > 0$

↳ MORE & MORE DIVERGENCES APPEAR
WHEN CALCULATING HIGHER LOOPS



NON-RENORMALIZABLE THEORY

- IF $\delta_i < 0$

↳ THEORY BECOMES MORE CONVERGENT WHEN CALCULATING
HIGHER LOOPS

SUCH THEORIES ARE TRIVIAL



SUPER-RENORMALIZABLE THEORY

- IF $\delta_i = 0$

↳ NO NEW DIVERGENCES APPEAR WHEN
CALCULATING HIGHER LOOP DIAGRAMS
COMPARED WITH 1 LOOP DIAGRAMS

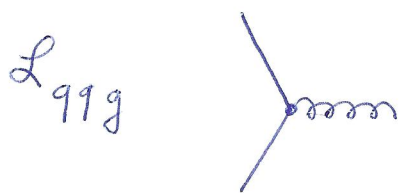


RENORMALIZABLE THEORY

'PHYSICAL' THEORIES

↳ QED, QCD, STANDARD ELECTROWEAK
GAUGE THEORY

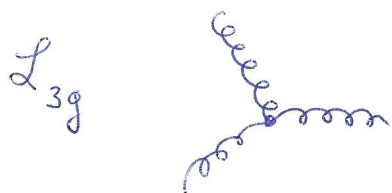
- e.g. FOR QCD



$$f_i = 2, \quad b_i = 1, \quad d_i = 0$$

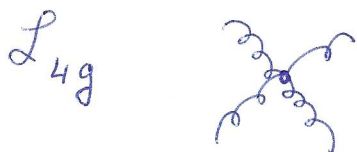
$$\delta_i = \frac{3}{2} f_i + b_i + d_i - 4$$

$$\rightarrow \delta_i = \frac{3}{2} \cdot 2 + 1 + 0 - 4 \stackrel{!}{=} 0$$



$$f_i = 0, \quad b_i = 3, \quad d_i = 1$$

$$\rightarrow \delta_i = 0 + 3 + 1 - 4 \stackrel{!}{=} 0$$



$$f_i = 0, \quad b_i = 4, \quad d_i = 0$$

$$\rightarrow \delta_i = 0 + 4 + 0 - 4 \stackrel{!}{=} 0$$

↳ QCD IS RENORMALIZABLE

⇒ RENORMALIZATION IN QCD

•
$$DD = 4 - \frac{3}{2} N_q - N_g$$

↑
DEGREE OF DIVERGENCE
OF ANY DIAGRAM

N_q : # EXTERNAL QUARK LINES

N_g : # EXTERNAL GLUON LINES

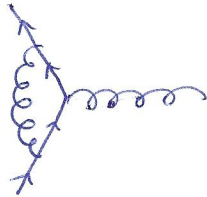
e.g.



$$N_q = 2, \quad N_g = 0$$

$$DD = 4 - \frac{3}{2} \cdot 2 - 0 = 1$$

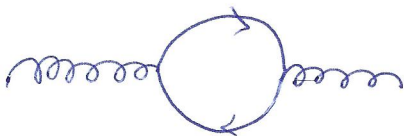
$DD = 1$ LINEARLY DIVERGENT



$$N_q = 2, \quad N_g = 1$$

$$DD = 4 - \frac{3}{2} \cdot 2 - 1 = 0$$

$DD = 0$ LOGARITHMICALLY DIVERGENT



$$N_q = 0, \quad N_g = 2$$

$$DD = 4 - 0 - 2 = 2$$

$DD = 2$ QUADRATICALLY DIVERGENT

- FOR A RENORMALIZABLE THEORY



ONLY A FINITE # OF DIVERGENCE TYPES APPEAR

IDEA

ABSORB DIVERGENCE

BY REDEFINING A FINITE NUMBER OF PARAMETERS IN THEORY

↳ PARAMETERS (m, g) AND FIELDS (φ, A_a^μ) IN \mathcal{L}

ARE TO BE CONSIDERED AS BARE (UNPHYSICAL) PARAMETERS

WHICH GET MODIFIED BY INTERACTION !

WRITE ORIGINAL $\mathcal{L} \rightarrow$ CALL \mathcal{L}_B (BARE)

$$\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{CT}$$

↑
RENORMALIZED \mathcal{L}
PHYSICAL
PARAMETERS / FIELDS

↑ COUNTER TERM : CANCELS DIVERGENCE

$$\begin{aligned} \varphi_B &= Z_2^{1/2} \varphi \\ A_{a,B}^\mu &= Z_3^{1/2} A_a^\mu \\ \chi_{a,B} &= \tilde{Z}_3^{1/2} \chi_a \\ g_B &= Z_g g \\ m_B &= Z_m m \\ \xi_B &= Z_\xi \xi \end{aligned}$$

φ_B : BARE FIELD

φ : RENORMALIZED FIELD

Z_2 : FIELD RENORM. CONSTANT
'ABSORBS' DIVERGENCE

(GAUGE PARAMETER)

↳ e.g. $\mathcal{L}_B^{qqq} = -g \bar{q}_B \gamma^\mu \frac{\lambda_a}{2} q_B A_{\mu,B}^a$

$$= -Z_g Z_2 Z_3^{1/2} g \bar{q} \gamma^\mu \frac{\lambda_a}{2} q A_\mu^a$$

$$= \underbrace{-g \bar{q} \gamma^\mu \frac{\lambda_a}{2} q A_\mu^a}_{\mathcal{L}_R} + \left(Z_g Z_2 Z_3^{1/2} - 1 \right) (-g) \bar{q} \gamma^\mu \frac{\lambda_a}{2} q A_\mu^a$$

$$+ \underbrace{\left(Z_g Z_2 Z_3^{1/2} - 1 \right) \cdot (-g) \bar{q} \gamma^\mu \frac{\lambda_a}{2} q A_\mu^a}_{\mathcal{L}_{CT}}$$

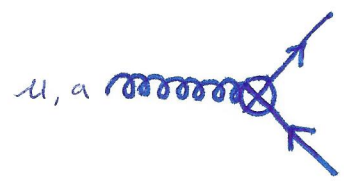
∴ INSTEAD OF CALCULATING WITH \mathcal{L}_B WITH UNPHYSICAL PARAMETERS,

WE CALCULATE WITH \mathcal{L}_R WITH PHYSICAL PARAMETERS

⊗ SUPPLEMENT THIS WITH $\mathcal{L}_{CT} \Rightarrow$ CHOSEN IN SUCH A WAY THAT IT ABSORBS ALL DIVERGENCES



ADDITIONAL FEYNMAN RULES FOR \mathcal{L}_{CT}



$$\boxed{-ig \left(Z_g Z_2 Z_3^{1/2} - 1 \right) \gamma^\mu \frac{\lambda_a}{2}}$$

$$\begin{aligned}
 \hookrightarrow \text{e.g. } \mathcal{L}_{q,B} &= \bar{q}_B (i\gamma^\mu \partial_\mu - m_B) q_B \\
 &= Z_2 \bar{q} (i\gamma^\mu \partial_\mu - Z_m m) q \\
 &= \underbrace{\bar{q} (i\gamma^\mu \partial_\mu - m) q}_{\mathcal{L}_R} \\
 &+ \underbrace{(Z_2 - 1) \bar{q} i\gamma^\mu \partial_\mu q - (Z_2 Z_m - 1) \bar{q} m q}_{\mathcal{L}_{CT}}
 \end{aligned}$$

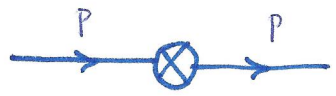
FEYNMAN RULE FOR \mathcal{L}_{CT}



$$i \left[(Z_2 - 1) \not{P} - (Z_2 Z_m - 1) m \right]$$

FEYNMAN RULES FOR COUNTERTERMS

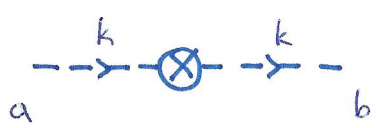
FEYNMAN RULE



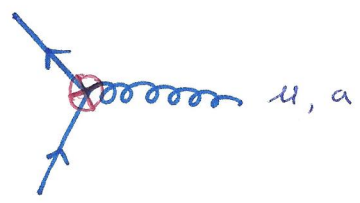
$$i \left[(Z_2 - 1) \not{P} - (Z_2 Z_m - 1) m \right]$$



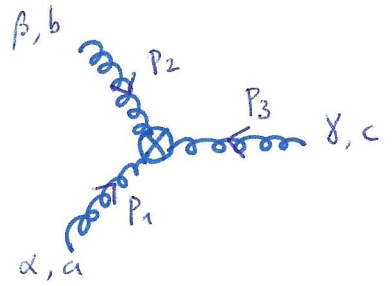
$$-i (Z_3 - 1) \delta_{ab} (k^2 g^{\mu\nu} - k^\mu k^\nu)$$



$$-i (\tilde{Z}_3 - 1) \delta_{ab} k^2$$

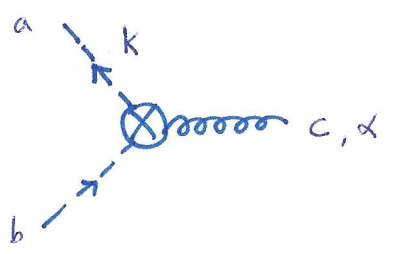


$$-ig (Z_{1F} - 1) \gamma^\mu \frac{\lambda_a}{2} \quad \text{WITH } Z_{1F} \equiv Z_g Z_2 Z_3^{1/2}$$

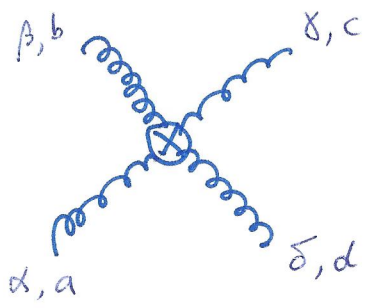


$$-g (Z_1 - 1) f_{abc} V^{\alpha\beta\gamma}(P_1, P_2, P_3)$$

$$\text{WITH } Z_1 \equiv Z_g Z_3^{3/2}$$



$$-g (\tilde{Z}_1 - 1) f_{abc} k^\alpha \quad \text{WITH } \tilde{Z}_1 = Z_g \tilde{Z}_3 Z_3^{1/2}$$



$$-ig^2 (Z_4 - 1) W_{abcd}^{\alpha\beta\gamma\delta} \quad \text{WITH } Z_4 = Z_g^2 Z_3^2$$

REGULARIZATION

CALCULATE LOOPS WITH \mathcal{L}_R



IN INTERMEDIATE STEP: REGULARIZE THEORY

↳ CUT-OFF METHODS $\int^{\Lambda} d^4 l$



VIOLATES LORENTZ-INVARIANCE

↳ PAULI-VILLARS REGULARIZATION

e.g.
$$\frac{1}{l^2 - m^2 + i\epsilon} \rightarrow \frac{1}{l^2 - m^2 + i\epsilon} \cdot \left(\frac{m^2 - \Lambda^2}{l^2 - \Lambda^2 + i\epsilon} \right)$$

↳ IMPROVES CONVERGENCE BY 2 POWERS
 ↳ $\Lambda \rightarrow \infty$ IN END
 ↳ 1

↳ DIMENSIONAL REGULARIZATION (T'HOOFT, VELTMAN)

CONSIDER
$$\int d^4 l \rightarrow \int d^D l$$

CALCULATE LOOPS IN D-DIMENSIONS

$D < 4$ ⇒ IMPROVES CONVERGENCE

DIVERGENCE WILL APPEAR AS POLE IN $\left(\epsilon \equiv 2 - \frac{D}{2} \right)$
 " $\frac{1}{\epsilon}$ " $D < 4 \Rightarrow \epsilon > 0$

AT THE END: WHEN DIVERGENCE IS ABSORBED IN COUNTERTERM



ONE CAN LET $\epsilon \rightarrow 0$ (ANALYTICAL CONTINUATION)

DIRAC ALGEBRA IN N-DIM.

$$\Rightarrow \left\| \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} I_N \right.$$

$$\Rightarrow \left\| \gamma_\mu \gamma^\mu = N I_N \right.$$

$$\gamma_\mu \gamma_\alpha \gamma^\mu = (2 - N) \gamma_\alpha$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4 g_{\alpha\beta} I_N + (N - 4) \gamma_\alpha \gamma_\beta$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\lambda \gamma^\mu = -2 \gamma_\lambda \gamma_\beta \gamma_\alpha - (N - 4) \gamma_\alpha \gamma_\beta \gamma_\lambda$$

$$\Rightarrow \text{Tr}(I_N) = 4 + f(N)$$

↳ DOES NOT AFFECT LIMIT $N \rightarrow 4$

UV DIVERGENCES

⇒ FEYNMAN PARAMETRIZATION

$$\frac{1}{A_0 A_1 \dots A_m} = \Gamma(m+1) \int_0^1 dz_1 \int_0^{z_1} dz_2 \dots \int_0^{z_{m-1}} dz_m \frac{1}{[A_0 + (A_1 - A_0)z_1 + \dots + (A_m - A_{m-1})z_m]^{m+1}}$$

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1} (1-x)^{\beta-1}}{[B + (A-B)x]^{\alpha+\beta}}$$

⇒ INTEGRALS

- $\int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 - \Delta + i\epsilon]^m} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(m-D/2)}{\Gamma(m)} \frac{(-1)^m}{(\Delta - i\epsilon)^{m-D/2}}$
- $\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{[k^2 - \Delta + i\epsilon]^m} = 0$
- $\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{[k^2 - \Delta + i\epsilon]^m} = \frac{i g^{\mu\nu}}{(4\pi)^{D/2}} \frac{\Gamma(m - \frac{D}{2} - 1)}{2\Gamma(m)} \frac{(-1)^{m-1}}{(\Delta - i\epsilon)^{m - \frac{D}{2} - 1}}$

⇒ Γ -FUNCTION

$$\Gamma(-m + \epsilon) = \frac{(-1)^m}{m!} \left\{ \frac{1}{\epsilon} + \psi(m+1) + O(\epsilon) \right\}$$

$$\psi(m+1) = 1 + \dots + \frac{1}{m} - \gamma_E$$

$$\psi(1) = -\gamma_E \quad (\gamma_E \approx 0.577)$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E$$

$$\Gamma(1 + \epsilon) = 1 - \epsilon \gamma_E + \frac{1}{2} (\gamma_E^2 + \frac{\pi^2}{6}) \epsilon^2 + O(\epsilon^3)$$

* MEASURE IN D DIM EUCLIDEAN SPACE

$$\theta_1, \dots, \theta_{D-2} : 0 \rightarrow \pi$$

$$\phi : 0 \rightarrow 2\pi$$

$$d^D \bar{P} = |\bar{P}|^{D-1} \sin^{D-2} \theta_1 \dots \sin \theta_{D-2} d|\bar{P}| d\theta_1 \dots d\theta_{D-2} d\phi$$

$$\int_0^\pi d\theta \sin^n \theta = \sqrt{\pi} \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}$$

$$\delta^D(\bar{P} - \bar{P}_0) = \frac{1}{|\bar{P}|^{D-1}} \delta(|\bar{P}| - |\bar{P}_0|) \delta^{D-1}(\hat{\Omega}_{\bar{P}} - \hat{\Omega}_{\bar{P}_0})$$

'HYPERSURFACE'

$$\int_{(D-1)} d\Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

* BETA - FUNCTION

$$B(n, m) = \int_0^1 dx x^{n-1} (1-x)^{m-1} = \frac{\Gamma(n) \Gamma(m)}{\Gamma(n+m)}$$

* α - PARAMETRIZATION

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty dx e^{-ax} x^{n-1}$$