

# QUANTIZATION OF NON-ABELIAN GAUGE THEORY (QCD): PATH INTEGRAL FORMALISM

⇒ RECALL: PATH INTEGRAL QUANTIZATION FOR SCALAR FIELD THEORY

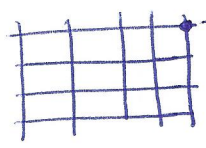
- GENERATING FUNCTIONAL FOR GREEN'S FUNCTIONS

$$Z[J] = N \int [d\phi] \exp\left\{i \int d^4x (\mathcal{L} + \phi(x) J(x))\right\}$$

$\uparrow$   
 NORMALIZATION FACTOR  $\rightarrow$  DETERMINED FROM  $Z[0]$

↳  $[d\phi]$  IMPLIES AN INTEGRAL OVER ALL VALUES  $\phi(x)$  CAN TAKE IN EVERY SPACE-TIME POINT  
 $\Rightarrow$  INFINITE DIM  $\int$

↳ TO EVALUATE SUCH INTEGRALS NUMERICALLY:  
 DISCRETIZE SPACE TIME (EUCLIDEAN)  $\Rightarrow$  LATTICE



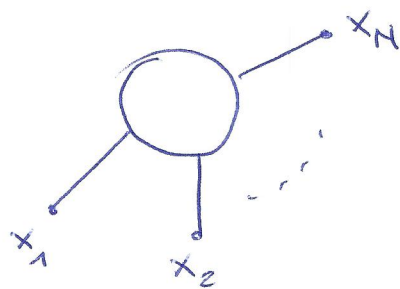
DEGREES OF FREEDOM  $\phi(x_i)$  ON EACH LATTICE POINT

↳  $e^{iS}$  ACTION  $S = \int d^4x \mathcal{L}$

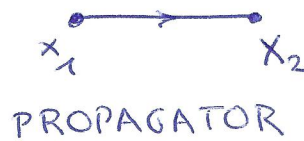
EACH FIELD CONFIGURATION IS WEIGHTED BY  $e^{iS}$

↳  $\phi J$  TERM : COUPLING TO EXTERNAL SOURCES  
 $\downarrow$   
 EXTERNAL PARTICLES IN PROCESS

• N-POINT GREEN'S FUNCTION (CORRELATION FUNCTION)



e.g.  $N = 2$



↳ IN SECOND QUANTIZATION

$$G^{(N)}(x_1 \dots x_N) = \langle 0 | T [\hat{\Phi}(x_1) \dots \hat{\Phi}(x_N)] | 0 \rangle$$

$\hat{\Phi}(x_i)$  : FIELD OPERATORS  
CREATES/ANNIHILATES SCALAR  
PARTICLE IN SPACE-TIME POINT  $x_i$

↳ IN PATH INTEGRAL QUANTIZATION

$$G^{(N)}(x_1 \dots x_N) = \frac{1}{Z[0]} \cdot (-i)^N \frac{\delta^N Z[J]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0}$$

$$= \frac{\int [d\Phi] \Phi(x_1) \dots \Phi(x_N) e^{iS}}{\int [d\Phi] e^{iS}}$$

$\Phi(x)$  : REAL FUNCTION  
(OR COMPLEX FOR CHARGED  
SPIN-0 FIELD)

• RELATION  $Z[J]$  AND  $G^{(N)}(x_1 \dots x_N)$

$$\frac{Z[J]}{Z[0]} = \sum_N \frac{i^N}{N!} \int d^4x_1 \dots d^4x_N G^{(N)}(x_1 \dots x_N) J(x_1) \dots J(x_N)$$

⇒ EXTENSION OF PATH INTEGRAL QUANTIZATION  
TO NON-ABELIAN GAUGE THEORY: ISSUES

$$\hookrightarrow \mathcal{L}_{\text{QCD}} = \bar{q} (i \gamma^\mu D_\mu - m) q - \frac{1}{4} F_{a\mu\nu} F_a^{\mu\nu}$$

$$\begin{aligned} \parallel D_\mu &= \partial_\mu + ig \frac{\lambda_a}{2} A_{a\mu} & a=1 \dots 8 \\ \parallel F_a^{\mu\nu} &= \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f_{abc} A_b^\mu A_c^\nu \end{aligned}$$

↳ THEORY HAS  $SU(3)_c$  (COLOR) GAUGE SYMMETRY

$\mathcal{L}_{\text{QCD}}$  INV. UNDER LOCAL TF:

$$\begin{aligned} \parallel q(x) &\xrightarrow{SU(3)_c} \exp\left\{i\theta_a(x) \frac{\lambda_a}{2}\right\} q(x) \\ \parallel A_a^\mu(x) &\xrightarrow{SU(3)_c} A_a^\mu - \frac{1}{g} \partial^\mu \theta_a - f_{abc} \theta_b A_c^\mu \end{aligned}$$

$$\hookrightarrow \mathcal{L}_{\text{YM}} = - \frac{1}{4} F_{a\mu\nu} F_a^{\mu\nu} \quad \text{YANG-MILLS TERM}$$

CONTAINS  $3g$  AND  $4g$  INTERACTIONS

LET'S CONSIDER THIS TERM

AND NAIVELY APPLY PATH INTEGRAL FORMALISM

$$\hookrightarrow \tilde{Z}[0] \sim \int [dA] \exp \{ i \int d^4x \mathcal{L}_{YM} \}$$

$$[dA] \equiv \prod_{\mu=0}^3 \prod_{a=1}^8 [dA_a^\mu]$$

WHERE  $\int [dA_a^\mu]$  IMPLIES AN INTEGRAL OVER ALL VALUES WHICH COMPONENT  $\mu, a$  OF  $A_a^\mu(x)$  CAN TAKE IN EVERY SPACE-TIME POINT (IN COMPARISON TO SCALAR FIELD CASE  $4 \cdot 8 = 32$  DEGREES OF FREEDOM IN EVERY SPACE-TIME POINT).

$\hookrightarrow \mathcal{L}_{YM}$  IS INV. UNDER GAUGE TF

$$A_a^\mu \xrightarrow{SU(3)_c} A_a'^\mu \equiv A_a^{\mu(\theta)} = A_a^\mu - \frac{1}{g} \partial^\mu \theta_a - f_{abc} \theta_b A_c^\mu$$

START FROM ONE FIELD CONFIGURATION IN PATH INTEGRAL

$$A_a^{\mu(\theta=0)} \xrightarrow{SU(3)_c} A_a^{\mu(\theta)} \quad \theta \text{ DEFINES SUBSET OF FIELD CONFIGURATIONS}$$

ACTION S IS CONSTANT FOR ALL  $A_a^{\mu(\theta)}$  IN THIS SUBSET OF FIELD CONFIGURATION

ISSUE: FUNCTIONAL INTEGRAL OVER THIS SUBSET DIVERGES AS REGION OF INTEGRAL OVER SUBSET IS INFINITE

## ORIGIN OF PROBLEM

$\forall a: A_a^\mu$ : ONLY 2 TRANSVERSE COMPONENTS  
ARE DYNAMICAL DEGREES OF FREEDOM  
(cf. SECOND QUANTIZATION  
OF PHOTON FIELD)

BUT IN PATH INTEGRAL:

WE INTEGRATE OVER ALL FIELD CONFIGURATIONS  
FOR 4 COMPONENTS  $\mu = 0, 1, 2, 3$



ILL DEFINED INTEGRAL

⇒ SOLUTION : A MATHEMATICAL ANALOGY

↳ CONSIDER A WELL-DEFINED (CONVERGENT)  
m-DIM INTEGRAL OVER VARIABLES  $z_{k+1}, \dots, z_{k+m}$

$$Z = \int_{-\infty}^{+\infty} dz_{k+1} \dots \int_{-\infty}^{+\infty} dz_{k+m} e^{-X}$$

WITH 'ACTION'  $X = X(z_{k+1}, \dots, z_{k+m})$

e.g.  $X = \frac{1}{2} (z_{k+1}^2 + \dots + z_{k+m}^2)$

GAUSSIAN ACTION

↳ ADD k VARIABLES  $z_1 \dots z_k$

X DOES NOT DEPEND UPON  $z_1 \dots z_k$

$$\text{INTEGRAL } \tilde{Z} = \underbrace{\int_{-\infty}^{+\infty} dz_1 \dots \int_{-\infty}^{+\infty} dz_k}_{\infty \text{ DIVERGENT RANGE}} \underbrace{\int_{-\infty}^{+\infty} dz_{k+1} \dots \int_{-\infty}^{+\infty} dz_{k+m}}_Z e^{-X}$$

↳ QUESTION : CAN WE FIND A WELL-DEFINED k+m DIM INTEGRAL WHICH IS EQUIVALENT TO Z ?

INTRODUCE k CONSTRAINTS :

$$f_i(z_1, \dots, z_{k+m}) = 0 \quad i = 1 \dots k$$

$$Z = \int_{-\infty}^{+\infty} df_1 \dots \int_{-\infty}^{+\infty} df_k \int_{-\infty}^{+\infty} dz_{k+1} \dots \int_{-\infty}^{+\infty} dz_{k+m} e^{-X} \underbrace{\prod_{i=1}^k \delta(f_i)}_{k \text{ } \delta\text{-FUNCTIONS}}$$

↳ CHANGE VARIABLES  $f_1 \dots f_k \Rightarrow z_1 \dots z_k$

$$Z = \int dz_1 \dots dz_k dz_{k+1} \dots dz_{k+m} \underbrace{\det\left(\frac{\partial f}{\partial z}\right)}_{\text{JACOBIAN}} e^{-X} \prod_{i=1}^k \delta(f_i)$$

↳ USE GRASSMANN AUXILIARY VARIABLES TO EXPONENTIATE THE DETERMINANT

GRASSMANN VARIABLES  $\chi_i, \bar{\chi}_i \quad i=1 \dots k$

$$\chi_i^2 = 0, \quad \bar{\chi}_i^2 = 0$$

$$\int d\chi_i = 0, \quad \int d\chi_i \chi_j = \delta_{ij}$$

$$\int d\bar{\chi}_i = 0, \quad \int d\bar{\chi}_i \bar{\chi}_j = \delta_{ij}$$

$$\begin{aligned} & \exp\{-\bar{\chi}_i A_{ij} \chi_j\} \\ &= 1 - (\bar{\chi}_i A_{ij} \chi_j) + \frac{1}{2!} (\bar{\chi}_i A_{ij} \chi_j)^2 + \dots + \frac{(-1)^k}{k!} (\bar{\chi}_i A_{ij} \chi_j)^k \end{aligned}$$

SERIES TERMINATES  
AS  $\chi_i^2 = 0, \bar{\chi}_i^2 = 0$

IDENTITY:

$$\| \det A = \int \prod_{\ell=1}^k (d\bar{\chi}_\ell d\chi_\ell) \exp\{-\bar{\chi}_i A_{ij} \chi_j\}$$

$$\hookrightarrow \det \left( \frac{\partial f}{\partial z} \right) = \int d\bar{\chi}_1 d\chi_1 \dots d\bar{\chi}_k d\chi_k \exp \left\{ - \bar{\chi}_i \left( \frac{\partial f_i}{\partial z_j} \right) \chi_j \right\}$$

$$\| Z = \int \left( \prod_{i=1}^{k+n} dz_i \right) \left( \prod_{j=1}^k d\bar{\chi}_j d\chi_j \right) e^{-\tilde{X}} \underbrace{\prod_{l=1}^k \delta(f_l)}_{k \text{ } \delta\text{-FUNCTIONS}}$$

$\uparrow$  INTEGRAL OVER  $k+n$  VARIABLES  
 $\uparrow$  INTEGRAL OVER AUXILIARY GRASSMANN (GHOST) VARIABLES  
 $\uparrow$   $\tilde{X}$ : MODIFIED ACTION

$$\| \tilde{X} = X + \sum_{i,j=1}^k \bar{\chi}_i \left( \frac{\partial f_i}{\partial z_j} \right) \chi_j$$



⇒ SOLUTION : FADDEEV-POPOV PROCEDURE

↳ CONSTRAINTS : GAUGE-FIXING CONDITIONS

$$\underline{\underline{f_a(A_a^{(0)}(x)) = B_a(x) \quad a = 1 \dots 8}}$$

EXAMPLES :

1) COULOMB GAUGE

$$f_a = \bar{\nabla} \cdot \bar{A}_a, \quad B_a = 0$$

2) LORENZ GAUGE

$$f_a = \partial_\mu A_a^\mu, \quad B_a = 0$$

3) AXIAL GAUGE

$$f_a = n_\mu A_a^\mu, \quad B_a = 0$$

$$n^\mu (0, 0, 0, 1)$$

4) TEMPORAL GAUGE

$$f_a = \tilde{m}_\mu A_a^\mu, \quad B_a = 0$$

$$\tilde{m}^\mu (1, 0, 0, 0)$$

$$\hookrightarrow \tilde{Z}[0] \sim \int [dA] \exp \left\{ i \int d^4x \mathcal{L}_{YM} \right\}$$

$$\downarrow \int \prod_{a=1}^8 [d\theta_a] \delta \left( f_a(A^{(0)}) - B_a \right) = \frac{1}{\det M_f}$$

$$\text{WITH } \underline{\underline{(M_f)_{ab}^{(x,y)} = \frac{\partial f_a(x)}{\partial \theta_b(y)}}}$$

$$\tilde{Z}[0] \sim \int \prod_a [d\theta_a] \int [dA] \det M_f \left( \prod_a \delta \left( f_a(A^{(0)}) - B_a \right) \right) \cdot \exp \left\{ i \int d^4x \mathcal{L}_{YM} \right\}$$

$\mathcal{L}_{YM}$ ,  $\det M_f$ ,  $[dA]$  INV. UNDER GAUGE TF

$\Rightarrow$  IN  $\delta$ -FUNCTION: WE MAY REPLACE

$$A_a^{\mu(0)} \rightarrow A_a^\mu$$

$$\tilde{Z}[0] \sim \underbrace{\left( \int \prod_a [d\theta_a] \right)}_{\text{INFINITE CONSTANT}} \int [dA] \det M_f \left( \prod_{a,x} \delta \left( f_a(A) - B_a \right) \right) \cdot \exp \left\{ i \int d^4x \mathcal{L}_{YM} \right\}$$

INSTEAD OF USING  $\tilde{Z}[0]$

WE DEFINE A FUNCTIONAL  $Z[0]$

BY REMOVING INFINITE CONSTANT

$$Z[0] = \int [dA] \det M_f \prod_{a,x} \delta(f_a(A) - B_a) e^{i \int d^4x \mathcal{L}_{YM}}$$

QCD  
PT  $\pi$   
 $i \int d^4x \mathcal{L}_{YM}$

AND LIKEWISE IN PRESENCE OF EXTERNAL SOURCES

$$Z[J] = \int [dA] \det M_f \left( \prod_{a,x} \delta(f_a(A) - B_a) \right) \cdot \exp \left\{ i \int d^4x [\mathcal{L}_{YM} + A_a^\mu J_{a\mu}] \right\}$$

↳ AVERAGE OVER  $B_a(x)$ : GAUGE FIXING

SINCE  $B_a(x)$  IS ARBITRARY

WE AVERAGE  $Z[J]$  OVER  $B_a$

WITH EXPONENTIAL WEIGHT

$$\exp \left\{ - \frac{i}{2\xi} \int d^4x (B_a(x))^2 \right\}$$

WHERE GAUGE PARAMETER  $\xi$  IS AN ARBITRARY CONSTANT

$$\therefore \left\| Z[J] = \int [dA] \det M_f \cdot \exp \left\{ i \int d^4x \left[ \mathcal{L}_{YM} - \underbrace{\frac{1}{2\xi} (f_a(A))^2}_{\text{GAUGE-FIXING TERM}} + A_a^\mu J_{a\mu} \right] \right\} \right.$$

↳

FADDEEV - POPOV GHOST FIELDS

• IN COVARIANT GAUGE

e.g. LORENZ GAUGE

$$f_a(A^{(0)}) = \partial_\mu A_a^{(0)\mu} = \partial_\mu \left[ A_a^\mu - \frac{1}{g} \partial^\mu \Theta_a - f_{abc} \Theta_b A_c^\mu \right]$$

$$(M_f)_{ab}(x,y) = \frac{\partial f_a(x)}{\partial \Theta_b(y)}$$

$$(M_f)_{ab}(x,y) = -\frac{1}{g} \partial_\mu \left[ \partial^\mu \delta_{ab} + g f_{abc} A_c^\mu \right] \delta^4(x-y)$$

$$\det M_f \sim \int [d\bar{\chi} d\chi] \exp \left\{ ig \int d^4x \int d^4y \bar{\chi}_a(x) M_f(x,y) \chi_b(y) \right\}$$

↑  
YIELDS CONSTANT  
OVERALL FACTOR (IRRELEVANT)

$$\det M_f \sim \int [d\bar{\chi} d\chi] \exp \left\{ i \int d^4x \left[ (\partial_\mu \bar{\chi}_a) (\partial^\mu \chi_a) + (\partial_\mu \bar{\chi}_a) g f_{abc} \chi_b A_c^\mu \right] \right\}$$

↑  
AUXILIARY  
GHOST FIELDS  
(GRASSMANN VARIABLES)

• NOTICE: IN ABELIAN CASE  $f_{abc} = 0$

⇒  $\det M_f$  IS CONSTANT ⇒ NO GHOSTS!

↳ IN TOTAL.

INTRODUCE ALSO SOURCES  $\sigma$  FOR  $\bar{\chi}$   
 $\bar{\sigma}$  FOR  $\chi$

$$Z[J, \sigma, \bar{\sigma}] = \int [dA] \int [d\bar{\chi} d\chi] \exp \left\{ i \int d^4x \left[ \mathcal{L}_{YM} + \mathcal{L}_{GF} + \mathcal{L}_{GHOST} + J_a^\mu A_{a\mu} + \bar{\sigma} \chi + \bar{\chi} \sigma \right] \right\}$$

⇒  $\mathcal{L}_{GF}$  : GAUGE-FIXING

$$\mathcal{L}_{GF} = - \frac{1}{2\alpha} (\partial_\mu A_a^\mu)^2$$

FEYNMAN RULE  
GLUON PROP.

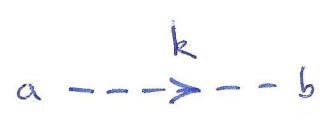


$$\delta_{ab} \frac{i}{k^2 + i\epsilon} \left[ -g^{\mu\nu} + (1-\xi) \frac{k^\mu k^\nu}{k^2} \right]$$

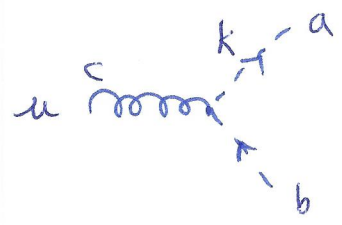
⇒  $\mathcal{L}_{GHOST}$  : GHOST TERM

$$\mathcal{L}_{GHOST} = (\partial_\mu \bar{\chi}_a) (\partial^\mu \chi_a) + (\partial_\mu \bar{\chi}_a) g f_{abc} \chi_b A_c^\mu$$

FEYNMAN RULES GHOSTS



$$\frac{i}{k^2 + i\epsilon} \delta_{ab}$$



$$-g f_{abc} k^\mu$$

↳ ABSENCE OF GHOSTS IN AXIAL / TEMPORAL GAUGES

e.g. AXIAL GAUGE

$$n_\mu A_a^\mu = 0$$

$$\mathcal{L}_a(A^{(0)}) = n_\mu A_a^{\mu(0)} = n_\mu \left[ A_a^\mu - \frac{1}{g} \partial^\mu \Theta_b - f_{abc} \Theta_b A_c^\mu \right]$$

$$\begin{aligned} (M_f)_{ab}(x, y) &= \frac{\partial \mathcal{L}_a(x)}{\partial \Theta_b(y)} \\ &= -\frac{1}{g} \left[ \delta_{ab} n_\mu \partial^\mu + g f_{abc} \underbrace{n_\mu A_c^\mu}_{\substack{= \\ 0}} \right] \delta^4(x-y) \\ &= -\frac{1}{g} \delta_{ab} (n_\mu \partial^\mu) \delta^4(x-y) \end{aligned}$$

∴  $\det M_f$  DOES NOT DEPEND ON  $A^\mu$

↳  $\det M_f$  IS CONSTANT

⇓  
NO NEED FOR GHOSTS IN AXIAL GAUGE

→ IN PRACTISE ONE OFTEN USES COVARIANT GAUGES (SUCH AS LORENZ GAUGE) IN CALCULATIONS AND ADDS GHOSTS