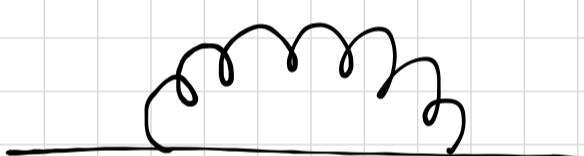
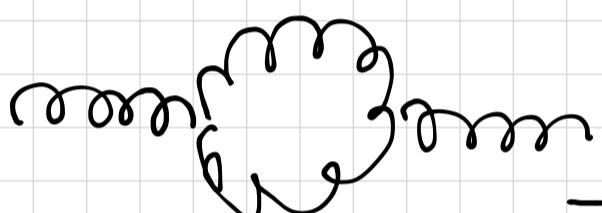
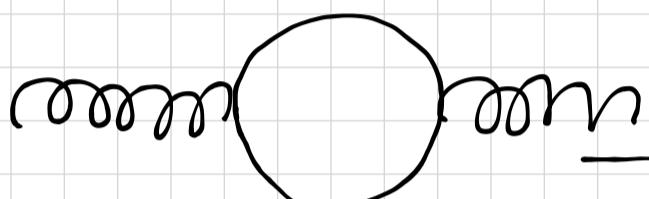


Lecture 9

What we computed last week:

Renormalization of QCD

Counterterms:



$$\delta_3 = \frac{1}{\epsilon} \frac{\alpha_s}{4\pi} \left[\frac{10}{3} C_A - \frac{8}{3} n_f T_F \right]$$

$$p \delta_2 - (\delta_2 + \delta_m) m_R$$

$$\delta_2 = \frac{1}{\epsilon} \frac{\alpha_s}{4\pi} (-2 C_F)$$

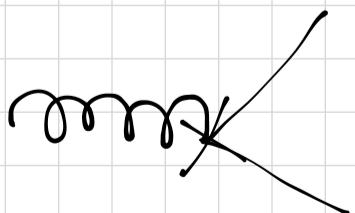
$$\delta_m = \frac{1}{\epsilon} \frac{\alpha_s}{4\pi} (-G C_F)$$

Color factors: $C_A = N_c$

$$T_F = \frac{1}{2}$$

$$C_F = \frac{N_c^2 - 1}{2N_c}$$

To compute the renormalization of the strong coupling need $Z_g = \frac{Z_1}{Z_2 \sqrt{Z_3}}$

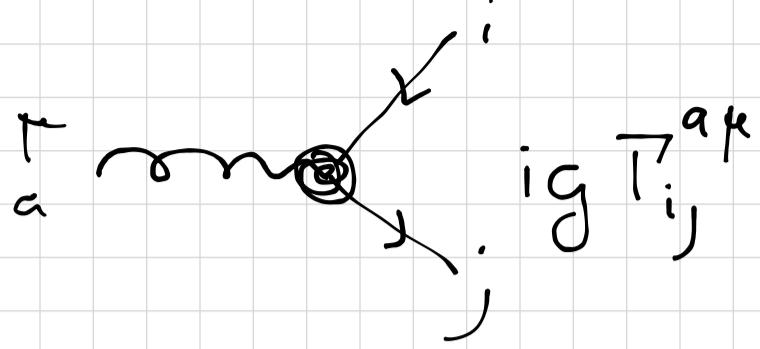


$$Z_1 i g_R \bar{\gamma}^\mu \gamma_i \gamma_\mu T_{ij}^\alpha \gamma_j$$

$$Z_i = 1 + \delta_i$$

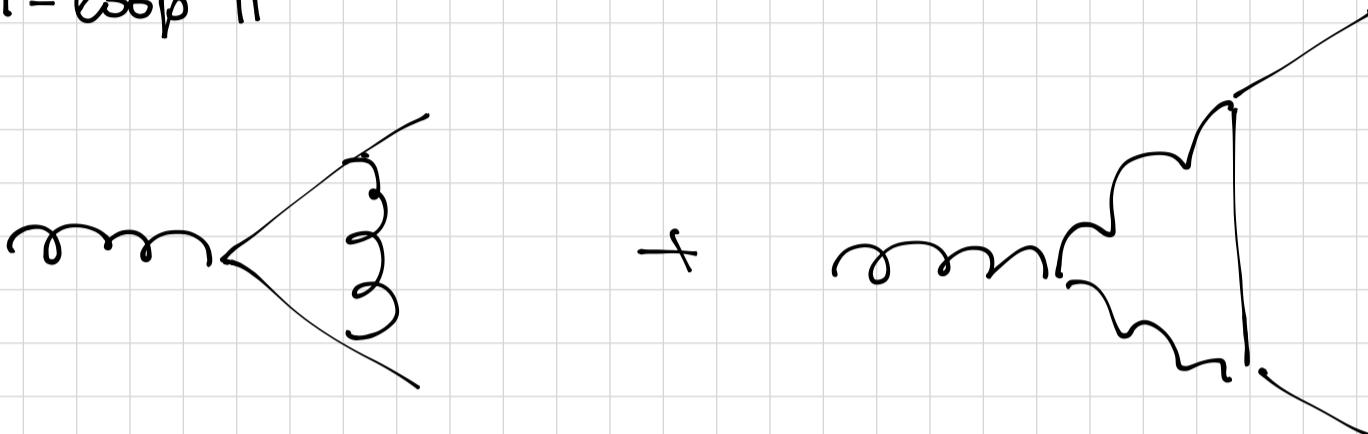
$$\rightarrow \delta_g = \delta_1 - \delta_2 - \frac{1}{2} \delta_3$$

Final missing piece : 3-point fn.

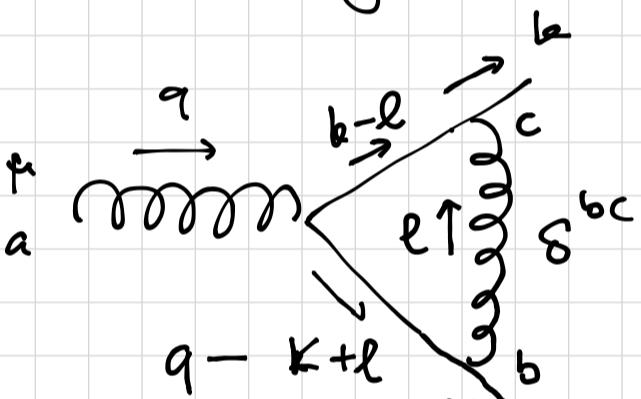


$$(T_{ij}^{ak})_{\text{tree}} = g^k \frac{\epsilon^a}{T_{ij}}$$

1-loop II



Left diagram :



Color factor

$$(\bar{T}^c \bar{T}^a \bar{T}^b)_{;j} \delta^{bc}$$

$$\bar{T}^b \bar{T}^a \bar{T}^b = \bar{T}^b \bar{T}^b \bar{T}^a + \bar{T}^b [\bar{T}^a, \bar{T}^b]$$

$$= C_F \bar{T}^a + i \bar{T}^b f^{abc} \bar{T}^c$$

$$= C_F \bar{T}^a + i f^{abc} \cdot \frac{1}{2} [\bar{T}^b, \bar{T}^c] = i f^{bcd} \bar{T}^d$$

$$= [C_F - \frac{1}{2} C_A] \bar{T}^a$$

Only need the most divergent part →
can neglect masses and ext. momenta
→ always the same integral

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 + x(1-x)q^2 + i\varepsilon]^2}$$

$$= \frac{i}{(4\pi)^2} \left[\frac{2}{\varepsilon} + \ln \frac{\mu^2}{-q^2} + \text{finite} \right]$$

$$\rightarrow ig \Gamma_f^{a\mu} = ig \frac{ds}{4\pi} T^a \gamma^\mu \left[\frac{2}{\varepsilon} + \ln \frac{\mu^2}{-q^2} + \dots \right]$$

$$ig \Gamma_g^{a\mu} = \text{mnemonic} \quad \boxed{\text{Color factor}}$$

$$f^{abc} [T^c T^b] = \frac{1}{2} f^{abc} [T^c, T^b]$$

$$= - f^{abc} \cdot \frac{i}{2} f^{dbc} T^d = - \frac{i}{2} C_A T^a$$

$$\hookrightarrow ig \Gamma_g^{a\mu} = ig C_A T^a \gamma^\mu \frac{ds}{4\pi} \left[\frac{3}{\varepsilon} + \frac{3}{2} \ln \frac{\mu^2}{-q^2} + \dots \right]$$

$$\text{Counterterm: } \delta_1 = \frac{1}{\varepsilon} \frac{ds}{4\pi} [-2C_F - 2C_A]$$

Finally (in Feynman gauge)

$$\begin{aligned} \delta_1 &= \frac{1}{\varepsilon} \frac{ds}{4\pi} [-2C_F - 2C_A] \\ \delta_2 &= \frac{1}{\varepsilon} \frac{ds}{4\pi} [-2C_F] \\ \delta_3 &= \frac{1}{\varepsilon} \frac{ds}{4\pi} \left[\frac{10}{3} C_A - \frac{8}{3} n_f T_F \right] \end{aligned} \quad \begin{aligned} &\rightarrow \delta_1 - \delta_2 - \frac{1}{2} \delta_3 \\ &= \frac{1}{\varepsilon} \frac{ds}{4\pi} \left[-\frac{11}{3} C_A + \frac{4}{3} n_f T_F \right] \end{aligned}$$

Now we can look for solution for RGE for the renormalized strong coupling ($\overline{\text{MS}} \text{ scheme}$)

$$\mathcal{L} = \mu^{2-d/2} g_R Z_1 A^\mu \bar{\psi}_i \gamma^\mu \bar{T}_{ij}^\alpha \psi_j$$

$$= \mu^{2-d/2} g_R \frac{Z_1}{Z_2 \sqrt{Z_3}} \left[A^\mu \bar{\psi} \gamma^\mu \bar{T}^\alpha \psi \right]^{(0)}$$

Bare charge

$$g_0 = g_R \frac{Z_1}{Z_2 \sqrt{Z_3}} \mu^{\frac{\varepsilon}{2}} \quad \text{independent of } \mu$$

$$\mu \frac{d}{d\mu} g_0 = 0 = \left(\mu \frac{d}{d\mu} g_R \right) \cdot Z g_R^{\frac{\varepsilon}{2}}$$

$$+ g_R \mu^{\frac{\varepsilon}{2}} \mu \frac{d}{d\mu} (\delta_1 - \delta_2 - \frac{1}{2} \delta_3)$$

$$+ g_R \frac{Z_1}{Z_2 \sqrt{Z_3}} \frac{\varepsilon}{2} \mu^{\frac{\varepsilon}{2}}$$

$$= \mu \frac{d}{d\mu} g_R = g_R \left[-\frac{\varepsilon}{2} - \mu \frac{d}{d\mu} (\delta_1 - \delta_2 - \frac{1}{2} \delta_3) \right]$$

Now, δ_i only depend on μ via g_R !

$$\hookrightarrow \mu \frac{d}{d\mu} \delta_i = \frac{d}{dg_R} \delta_i \cdot \underbrace{\mu \frac{d}{d\mu} g_R}_{=} = -g_R \frac{\varepsilon}{2}$$

$$\Rightarrow \mu \frac{d}{d\mu} g_R = g_R \left[-\frac{\varepsilon}{2} + \frac{\varepsilon}{2} g_R \frac{\partial}{\partial g_R} (\delta_1 - \delta_2 - \frac{1}{2} \delta_3) \right]$$

$$\delta_1 - \delta_2 - \delta_3/2 = -\frac{1}{\varepsilon} \frac{\alpha_s}{4\pi} \left[\frac{11}{3} C_A - \frac{4}{3} n_f T_F \right]$$

$$g_R \frac{\partial}{\partial g_R} (\delta_1 - \delta_2 - \frac{\delta_3}{2}) = -\frac{1}{\varepsilon} \frac{2\alpha_s}{4\pi} \left[\frac{11}{3} C_A - \frac{4}{3} n_f T_F \right]$$

$$\beta(g_R) = g_R \left[-\frac{\epsilon}{2} - \frac{g_R^2}{16\pi^2} \left(\frac{11}{3} C_A - \frac{4}{3} n_f T_F \right) \right]$$

o back to 4 dimensions

$$k \frac{d}{d\mu} \alpha_s(\mu) = - \frac{\alpha_s^2}{2\pi} \beta_0, \quad \beta_0 = \frac{11}{3} C_A - \frac{4}{3} n_f T_F$$

Integrate the diff. eq.

$$\text{QCD} \rightarrow \beta_0 = 11 - \frac{2}{3} n_f$$

$$\frac{d\alpha_s}{\alpha_s^2} = - \frac{\beta_0}{2\pi} d \ln \mu$$

$$\int_{\lambda}^{\mu} \left(-\frac{d}{d\mu} \frac{1}{\alpha_s(\mu)} \right) = \frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\lambda)} = \frac{\beta_0}{2\pi} \ln \frac{\mu}{\lambda}$$

↓

$$\alpha_s(\mu) = \frac{\alpha_s(\lambda)}{1 + \frac{\beta_0}{4\pi} \ln \frac{\mu^2}{\lambda^2}}$$

$$\text{QCD: } \beta_0 = 11 - \frac{2}{3} n_f > 0 \text{ for } \underline{n_f < 17}$$

$$\text{QED: } \beta_0 = -\frac{4}{3} < 0$$

From the calculation above (done in Feynman gauge for simplicity) it is not obvious that gauge dependence drops from β -function.

Consult Schwartz's Eqs. (26, 80-87)

In QED we had $Z_1 = Z_2$, so that

$$\mathcal{L} = Z_2 \bar{\psi} (i\gamma - e_R \frac{Z_1}{Z_2} A) \psi$$

$$\text{In QCD } \mathcal{L} = \sum_f Z_{2f} \bar{\psi}_f (i\gamma + g_R \frac{Z_{1f}}{Z_{2f}} A^a T^a) \psi_f$$

Rather than strictly $Z_1 = Z_2$ enough to have

$\frac{Z_1}{Z_2}$ is equal for all f

→ universal renorm. coupling g_R and covariant der. $D^\mu = \partial^\mu - i g_R A^a T^a$

The same coupling appears in other places

$$\rightarrow \frac{Z_1}{Z_2} = \frac{Z_{1c}}{Z_{3c}} = \frac{Z_{A^3}}{Z_3} = \frac{\sqrt{Z_A}}{\sqrt{Z_3}}$$

$$\delta_1 - \delta_2 = \delta_{1c} - \delta_{3c} = \delta_{A^3} - \delta_3 = \frac{1}{2}(\delta_{A^4} - \delta_3) = \frac{1}{\epsilon} \frac{\alpha}{g_R} (\xi + 3)$$

to order g^2

$$\text{For large } \mu \gg \Lambda \rightarrow \alpha_s(\mu) = \frac{1}{\frac{\beta_0}{2\pi} \ln \mu/\Lambda}$$

Importantly: the value of the coupling

constant drops completely!

(trade coupling for scale Λ)

In this form \log' has a pole at $\mu = 1$

\rightarrow Landau/pole

Rather than a physical pole, it indicates the range of validity of renormalized theory

QED is defined for $\mu < \Lambda_{\text{QED}}$

QCD is defined for $\mu > \Lambda_{\text{QCD}}$

Measurement: $\alpha_s(M_Z) = 0.1184$, $n_f = 5$

$$\hookrightarrow \overset{\text{1-loop}}{\Lambda_{\text{QCD}}} \approx 90 \text{ MeV}$$

If including β up to 4-loop

$$\beta(\alpha_s) = -2\alpha_s \sum_{n=1}^4 \left(\frac{\alpha_s}{4\pi} \right)^n \beta_{n-1}$$

$$\Lambda_{\text{QCD}} = 215 \text{ MeV}$$

$$\text{Numerically: } \beta_0(N_c=3, n_f=5) = 7\frac{2}{3}$$

$$\beta_1 = 38\frac{2}{3}$$

$$\beta_2 = 180.9$$

...

All coefficients are positive and grow

For $\mu \gg 1$ $\alpha_s(\mu) \rightarrow 0$

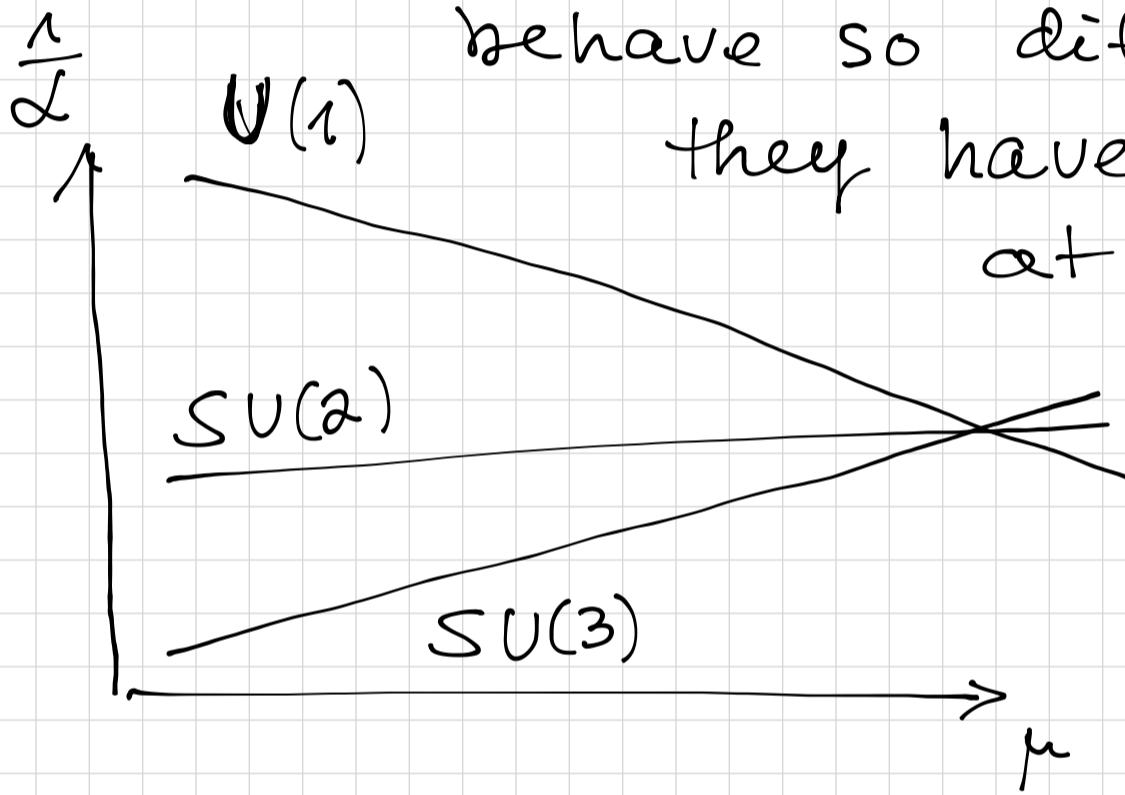
Asymptotic freedom

In comparison: QED $\alpha(\mu \gg) \uparrow$

Landau pole at $\lambda = m_e \exp\left[-\frac{2\pi}{\beta_0 \alpha_{em}(m_e)}\right]$
 $\simeq m_e \cdot e^{645}$

No problems practically; but QED is not well-defined in the UV

The observation: if couplings behave so differently they have to cross at some point



$U(1)$ — QED

$SU(2)$ — weak interaction

$SU(3)$ — strong — //

Grand unification:

Is there only one coupling?

Then, just one theory that just looks differently at low energies

$$\Lambda_{GUT} \sim 10^{16} \text{ GeV}$$

Unfortunately, the 3 lines do not meet in the same point

Need to assume additional physics between electroweak scale 250 GeV and GUT scale (SUSY, ...)
No simple solution is known.

Asymptotically free QCD \rightarrow justifies our speculative treatments of DIS process and $(e^+ e^- \rightarrow \text{hadrons}) \cong e^+ e^- \rightarrow q\bar{q}$

This is the basis of operator product expansion:

typical correlation fns. $\sim T \{ \hat{O}_1(x) \hat{O}_2(y) \}$

For short distances x (high energy)
the non-local product can be represented by a Taylor expansion in powers of $x = \text{operator product exp.}$

$$\hat{O}_1(x) \hat{O}_2(z) = \sum_i c_i(x-z)^i \hat{B}_i(z)$$

With the operators \hat{B}_i now local.

Another observation: $\beta_0 \sim 11 - \frac{2}{3} n_f$

"11" \rightarrow comes from SU(3)

$n_f \rightarrow$ external parameter, not related

to $SU(3)$; it is 6 but could be more
 Also, there are hypothetic models of
 physics beyond the SM with strongly
 interacting sectors. E.g. composite
 Dark Matter models, \rightarrow elementary
 fermions (charged under $U(1)$ of SM)
 are strongly bound into Dark mesons
 or "atoms" that are charge-neutral.

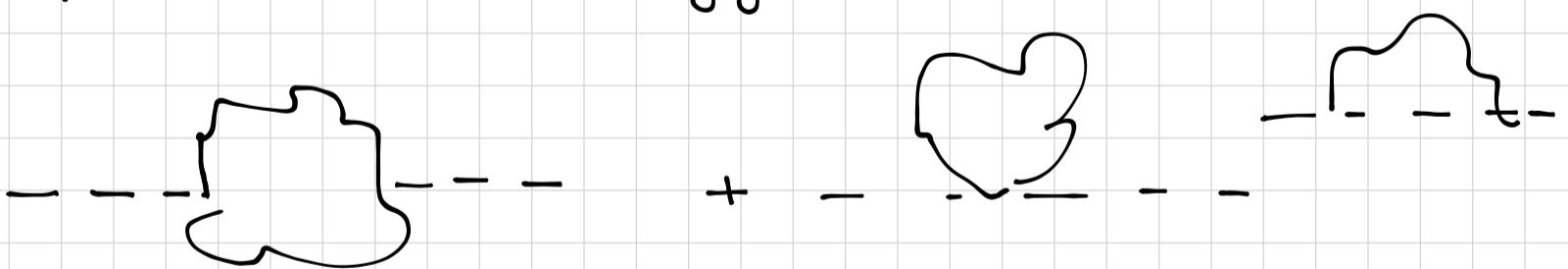
The behavior of ρ -function may
 change if adding extra flavors!

Another obs: generally, boson loops have
 an opposite sign w.r.t. fermion loops.

For QCD the bosonic loops win.

But there could be an exact cancellation
 if a symmetry is imposed.

The problem of Higgs mass:



The Higgs SE diverges due
 to boson loops; this divergence cannot
 be cured in SM \rightarrow no reason for M_H to
 be 126 GeV \rightarrow could go up to $M_P = 10^{19}$ GeV.

Solution proposed: SuperSymmetry

Each boson has a superpartner fermion, and vice versa.

Above some scale they are exactly degenerate \rightarrow so s-fermion loops cancel the boson loops exactly.

fermions
electron $\overset{\text{SM}}{\circ}$
quark

wino

Zino
photino) neutralino

gluino

higgsino

etc

bosons
selectron
squark

W/Z -bosons

photon

gluon

Higgs

$\overset{\text{SM}}{\circ}$

Finally: what if β function has a zero?

Either identically $\beta_0 = 0 \rightarrow$
theory is scale invariant
 \rightarrow Conformal — there are no scales
whatsoever

What if β_0 is accidentally suppressed,

like e.g. $\beta_0 = 11 - \frac{2}{3} n_f$ for $n_f = 16$

then

$$\beta = \beta_0 \left(\frac{\alpha}{4\pi} \right) + \beta_1 \left(\frac{\alpha}{4\pi} \right)^2 + \dots$$

If $\beta(\alpha) < 0$ for $\alpha < \tilde{\alpha}$

$\beta(\alpha)$ could go positive for $\alpha > \tilde{\alpha}$,

so that it can have a dynamical zero $\beta(\tilde{\alpha}) = 0$ — fixed point

$\frac{\beta}{\alpha}$

$$\beta = -\frac{\alpha^2}{2\pi} (\beta_0 + \frac{\alpha}{4\pi} \beta_1 \dots)$$

