

## Lecture 3

$$\text{In QFT : } \hat{\phi} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left( \hat{a}_p e^{i\vec{p}\vec{x}} + \hat{a}_p^\dagger e^{-i\vec{p}\vec{x}} \right)$$

$$\frac{\hat{\pi}}{\hbar} = -i \int \frac{d^3 p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} \left( \hat{a}_p e^{i\vec{p}\vec{x}} - \hat{a}_p^\dagger e^{-i\vec{p}\vec{x}} \right)$$

$$\text{ETCR} \quad [\phi(x), \pi(y)] = i \delta^3(\vec{x} - \vec{y})$$

Complete set of eigenstates :  $\hat{\phi} |\Phi\rangle = \phi(\vec{x}) |\Phi\rangle$   
 $\hat{\pi} |\Pi\rangle = \pi(\vec{x}) |\Pi\rangle$

$$\langle \Pi | \Phi \rangle = e^{-i \int d^3 \vec{x} \pi(\vec{x}) \phi(\vec{x})}$$

$$\langle \Phi' | \Phi \rangle = \int \mathcal{D}\Pi \langle \Phi' | \Pi \rangle \langle \Pi | \Phi \rangle = \int \mathcal{D}\Pi e^{-i \int d^3 \vec{x} \pi(\vec{x}) (\Phi - \Phi')}$$

$$H = \frac{1}{2} \hat{\pi}^2 + V(\hat{\phi})$$

$\pi(x), \phi(x) \rightarrow$  classical fields  
functions, not operators

Now we wish to compute  $\langle 0, t_f | 0, t_i \rangle$

Break  $T = t_f - t_i$  in  $n+1$  equal pieces

$$\langle 0, t_f | 0, t_i \rangle = \int \mathcal{D}\Phi_1 \dots \mathcal{D}\Phi_n \langle 0 | e^{-i\hat{H}(t_n)\delta t} | \Phi_n \rangle \dots \langle \Phi_1 | e^{-i\hat{H}(t_1)\delta t} | 0 \rangle$$

$$\langle \Phi_{j+1} | e^{-i\delta t \hat{H}(t_j)} | \Phi_j \rangle = \int \mathcal{D}\Pi_j \langle \Phi_{j+1} | \Pi_j \rangle \cdot \langle \Pi_j | e^{-i\delta t \int d^3x \left[ \frac{\hat{\Pi}^2}{2} + V(\hat{\Phi}) \right]} | \Phi_j \rangle$$

$$= \int \mathcal{D}\Pi_j e^{i \int d^3x \Pi_j(x) [\Phi_{j+1}(x) - \Phi_j(x)]} \cdot e^{-i\delta t \int d^3x \left[ \frac{\Pi_j^2}{2} + V(\Phi_j) \right]}$$

$$= \int \mathcal{D}\Pi_j e^{i\delta t \int d^3x \left[ -\frac{\Pi_j^2}{2} + \Pi_j(x) \dot{\Phi}_j(x) - V(\Phi_j) \right]}$$

\* Gauss  $\int_{-\infty}^{\infty} dp e^{-\frac{p^2}{2} + p\varphi} = \int_{-\infty}^{\infty} dp e^{-\frac{(p-\varphi)^2}{2} + \frac{\varphi^2}{2}} = N e^{\varphi^2/2}$

$$= N e^{i\delta t \int d^3x \left[ \frac{\dot{\Phi}_j^2}{2} - V(\Phi_j) \right]}$$

$$= N e^{i\delta t \int d^3x \mathcal{L}(\dot{\Phi}_j, \Phi_j)}$$

$\Downarrow$   $\delta t \rightarrow 0$  ( $n \rightarrow \infty$ ) Continuum limit

$$\langle 0, t_f | 0, t_i \rangle = N \int \mathcal{D}\Phi(\vec{x}, t) e^{iS[\Phi]}$$

$\int$  over all classical configurations  $\Phi(x)$

Classical limit:  $[S] = \hbar$

$$\hookrightarrow \int \dots e^{i \frac{S[\phi]}{\hbar}} \Big|_{\hbar \rightarrow 0} = e^{i \frac{S[\phi_0]}{\hbar}} \delta S[\phi_0] =$$

OK

T-ordered products

$$\int \mathcal{D}\bar{\Phi} e^{i S[\bar{\Phi}]} \bar{\Phi}(\vec{x}_j, t_j)$$

$$= \int \mathcal{D}\bar{\Phi}_1 \dots \mathcal{D}\bar{\Phi}_n \dots e^{-i \delta t H} \underbrace{|\bar{\Phi}_j\rangle \bar{\Phi}(\vec{x}_j) \langle \bar{\Phi}_j|}_{\hat{\psi}(\vec{x}_j) |\bar{\Phi}_j\rangle}$$

$$= \langle 0 | \hat{\psi}(\vec{x}_j, t_j) | 0 \rangle$$

Similarly,  $\int \mathcal{D}\bar{\Phi} e^{i S[\bar{\Phi}]} \bar{\Phi}(\vec{x}_k, t_k) \bar{\Phi}(\vec{x}_l, t_l)$

$$= \langle 0 | T \{ \hat{\psi}(\vec{x}_k) \hat{\psi}(\vec{x}_l) \} | 0 \rangle$$

$$\langle \Omega | T \{ \hat{\psi}(x_1) \hat{\psi}(x_2) \} | \Omega \rangle$$

$$= \frac{\int \mathcal{D}\bar{\Phi}(x) e^{i S[\bar{\Phi}]} \bar{\Phi}(x_1) \bar{\Phi}(x_2)}{\int \mathcal{D}\bar{\Phi}(x) e^{i S[\bar{\Phi}]}}$$

T-ordering for free!

# Generating functionals

$$Z[J] = \int \mathcal{D}\Phi \exp[iS[\Phi] + i \int d^4x J(x)\Phi(x)]$$

$$Z[0] = \int \mathcal{D}\Phi \exp[iS[\Phi]] \quad (\text{vacuum ampl.})$$

$$\rightarrow \frac{\partial J(x)}{\partial J(y)} = \delta^4(x-y) \quad \left( J(y) = \int d^4x \delta^4(x-y) J(x) \right)$$

$$\frac{\partial}{\partial J(x_1)} \int d^4x J(x)\Phi(x) = \Phi(x_1)$$

$\Downarrow$

$$-i \frac{1}{Z[0]} \frac{\partial}{\partial J(x_1)} Z[J] \Big|_{J=0} = \langle \Omega | \hat{\phi}(x_1) | \Omega \rangle$$

$$(-i)^n \frac{1}{Z[0]} \frac{\partial}{\partial J(x_1)} \dots \frac{\partial}{\partial J(x_n)} Z[J(x)] \Big|_{J=0}$$

$$= \langle \Omega | T \{ \hat{\phi}^a(x_1) \dots \hat{\phi}^a(x_n) \} | \Omega \rangle$$

Generating functional has a complete information on a theory!

Free theory

$$\mathcal{L} = -\frac{1}{2} \hat{\phi} (\partial^2 + m^2) \hat{\phi}$$

$$\downarrow$$

$$Z_0[J] = \int \mathcal{D}\underline{\Phi} e^{i \int d^4x \left[ -\frac{1}{2} \phi (\partial^2 + m^2) \phi + J\phi \right]}$$

$$\text{Gauss int} \int_{-\infty}^{\infty} d\vec{p} e^{-\frac{1}{2} \vec{p} A \vec{p} + \vec{J} \vec{p}}$$

$$= \sqrt{\frac{(2\pi)^n}{\det A}} e^{\frac{1}{2} \vec{J} A^{-1} \vec{J}}$$

$$\text{Here } A = \partial_x^2 + m^2$$

$$\hookrightarrow A^{-1} = -\Pi(x-y):$$

$$(\partial_x^2 + m^2) \Pi(x-y) = -\delta^4(x-y)$$

$$\Pi(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} e^{ip(x-y)}$$

$$\hookrightarrow Z_0[J] = N e^{i \int d^4x d^4y \frac{1}{2} J(x) \Pi(x-y) J(y)}$$

$$\hookrightarrow \langle 0 | T \{ \hat{\phi}(x_1) \hat{\phi}(x_2) \} | 0 \rangle$$

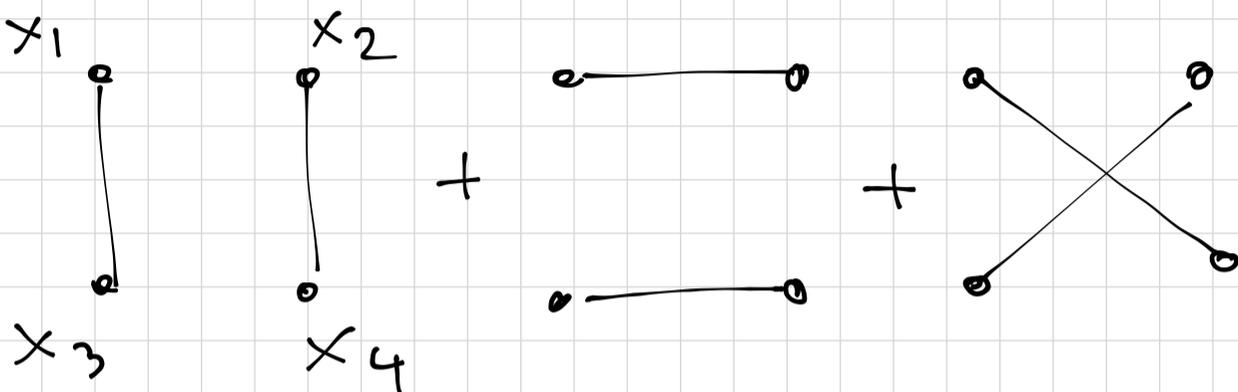
$$= (-i)^2 \frac{1}{Z_0[0]} \frac{\partial^2 Z_0}{\partial J(x_1) \partial J(x_2)} \Big|_{J=0}$$

$$= i \Pi(x_1 - x_2)$$



Similarly, 4-p. fn.

$$\langle 0 | T \{ \varphi(x_1) \varphi(x_2) \varphi(x_3) \varphi(x_4) \} | 0 \rangle$$



↪ Bubble diagrams drop w.  $\frac{1}{Z[0]}$   
 ↪ Feynman rules

### Interactions

$$\mathcal{L}_{\phi^3} = -\frac{1}{2} \varphi (\partial^2 + m^2) \varphi + \frac{g}{3!} \varphi^3$$

$$\begin{aligned} Z[J] &= \int \mathcal{D}\varphi e^{i \int d^4x \left[ -\frac{1}{2} \varphi (\partial^2 + m^2) \varphi + J\varphi \right]} e^{i \int d^4x \frac{g}{3!} \varphi^3} \\ &= \int \mathcal{D}\varphi e^{i \int d^4x \left[ -\frac{1}{2} \varphi (\partial^2 + m^2) \varphi + J\varphi \right]} \end{aligned}$$

$$\times \left[ 1 + \frac{ig}{3!} \int d^4z \varphi^3(z) + \left( \frac{ig}{3!} \right)^2 \frac{1}{2!} \int d^4z d^4w \varphi^3(z) \varphi^3(w) + \dots \right]$$

$$\begin{aligned} \Rightarrow Z[J] &= Z_0[J] + \frac{ig}{3!} \int d^4z (-i)^3 \frac{\partial^3 Z_0[J]}{(\partial J(z))^3} \\ &\quad + \left( \frac{ig}{3!} \right)^2 \frac{1}{2!} \int d^4z d^4w (-i)^6 \frac{\partial^6 Z_0[J]}{[\partial J(z)]^3 [\partial J(w)]^3} + \dots \end{aligned}$$

Each term is a path integral in free theory



$$\langle \Omega | T \{ \hat{\psi}(x_1) \hat{\psi}(x_2) \} | \Omega \rangle$$

$$= \frac{1}{Z[0]} \langle 0 | T \{ \} | 0 \rangle$$

$$+ \frac{i g}{3!} \frac{1}{Z[0]} \int d^4 z \langle 0 | T \{ \dots \phi^3(z) \} | 0 \rangle + \dots$$

$$= \frac{\langle 0 | T \{ \hat{\psi}(x_1) \hat{\psi}(x_2) e^{i \int d^4 z \frac{g}{3!} \phi^3(z)} \} | 0 \rangle}{\langle 0 | T \{ e^{i \int d^4 z \frac{g}{3!} \phi^3(z)} \} | 0 \rangle}$$

↳ Feynman rules reproduced

How is  $i\varepsilon$  prescription arises → exercises on Tue?

---

# Fermionic path integral

Bosonic fields  $\rightarrow$  commutators

Fermionic fields  $\rightarrow$  anticommut.

Path integral  $\rightarrow$   $\int$  over classical field configurations (= numbers)

Anticommuting numbers  $\rightarrow$  Grassmann algebra  $G$

Basis  $\{\theta_i\} \rightarrow G$  numbers

$$\theta_i \theta_j = -\theta_j \theta_i$$

Since  $\theta^2 = 0$  ( $\theta_i = \theta_j$ )

Any element at most linear  $g = a + b\theta$   $a, b \in \mathbb{C}$

2 distinct  $\theta$ :  $g = a + b\theta_1 + c\theta_2 + d\theta_1\theta_2$

Terms  $\sim \theta_1\theta_2 \rightarrow$  bosonic (commute)

$\sim \theta_1$  or  $\theta_2 \rightarrow$  fermionic (anti)

Lagrangian  $\rightarrow$  bosonic (bilinear in fields)

$$\theta_1 = \psi(x_1); \theta_2 = \psi(x_2); \dots$$

How  $\int d\theta$  works?  $\int$  operator:  $G \rightarrow \mathbb{C}$

$$\int d\theta (a + b\theta) = \underbrace{a \int d\theta}_{\in G \Rightarrow = 0} + b \underbrace{\int d\theta \theta}_{\text{Define} \equiv 1}$$

Integration and differentiation are the same!

$$\int d\theta (a + b\theta) = b = \frac{d}{d\theta} (a + b\theta)$$

$$\int d\theta_1 \dots d\theta_n X = \frac{\partial}{\partial \theta_1} \dots \frac{\partial}{\partial \theta_n} X$$

$$\int d\theta_1 \dots d\theta_n \theta_n \dots \theta_1 = 1$$

order important!

$$\iint d\theta_1 d\theta_2 \theta_2 \theta_1 = - \int d\theta_1 d\theta_2 \theta_1 \theta_2 = 1$$

Bosonic  $\int$ :  $\int_{-\infty}^{\infty} dx f(x) = \int_{-\infty}^{\infty} dx f(x+a)$   
 any  $\frac{\partial a}{\partial x} = 0$

Fermionic:  $\int d\theta (A+B\theta) = \int d\theta (A+B(\theta+X))$   
 any  $X$  w.  $\frac{\partial X}{\partial \theta} = 0$

Gaussian  $\int$ :

$$\int d\theta_1 d\theta_2 e^{-\theta_1 A_{12} \theta_2}$$

$$= \int d\theta_1 d\theta_2 (1 - A_{12} \theta_1 \theta_2) = A_{12}$$

n-dim.

$$\int d\bar{\theta}_1 \dots d\bar{\theta}_n d\theta_1 \dots d\theta_n e^{-\bar{\theta}_i A_{ij} \theta_j}$$

$$= \int \dots \left[ 1 - \bar{\theta}_i A_{ij} \theta_j + \frac{1}{2} (\bar{\theta}_i A_{ij} \theta_j) (\bar{\theta}_k A_{kl} \theta_l) \right.$$

Only non-zero for the term w.  $n \bar{\theta}_i$  and  $n \theta_j$

$$\hookrightarrow \int d\bar{\theta}_1 \dots d\bar{\theta}_n d\theta_1 \dots d\theta_n e^{-\bar{\theta}_i A_{ij} \theta_j}$$

$$= \frac{1}{n!} \sum (\pm) A_{i_1 i_2} \dots A_{i_{n-1} i_n} \equiv \det A$$

perm.  $\{i_n\}$

Compare to normal numbers

$$\int dx_1 \dots dx_n e^{-\frac{1}{2} x_i A_{ij} x_j} = \sqrt{\frac{(2\pi)^n}{\det A}}$$

Now add external currents  $\eta_i, \bar{\eta}_i$

$$\begin{aligned} & \int d\bar{\theta}_1 \dots d\bar{\theta}_n d\theta_1 \dots d\theta_n e^{-\bar{\theta}_i A_{ij} \theta_j + \bar{\eta}_i \theta_i + \bar{\theta}_i \eta_i} \\ &= \int d\vec{\bar{\theta}} d\vec{\theta} e^{-\left(\vec{\bar{\theta}} - \vec{\bar{\eta}} A^{-1}\right) A \left(\vec{\theta} - A^{-1} \vec{\eta}\right)} e^{\vec{\bar{\eta}} A^{-1} \vec{\eta}} \\ &= \det A \cdot e^{\vec{\bar{\eta}} A^{-1} \vec{\eta}} \end{aligned}$$

Now take  $\theta_i = \psi(x)$ ;  $\bar{\theta}_i = \bar{\psi}(x)$  functionals  
continuum limit (op.-valued fn)

$$\begin{aligned} Z[\bar{\eta}, \eta] &= \int D\bar{\psi}(x) D\psi(x) \\ &\times e^{i \int d^4x \left[ \bar{\psi}(i\not{\partial} - m + i\varepsilon)\psi + \bar{\eta}\psi + \bar{\psi}\eta \right]} \end{aligned}$$

$$\begin{aligned} A &= -i(i\not{\partial} - m + i\varepsilon) \Rightarrow \\ &= Z[\bar{\eta}, \eta] = N e^{i \int d^4x \int d^4y \bar{\eta}(y) (i\not{\partial} - m + i\varepsilon)^{-1} \eta(x)} \end{aligned}$$

2-p. fn in free theory:

$$\langle 0 | T \{ \psi(x) \bar{\psi}(y) \} | 0 \rangle = \frac{1}{Z[0]} \frac{\partial^2 Z[\bar{\eta}, \eta]}{\partial \bar{\eta}(x) \partial \eta(y)} \Big|_{\eta=0}$$

$$= \frac{i}{i\not{p} - m + i\varepsilon} \delta^4(x-y)$$

$$= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{\not{p} - m + i\varepsilon} e^{-ip(x-y)}$$

↳ Dirac propagator