

Lecture 2

LSE derivation

Free theory

1-p. state $|k\rangle = a_k^+ |0\rangle$

$$a_k^+ = +i \int d^3x \bar{e}^{-ikx} \overset{\leftrightarrow}{\partial}_0 \varphi(x)$$

$$a_k = -i \int d^3x e^{ikx} \overset{\leftrightarrow}{\partial}_0 \varphi(x)$$

$$\begin{aligned} & \int d^3x \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \left[a_k \bar{e}^{-ipx} + a_k^+ e^{ipx} \right] e^{-ikx} \\ &= \int \frac{d^3\vec{p}}{2\omega_p} \left[a_k^+ \delta(\vec{p} - \vec{k}) + a_k e^{2i\omega_{\text{tot}}} \delta(\vec{p} + \vec{k}) \right] + \\ &\quad \downarrow \qquad \qquad \qquad = \frac{a_k^+}{2\omega_k} + \frac{a_k}{2\omega_k} e^{2i\omega_k} \\ & \int d^3x \bar{e}^{-ikx} \overset{\leftrightarrow}{\partial}_0 \varphi(x) = \frac{i}{2} a_k^+ - \frac{i}{2} a_k e^{2i\omega_{\text{tot}}} \end{aligned}$$

$$\begin{aligned} & \int d^3x \bar{e}^{-ikx} \overset{\leftrightarrow}{\partial}_0 \varphi = -i\omega_k \left[\frac{a_k^+}{2\omega_k} + \frac{a_k}{2\omega_k} e^{2i\omega_{\text{tot}}} \right] \\ & \quad - \left[\frac{i}{2} a_k^+ - \frac{i}{2} a_k e^{2i\omega_{\text{tot}}} \right] = -i \overset{\cancel{k+}}{a_k} \end{aligned}$$

$$\begin{aligned} & \int d^3x \bar{e}^{ikx} \overset{\leftrightarrow}{\partial}_0 \varphi = i\omega_k \int d^3x \bar{e}^{ikx} \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \left[\hat{a}^- e^{-ipk} + \hat{a}^+ e^{ipk} \right] \\ & \quad - \int d^3x \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \left[-i\omega_p \hat{a}_p^- e^{-ipx} + i\omega_p \hat{a}_p^+ e^{ipx} \right] e^{ikx} \\ & \quad = +i \overset{\cancel{k+}}{a_k} \end{aligned}$$

OK

Vacuum state $|0\rangle$: $a_k |0\rangle = 0$; $\langle 0|0 \rangle = 1$

$$\begin{aligned}
 \langle k | k' \rangle &= \langle 0 | a_k a_{k'}^+ | 0 \rangle \\
 &= \langle 0 | [a_k, a_{k'}^+] | 0 \rangle + \cancel{\langle 0 | a_{k'}^+ a_k | 0 \rangle} \\
 &= (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}')
 \end{aligned}$$

Define a t-indep. operator (in free theory) that creates a particle near k_1 and near origin in x

$$\begin{aligned}
 a_1^+ &= \int d^3 \vec{k} f_1(k) a_{\vec{k}}^+ \\
 f_1(k) &\sim e^{-\frac{(k-k_1)^2}{4\sigma^2}}
 \end{aligned}$$

Time-evolving $e^{iHt} a_1^+ |0\rangle$

↪ wave packet spreads out

$a_1^+ a_2^+ |0\rangle \rightarrow$ separated far apart
 $k_1 \neq k_2$ for $t = \pm\infty$

↪ initial state $|i\rangle = a_1^+ a_2^+ |0\rangle \Big|_{t \rightarrow -\infty}$
 require $\langle i | i \rangle = 1$

Similar for $|f\rangle$: $t \rightarrow +\infty$, $k_1' \neq k_2'$

To compute $\langle f | i \rangle$: how $a^+(t=\infty)$ evolves with time into $a^+(t=+\infty)$?

$$a_1^+(+\infty) - a_1^+(-\infty) = \int_{-\infty}^{\infty} dt \partial_0 a_1^+(t)$$

$$= i \int d^3 k f_1(k) \int d^4 x \times \partial_0 (e^{-ikx} \overleftrightarrow{\partial}_0 \varphi(x))$$

$$= -i \int d^3 k f_1(k) \int d^4 x \bar{e}^{ikx} (\partial_0^2 + \omega^2) \varphi(x)$$

$$\omega^2 = k^2 + m^2 \\ = -\vec{k}^2 + m^2$$

$$= -i \int d^3 k f_1(k) \int d^4 x \bar{e}^{-ikx} (\partial_0^2 + \vec{k}^2 + m^2) \varphi(x)$$

$$= -i \int d^3 \vec{k} f_1(\vec{k}) \int d^4 x \bar{e}^{-ikx} (\partial_0^2 - \nabla^2 + m^2) \varphi(x)$$

by parts

$$= -i \int d^3 \vec{k} f_1(\vec{k}) \int d^4 x \bar{e}^{-ikx} (\partial_0^2 - \vec{\nabla}^2 + m^2) \varphi(x)$$

$$= -i \int d^3 \vec{k} f_1(\vec{k}) \int d^4 x \bar{e}^{-ikx} (\partial^2 + m^2) \varphi(x)$$

||

$$a_1^+(-\infty) = a_1^+(+\infty) + i \int d^3 \vec{k} f_1(\vec{k}) \int d^4 x \bar{e}^{-ikx} (\partial^2 + m^2) \varphi$$

$$a_1(+\infty) = a_1(-\infty) + i \int d^3 \vec{k} f_1(\vec{k}) \int d^4 x e^{ikx} (\partial^2 + m^2) \varphi(x)$$

↓

Can remove $f_i \rightarrow \delta$ fn. normalize.

$$\langle f | i \rangle = \langle 0 | T a_1(+\infty) a_2(+\infty) a_1^+(-\infty) a_2^+(-\infty) | 0 \rangle$$

↑ automatically

$$\Rightarrow i^{n+n'} \int d^4 x_1 e^{ik_1 x_1} (\partial_1^2 + m^2) \dots n$$

$$\int d^4 x_1' e^{ik_1' x_1'} (\partial_1'^2 + m^2) \dots n'$$

$$\times \langle 0 | T \varphi(x_1) \dots \varphi(x_{\ell'}) \dots | 0 \rangle$$

In free theory $(\partial_i^2 + m^2) \varphi(x_i) = 0$

interacting (e.g. $S_{\text{int}} = \frac{g}{3!} \varphi^3 \rightarrow (\partial_i^2 + m^2) \varphi = \frac{g}{2!} \varphi^2$)

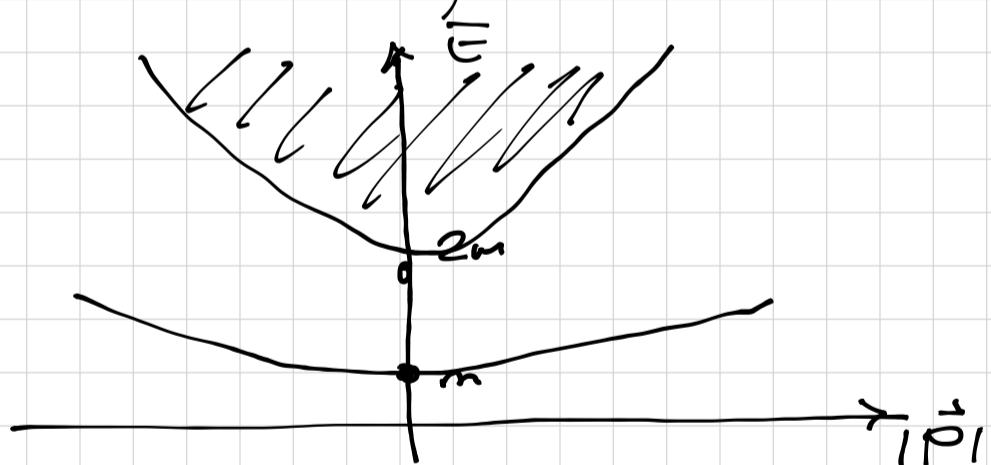
Almost there? assumed that a_i^+ of free theory would act similarly in interacting theory

1. assume that the full theory has unique g.s. $|0\rangle$

1st exc. state : 1 part. with mass m

2nd
(no bound st.)

mass $\geq 2m$
(continuum)



$$\langle 0 | \varphi(x) | 0 \rangle = \langle 0 | e^{iBx} \varphi(0) e^{-iBx} | 0 \rangle$$

$$= \langle 0 | \varphi(0) | 0 \rangle$$

→ To have $a_i^+(\pm\infty) |0\rangle = 1 \text{ p. state only}$

↪ require $\langle 0 | \varphi(0) | 0 \rangle = 0$

(can redefine $\tilde{\varphi} = \varphi - v$ in $\langle 0 | \varphi(0) | 0 \rangle = 0$)

$$\langle p | \varphi(x) | 0 \rangle = \langle p | e^{iPx} \varphi(0) e^{-iPx} | 0 \rangle$$

$$= e^{iPx} \langle p | \varphi(0) | 0 \rangle$$

In free theory $\langle p | \varphi(0) | 0 \rangle = 1$ (Why?)

Require $\langle p | \varphi(0) | 0 \rangle = 1$ in interacting theory!

↪ if $\langle p | \varphi(0) | 0 \rangle = c \neq 1 \rightarrow$ rescale φ as

$$\tilde{\varphi} = \varphi/c$$

Now consider $\langle p, n | \varphi(x) | 0 \rangle$

$$= e^{+iPx} A_n(p)$$

→ multiparticle
w. total mom p ,
 $n \rightarrow$ everything else
 $p^0 = \sqrt{\vec{p}^2 + M^2}, M \geq 2m$

(We need $\langle p, n | a_1^+(\pm\infty) | 0 \rangle = 0$)

because $a_1^+(\pm\infty)$ should create a 1-p state

$$\begin{aligned} \langle p, n | a_k^+(t) | 0 \rangle &= i \int d^3x \overset{-ikx \leftrightarrow}{\partial_0} \langle p, n | \varphi(x) | 0 \rangle \\ &= i \int d^3x \left(e^{-ikx \leftrightarrow} \partial_0 e^{iPx} \right) A_n(p) \\ &= \int d^3x (p^0 + k^0) e^{i(p^0 - k^0)t} e^{i(\vec{p} - \vec{k}) \cdot \vec{x}} A_n(\vec{p}) \\ &= (2\pi)^3 (p^0 + k^0) A_n(\vec{p}) e^{i(p^0 - k^0)t} \end{aligned}$$

$p^0 - k^0$ is always > 0 ($M \geq 2m$)

$$\Rightarrow \underbrace{\langle p, n | a_k^+(\pm\infty) | 0 \rangle}_{=} = 0$$

More precisely: introduce wave packet

$f_1(k)$ for 1-p state

$f_n(k)$ for n-p state

$$\hookrightarrow (2\pi)^3 \int d^3 \vec{p} (p^0 + k^0) f_1(\vec{p}) f_n(\vec{p}) A_n(\vec{p}) e^{i(p^0 - k^0)t}$$

1-p and n-p wave packets propagate differently \rightarrow the overlap becomes arbitrarily small for large enough t



$$\langle f | i \rangle = i^{n+n'} \int d^4 x_1 e^{ik_1 x_1} (\partial_1^2 + m^2) \dots n$$
$$\int d^4 x_1' e^{ik_1' x_1'} (\partial_1'^2 + m^2) \dots n'$$

$$\langle 0 | T \varphi(x_1) \dots \underset{n}{\varphi(x_1)} \dots \underset{n'}{\varphi(x_1')} \dots | 0 \rangle$$

Also in interacting theory

If embedded in a scattering amplitude:

each 1-p external line will have $\frac{1}{\partial^2 + m^2}$ propagator (multiparticles will not).



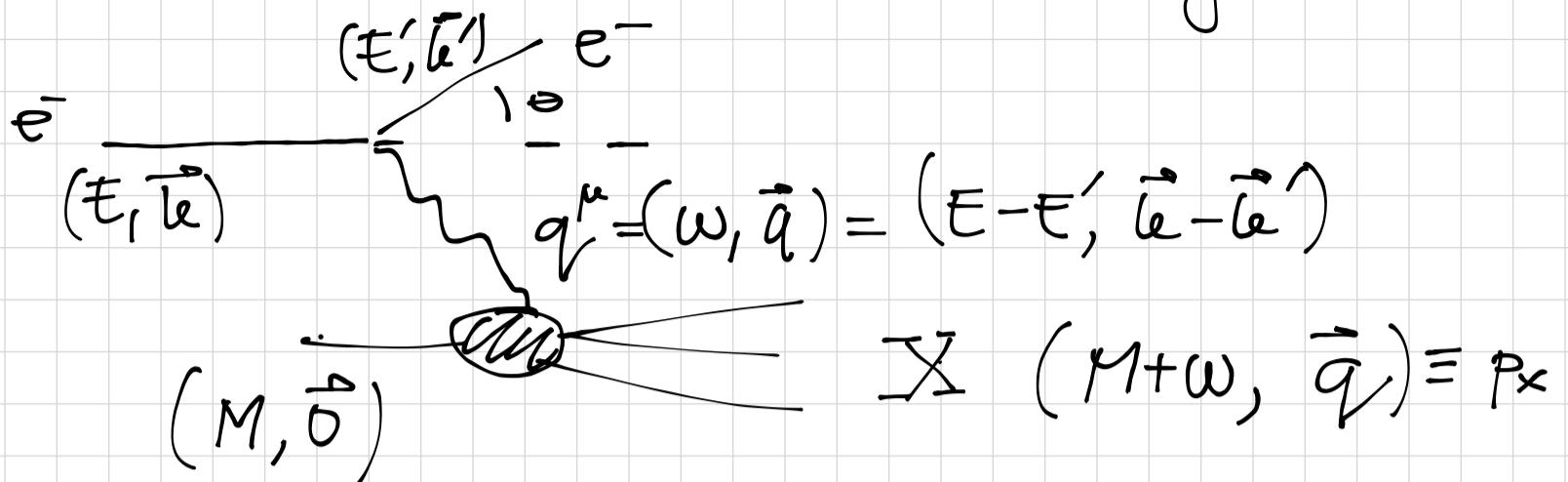
1 p will survive, everything else $\rightarrow 0$

LSE reduction allows to isolate the asymptotic state we need

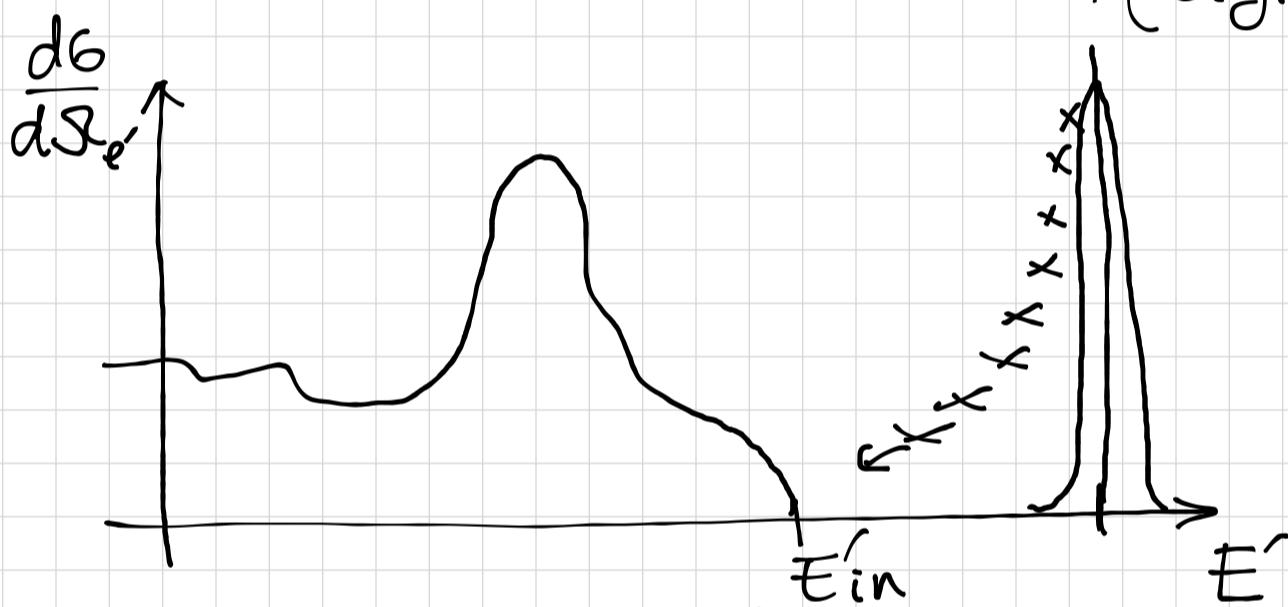
How does it work in experiments?

Example 1

Inclusive electron scattering



Electron spectrum : $\frac{d^6}{d\Omega_{e'} dE'}$ as fn. of E'
 (e.g. for fixed $q^2 < 0$)



Finite resolution of beam
and detector

$$E'_{\max} : P_x^2 = M^2 \text{ g.s.}$$

The next state is separated by
a finite threshold

$$M \rightarrow M_p, M \rightarrow M_\pi$$

No absorption in between!

What do actual data look like?

Why? \rightarrow The final state contains not just e^- and $p \rightarrow$ an ∞ # of j 's!

We assumed that the theory has a finite gap

but QED does not. How can we deduce the valuee of the proton FF if the elastic peak contains photons?

Gauge invariance: 0-energy photons

Another problem:

we want to interpret a scattering exp. and assume that the interaction vanishes fast enough. But it doesn't always do:

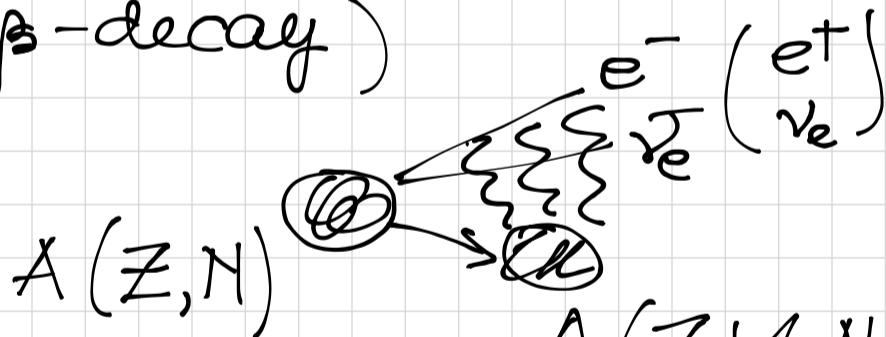
outgoing e^- in the Coulomb field of a heavy nucleus (β -decay)

decouple (real vs. loops cancel)

LSZ: non-zero energy photons will generate a higher inv. mass

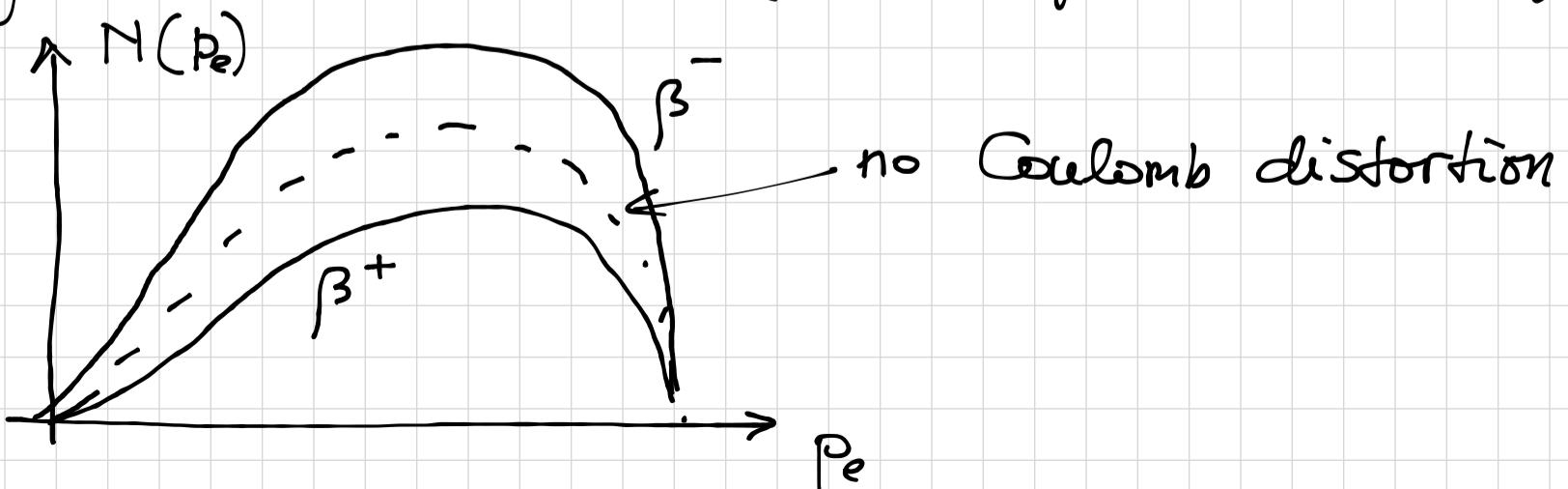
Measuring the height of the elastic peak \rightarrow determine FF

Subtract calculated corrections for emitted γ



$$E_{e^-} \sim \text{few MeV}$$

Enough time to interact (exchange Coulomb γ 's)



Without derivation:

Vacuum of interacting theory $\neq \dots$ of free th.

$$|\Omega\rangle \neq |0\rangle$$

$$\langle \Omega | T\phi(x_1) \dots \phi(x_n) | \Omega \rangle = \frac{\langle 0 | T\{ \phi \dots \phi e^{i \int d^4x L_{int}} \} | 0 \rangle}{\langle 0 | T e^{i \int d^4x L_{int}} | 0 \rangle}$$

all bubbles (no ext. lines) \longrightarrow see Schwartz

$$\boxed{\begin{aligned} \langle f | i \rangle &= i^{n+n'} \int d^4x_1 e^{i k_1 x_1} (\partial_1^2 + \omega^2) \dots n \\ &\quad \int d^4x_1' e^{i k_1' x_1'} (\partial_1'^2 + \omega^2) \dots n' \\ \langle 0 | T \underset{n}{\phi(x_1)} \dots \underset{n'}{\phi(x_1')} \dots | 0 \rangle \end{aligned}}$$

How do we compute correlation fn.

$$\langle 0 | T \phi(x_1) \dots \phi(x_i') \dots | 0 \rangle ?$$

Wick's theorem

$$T(\phi \dots \phi) = : \phi \dots \phi : +$$

all contractions

(one way)

The other way: path integral

$$\langle S | T \phi(x_1) \dots \phi(x_n) | S \rangle = \frac{\int \mathcal{D}\varphi \varphi \dots \varphi e^{i S[\varphi]}}{\int \mathcal{D}\varphi e^{i S[\varphi]}}$$

Recall QM

$$H(P, Q) = \frac{P^2}{2m} + V(Q) \quad [Q, P] = i$$

$|q_i\rangle \rightarrow$ eigenstates of Q

Probability amplitude $(q', t') \rightarrow (q'', t'')$

$$\langle q'' | e^{-iH(t_2 - t_1)} | q' \rangle$$

Heisenberg picture $Q(t) = e^{iHt} Q e^{-iHt}$

$$|q, t\rangle \equiv e^{iHt} |q\rangle, \quad Q(t) |q, t\rangle = q |q, t\rangle$$

$$Q(q) = q |q\rangle$$

Break (t', t'') in $N+1$ equal pieces $\delta t = \frac{t'' - t'}{N+1}$

Introduce N complete sets of q e.s. q_j :

$$\hookrightarrow \langle q'', t'' | q', t' \rangle = \int_{j=1}^N dq_j \langle q'' | e^{-iH\delta t} | q_N \rangle \langle q_N | e^{-iH\delta t} | q_{N-j} \rangle \dots \langle q_1 | e^{-iH\delta t} | q' \rangle$$

$$e^{-iH\delta t} = e^{-i \frac{P^2}{2m} \delta t} e^{-i V(Q) \delta t} \xrightarrow{\text{e } O(\delta t^2)}$$

Insert complete set of mom. states.

$$\begin{aligned} \langle q_2 | e^{-iH\delta t} | q_1 \rangle &= \int dp_1 \langle q_2 | e^{-i \frac{\delta t}{2m} P^2} | p_1 \rangle \cdot \\ &\quad \langle p_1 | e^{-i V(Q) \delta t} | q_1 \rangle \\ &= \int dp_1 e^{-i \frac{\delta t}{2m} P_1^2} e^{-i \delta t V(q_1)} \langle q_2 | p_1 \rangle \langle p_1 | q_1 \rangle \end{aligned}$$

$$\frac{e^{ip_1 q_2}}{\sqrt{2\pi}} \quad \frac{e^{-ip_1 q_1}}{\sqrt{2\pi}}$$

$$= \int d\mathbf{p}_1 e^{-i\delta t H} e^{i\mathbf{p}_1(\mathbf{q}_2 - \mathbf{q}_1)}$$

$$\Rightarrow \langle \mathbf{q}'', t'' | \mathbf{q}', t' \rangle = \int \prod_k dq_k \prod_j \frac{d\mathbf{p}_j}{2\pi} e^{+i\delta t \left(\mathbf{p}_j \frac{\delta \mathbf{q}_k}{\delta t} - H \right)}$$

$$\hookrightarrow \langle \mathbf{q}'', t'' | \mathbf{q}', t' \rangle = \int Dq \ e^{i \int_{t'}^{t''} dt \mathcal{L}(\dot{q}, q)}$$

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