

Lecture 14

Leading order chiral Lagrangian

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4} \text{tr}(\partial_\mu U)(\partial^\mu U)^+$$

$$+ \frac{V^3}{2} \text{tr}(M U + \eta^+ U^+)$$

$$U = e^{i\pi^\alpha \tau^\alpha / F_\pi}$$

F_π — pion decay constant (from weak
 $\pi^\pm \rightarrow \mu^\pm \gamma$ decay)

Mass term for a pure Goldstone boson
 forbidden but is allowed if explicit breaking
 of chiral symmetry is present

→ represented by quark mass matr. M

$V^3 = \langle \bar{u}u \rangle = \langle \bar{d}d \rangle$ is quark condensate
 which is the sign of spontaneous symmetry
 breaking.

Can do explicit calculations by
 expanding the exponential in powers
 of pion field accompanied by powers
 of $1/F_\pi$.

Can include electromagnetic interac-

tion via covariant derivative,

$$\partial_\mu \pi^a \rightarrow \partial_\mu \pi^a - e A_{\mu}{}^c \epsilon^{abc} \pi^b$$

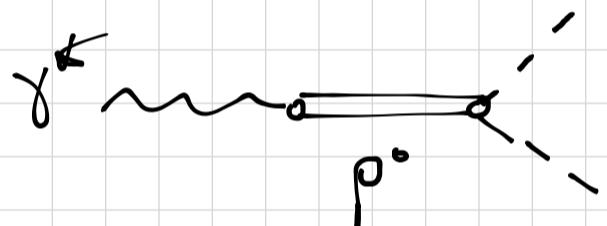
(π^\pm have opposite charge, $\pi^0 = \pi^3$ has none)

Can calculate e.m. properties of the pion

Form factor

$$\gamma^* \text{---} \pi^\pm = \text{---} \pi^\pm + \dots$$

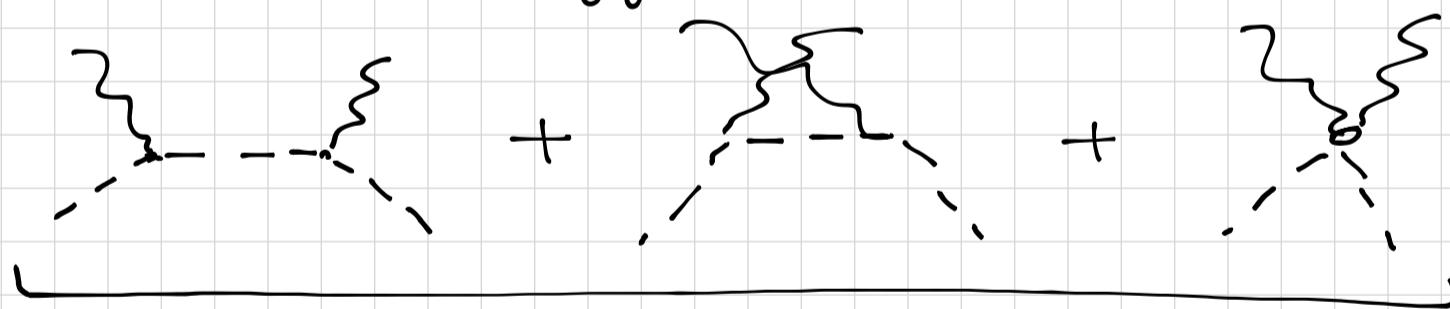
Unfortunately, the dominant contribution is not governed by chiral symmetry:



$$F_\pi(Q^2) \approx \frac{1}{1 + Q^2/\pi_\rho^2}$$

Electromagnetic polarizabilities

from low-energy expansion of Compton ampl.



"Born" contribution: depends on M_π , e only!

Polarizabilities \rightarrow response of internal structure to external e.m. field of low frequency
 \rightarrow from chiral loops

More interesting: interactions of pions with nucleons

How do pions couple to the nucleon?

$$\mathcal{L}_{N\pi} = \bar{N} (i\gamma^\mu - M_N + \gamma^5 + g_A \not{\alpha} \gamma_5) N$$

↪ again only from symmetries:

$\bar{N} \gamma^\mu N$ transforms as a vector

→ v^μ should be a vector

$\bar{N} \gamma^\mu \gamma_5 N$ and a^μ transform as axial v .

$$v_\mu = \frac{1}{2i} (u \partial_\mu u^+ + u^+ \partial_\mu u)$$

$$a_\mu = \frac{1}{2i} (u^+ \partial_\mu u - u \partial_\mu u^+)$$

$$\text{with } u = V^{-1/2} = e^{i\pi^\alpha \tau^\alpha / 2F_\pi}.$$

Exercise: expand to leading non-vanishing order v_μ , a_μ to obtain Feynman rules

! Even if you only take the leading chiral Lagrangian which may have dim. 4, the non-linear way the pion field enters (exp. fn.) leads to an infinite series $\sim (p/F_\pi)^n$ which is non-renormalizable

Nonetheless, even with this complication one can make predictions once the (finite number of) counterterms at each

chiral order are fixed.

Systematic expansion external momenta

over F_π (or, rather $4\pi F_\pi \sim 1 \text{ GeV}$)

$\frac{1}{4\pi}$ a typical loop factor

Goldberger - Treiman relation

Consider nucleon axial vector current

$$J_{\mu 5}^a = \bar{N} \gamma^a \left[G_A(Q^2) \gamma_\mu \gamma_5 + q_\mu F_P(Q^2) \gamma_5 + i g_{\mu 5} \gamma_5 \frac{q^2}{2\pi} \bar{F}_T(Q^2) \right] N$$

This form is only dictated by symmetries

Compute the divergence of this current

$$\begin{array}{c} \uparrow \{ q^\mu = (p - p')^\mu \\ N \quad \quad \quad N \\ \hline p^\mu \quad \quad \quad p'^\mu \end{array}$$

$$q^\mu J_{\mu 5}^a = \bar{N}(p') \gamma^a \left[G_A(Q^2) (p - p') \gamma_5 + q^2 F_P(Q^2) \gamma_5 \right]$$

$$[\dots] = -2M G_A(Q^2) - Q^2 F_P(Q^2)$$

Consider the limit $q^2 = -Q^2 \rightarrow 0$

$G_A(0) = g_A \approx 1.2765$ from neutron
 β -decay (I use $g_A > 0$ here)

If chiral symmetry were exact \rightarrow the

axial current would be conserved

$$\Rightarrow F_p(0) = \frac{2M}{q^2} g_A \rightarrow \text{has to contain a pole } \frac{1}{q^2}$$

Such a pole could only be due to an exchange of a massless particle
→ Goldstone boson — pion

Because $\langle S_L | J_5^{ta} | \pi^b \rangle = \delta^{ab} (-iq^\mu) F_\pi e^{-iqx}$
and pseudoscalar πN coupling

$$-G_{\pi N} \bar{N} \gamma_5 \tau^a N \pi^a$$

$\sim \frac{q^\mu}{q^2}$

↪ obtain a relation

$$\boxed{M_N g_A = F_\pi G_{\pi N}}$$

This relation involves experimentally measurable quantities

F_π : pion decay 92.4 MeV

g_A : neutron β -decay 1.2756

$G_{\pi N}$: nucleon-nucleon scatt; 13.4
pion-nucleon scatt;

$$\frac{M_N}{F_\pi} g_A \approx 13 \quad \text{good agreement!}$$

Corrections can be computed

In reality pion is not massless

$$\hookrightarrow J_5^{\mu\alpha} = \bar{N} [g_A (\gamma^\mu \gamma_5 + \frac{2Mq^\mu}{q^2 - m_\pi^2} \gamma_5) + \dots] N$$

and $\partial^\mu J_5^{\mu\alpha} \sim -2G_{\pi N} \frac{m_\pi^2 F_\pi}{q^2 - m_\pi^2} \bar{N} \gamma_5 N$

PCAC (partially conserved axial current) relation

$$J_{\mu 5}^a = F_\pi \partial^\mu \pi^a(x)$$

$$\langle S | \partial^\mu J_{\mu 5}^a | \bar{\pi}^b(q) \rangle \approx -m_\pi^2 F_\pi \delta^{ab}$$

Divergence of axial current is proportional to m_π^2

In principle, from pure Lorentz invariance one could also have heavier PS states parametrizing the axial current

What counts is, however, that the divergence is small $\sim m_\pi^2$ everywhere except in the vicinity of the pion pole where the latter gives the leading contribution to the axial current.

More generally, for any $\langle f | J_{\mu 5}^a | i \rangle$

$$\langle f | \partial^\mu J_{\mu 5}^a | i \rangle = -m_\pi^2 f_\pi \langle f | \pi^a | i \rangle + \dots$$

Soft pion theorems

(recommend books by Weinberg; Coleman;...)

Now, how can we deal with nucleons?

$$M \sim 4\pi F_\pi \sim 1 \text{ GeV}$$

\Rightarrow expansion in powers $\left(\frac{M}{4\pi F_\pi}\right)^n$ is bound to fail!

As long as the kick that a nucleon receives is small, $p^\mu = M v^\mu + k^\mu$

with $v^2 = 1$ (e.g., $v^\mu = (1, \vec{v})$) and $\frac{k}{M} \ll 1$ can still describe structure and interactions of nucleons with chiral Lagrangian

\rightarrow heavy baryon formalism

Define $B_\nu(x) = \exp[iM\cancel{x} v_\mu x^\mu] N(x)$

$$(i\cancel{x} + M)N(x) = 0$$

$$\hookrightarrow i\cancel{x} B_\nu(x) = 0$$

Allows to simplify Dirac algebra

Can define velocity projection op.

$$P_\nu = \frac{1}{2}(1 + \cancel{x}) : P_\nu B_\nu = B_\nu$$

Spin operator S_ν^μ with properties

$$v \cdot S_\nu = 0 ; S_\nu^2 B_\nu = -\frac{3}{4} B_\nu ;$$

$$\{S_\nu^\alpha, S_\nu^\beta\} = \frac{1}{2}(\nu^\alpha\nu^\beta - g^{\alpha\beta})$$

$$[S_\nu^\alpha, S_\nu^\beta] = : \varepsilon^{\alpha\beta\delta} \nu_\delta S_\nu^\delta .$$

Then, $\bar{B}_\nu \gamma_5 B_\nu = 0$

$$\bar{B}_\nu \gamma^\mu B_\nu = \nu^\mu \bar{B}_\nu B_\nu$$

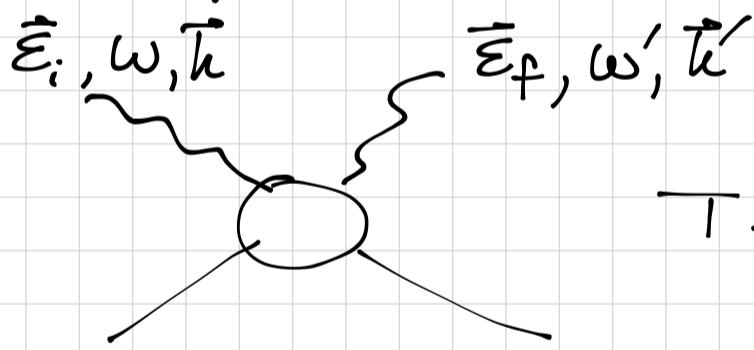
$$\bar{B}_\nu \gamma^\mu \gamma_5 B_\nu = 2 \bar{B}_\nu S_\nu^\mu B_\nu$$

$$\bar{B}_\nu G^{\mu\nu} B_\nu = 2 \varepsilon^{\mu\nu\alpha\beta} \nu_\alpha \bar{B}_\nu S_{\nu\beta} B_\nu$$

$$\bar{B}_\nu G^{\mu\nu} \gamma_5 B_\nu = 2i \bar{B}_\nu (\nu^\mu S_\nu^\nu - \nu^\nu S_\nu^\mu) B_\nu$$

A classical example for the use of HB ChPT (heavy-baryon chiral perturbation theory)

→ computation of nucleon polarizabilities



$$T = T_B + \alpha_E \omega \omega' (\vec{\epsilon}'^* \vec{\epsilon})$$

$$+ \beta_M [\vec{\epsilon}'^* \times \vec{k}'] \cdot [\vec{\epsilon} \times \vec{k}]$$

$$+ O(\omega^4)$$

$$T_B = -\frac{e^2}{4\pi M} (\vec{\epsilon}'^* \cdot \vec{\epsilon})$$

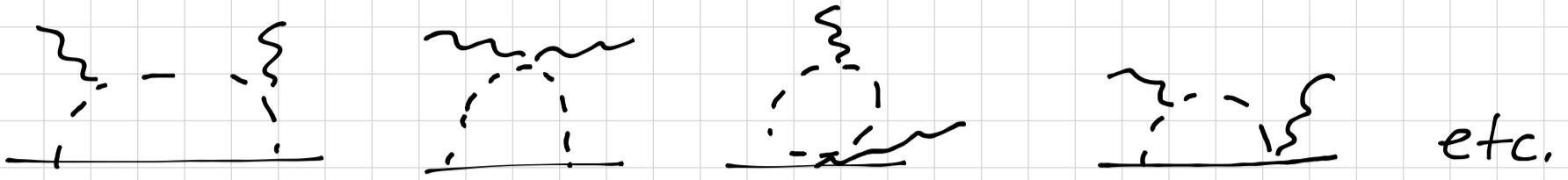
Thomson term

$$H \sim -\frac{1}{2} \alpha_E \vec{\epsilon}^2 - \frac{1}{2} \beta_M \vec{B}^2$$

ChPT allows to compute the leading non-analytic behavior in the pion mass

$$\sim \frac{1}{M_\pi} \quad \text{or} \quad \ln \frac{M_\pi}{M}$$

Need to compute the full set of 1-loop diags



The early prediction of HB χ PT:

$$\alpha_p = \frac{e^2 g_A^2}{192\pi^3 F_\pi^2} \cdot \frac{5\pi}{2M_\pi} \equiv 12.7 \cdot 10^{-4} \text{ fm}^3$$

$$\beta_p = \frac{e^2 g_A^2}{192\pi^3 F_\pi^2} \cdot \frac{\pi}{4M_\pi} = \frac{\alpha_p}{10} \cong 1.3 \cdot 10^{-4} \text{ fm}^3$$

Good agreement with data on low-energy Compton scattering + Baldin sum rule

$$\alpha + \beta = \frac{1}{2\pi^2} \int_{\omega_\pi}^{\infty} \frac{d\omega}{\omega^2} G_{p \rightarrow x}(\omega) \cong 14.2 \cdot 10^{-4} \text{ fm}^3$$

However... HB χ PT also gives subleading terms in the expansion $\frac{m_\pi}{M_N}$.

$$\alpha_p = \# \left\{ \frac{5\pi M_N}{2M_\pi} + 18 \ln \frac{M_\pi}{M_N} + \frac{33}{2} + O\left(\frac{m_\pi}{M_N}\right) \right\}$$

$$\beta_p = \# \left\{ \frac{\pi M_N}{4M_\pi} + 18 \ln \frac{M_\pi}{M_N} + \frac{63}{2} + O\left(\frac{m_\pi}{M_N}\right) \right\}$$

$$\alpha_p \rightarrow 8 \cdot 10^{-4} \text{ fm}^3 \quad \beta_p \rightarrow -2 \cdot 10^{-4} \text{ fm}^3$$

The reason : large contribution
of the low-lying Δ -resonance

$$\rightarrow \beta_p^\Delta \sim 5 \cdot 10^{-4} fm^3$$

It is formally higher chiral order, but
contributes at tree-level

In general : convergence pattern of the HB ·
expansion in the nucleon sector is not
very good : numerically large coefficients
at higher-order terms

Higgs mechanism

Consider Abelian case first

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + (\partial_\mu \varphi^*) (\partial^\mu \varphi) + m^2 |\varphi|^2 - \frac{\lambda}{4} |\varphi|^4$$

Again, the minimum of the potential is at

$$\langle \varphi \rangle = \frac{v}{\sqrt{2}} = \frac{\sqrt{2}m}{\sqrt{\lambda}}$$

Define $\varphi(x) = \frac{v + \delta(x)}{\sqrt{2}} e^{i \frac{\pi(x)}{F_\pi}}$

and integrate out δ

$$\hookrightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{v^2}{2} \left[\frac{(\partial_\mu \pi)^2}{F_\pi^2} + e^2 A^2 + 2e \frac{A_\mu \partial^\mu \pi}{F_\pi} \right]$$

! The gauge boson picks a mass term

$$m_A = e\sigma, \quad F_\pi = v$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_A^2 \left(A_\mu + \frac{1}{eF_\pi} \partial_\mu \pi \right)^2$$

Original \mathcal{L} was gauge invariant

It still is under $\begin{cases} A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \lambda \\ \pi \rightarrow \pi - F_\pi \lambda \end{cases}$

Next step: get rid of the crossed term by fixing gauge

1. Unitary gauge: $\lambda = \frac{\pi}{F_\pi}$, so that after transf. $\pi = 0$

$$\Rightarrow \mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_A^2 A^2$$

\rightarrow just massive gauge boson
3 d.o.f. (3 polarizations)

2. Lorenz gauge $\partial_\mu A^\mu = 0$

\hookrightarrow cross term vanishes (\int by parts)

$$\tilde{\mathcal{L}} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_A^2 A^2 + \frac{1}{2}(\partial_\mu \pi)^2$$

with $\partial_\mu A^\mu = 0$

gauge boson : 2 d.o.f.

pion : 1 d.o.f.

Higgs mechanism : gauge boson absorbs
the Goldstone boson

Abelian Higgs mechanism is realized in superconductors

Spontaneous symmetry breaking through
Boose-Einstein condensate $\langle \phi \rangle \sim \langle e^- e^- \rangle \neq 0$
Cooper pairs due to an attractive interaction between e^- (phonon exchange
- collective phenomenon leading to pairing)

A similar mechanism in neutron stars:

$e^+ e^-$ fluctuations in the thermal bath

with mostly n , some p and e^-

($n \rightarrow p e^- \bar{\nu} \rightarrow n e^+ e^- \nu \bar{\nu} \dots$) produce

plasmons \sim massive photons, $m \sim 1 \text{ MeV}$

Meissner effect: magnetic fields screened out of SC.

Can be extended to non-Abelian case