

Lecture 13

Spontaneous symmetry breaking
(Non) Linear sigma model
Higgs mechanism

Consider complex scalar field

$$\mathcal{L} = (\partial^\mu \varphi^*) (\partial_\mu \varphi) + m^2 \varphi \varphi^* - \frac{\lambda}{4} \varphi^2 \varphi^{*2}$$

Global $U(1)$ symmetry under $\varphi \rightarrow e^{i\alpha} \varphi$
 $\alpha = \text{const.}$

Potential $V(\varphi) = -m^2 |\varphi|^2 + \frac{\lambda}{4} |\varphi|^4$ has a local maximum at $\varphi = 0$ for $m^2 > 0$
→ theory is unstable

$$V \text{ is minimized at } |\varphi|^2 = \frac{2m^2}{\lambda}$$

An infinite number of equivalent vacua!

$$|\Omega_0\rangle : \langle \Omega_0 | \phi | \Omega_0 \rangle = \sqrt{\frac{2m^2}{\lambda}} e^{i\theta}$$

Choose Ω_0 as true vacuum

Classically : expect

$$\langle \Omega | \varphi | \Omega \rangle = \lim_{t \rightarrow 0} \int D\varphi e^{\frac{i}{\hbar} \int d^4x \mathcal{L}[\varphi]} \varphi = v$$

Called Vacuum Expectation Value (VEV)

Parametrize $\varphi(x)$ (complex!) in terms
of 2 real scalar fields $\sigma(x), \pi(x)$

$$\hookrightarrow \varphi(x) = [v + \frac{1}{\sqrt{2}}\sigma(x)] e^{i \frac{\pi(x)}{F_\pi}}$$

$\Rightarrow V(\varphi)$ only depends on σ .

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} (\partial_\mu \sigma)^2 + \left(v + \frac{1}{\sqrt{2}}\sigma(x)\right)^2 \frac{1}{F_\pi^2} (\partial_\mu \pi)^2 \\ & - \left[-\frac{m^4}{\lambda} + m^2 \sigma^2 + \frac{2\sqrt{m}}{2} \sigma^3 + \frac{\lambda \sigma^4}{16} \right] \end{aligned}$$

If choosing $F_\pi = \sqrt{2}v$ the kinetic term of the pion field

$$\frac{v^2}{F_\pi^2} (\partial_\mu \pi)^2 = \frac{1}{2} (\partial_\mu \pi)^2 \text{ is canonically normalized.}$$

- The Lagrangian describes a massless π -field and massive g -field

"Mexican hat" potential

1). "tachyon" mode around 0

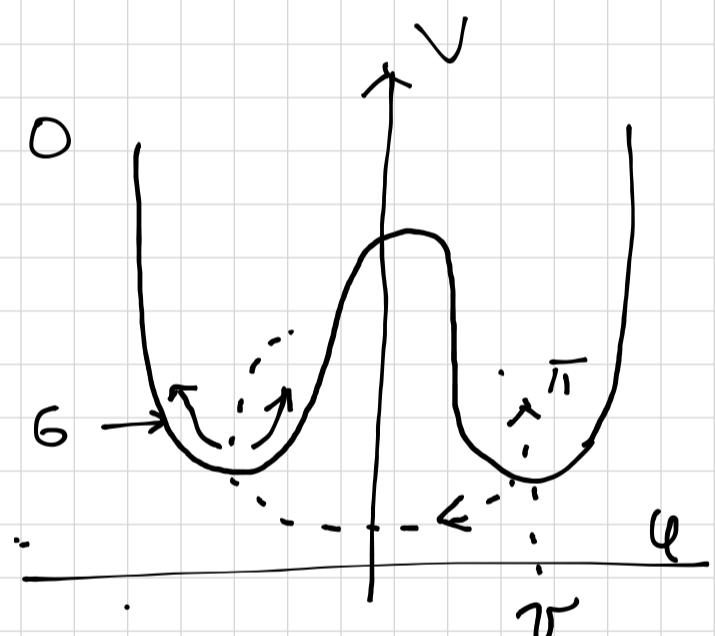
(wrong sign of $m^2 < 0$)

2). Two modes in the exp.

around stable vacuum

massless pion,

and massive g .



The Lagrangian above is "lineal g -model

The original L is invariant under

$$\varphi(x) \rightarrow e^{i\Theta} \varphi(x)$$

The VEV $\langle \varphi \rangle = v$ breaks this symmetry

The global $U(1)$ symmetry is now realized

$$\pi(x) \rightarrow \pi(x) + F_\pi \Theta;$$

$g(x)$ stays invariant

This symmetry forbids a mass term for $\pi(x)$!

Importantly: information about the symmetry breaking of L is entirely carried by the π field

ϕ is unrelated to breaking of $U(1)$ symmetry.

If ϕ field is very heavy ($m, \lambda \rightarrow \infty$, $v = 2m/\sqrt{\lambda}$ fixed) \rightarrow integrate out ϕ

$$\Rightarrow \mathcal{L} = \frac{1}{2} (\partial_\mu \pi)^2 \text{ (non-linear } \phi\text{-model)}$$

Noether current associated with the shift symmetry:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \pi)} \frac{\delta \pi}{\delta \theta} = F_\pi \partial_\mu \pi \quad \partial_\mu J^\mu = 0$$

Conserved charge

$$Q = \int d^3x J_0(x) = \int d^3x \frac{\partial \mathcal{L}}{\partial \dot{\pi}} \frac{\delta \pi}{\delta \theta} \equiv \int d^3x \varphi(x) \frac{\delta \pi}{\delta \theta}$$

π, φ — canonically conjugate fields

$$\Rightarrow \text{obey } [\pi(x), \varphi(y)] = i \delta^3(\vec{x} - \vec{y})$$

$$\Rightarrow [Q, \pi(y)] = \int d^3x [\varphi(x), \pi(y)] \frac{\delta \pi}{\delta \theta} = -i \frac{\delta \pi(x)}{\delta \theta}$$

Charge Q is generator of symm. transformation

Since $\partial_t Q = 0$, $[H, Q] = 0$

Acting with Q on vacuum $|\Omega\rangle$ creates a state which is degenerate w. $|\Omega\rangle$:

$$HQ|\Omega\rangle = \cancel{[H, Q]}|\Omega\rangle + Q\underbrace{H|\Omega\rangle}_{\text{!!! } E_0|\Omega\rangle} = E_0 Q|\Omega\rangle$$

\Downarrow

We can construct states of 3-mom \vec{p} out of vacuum state!

$$|\pi(\vec{p})\rangle = -\frac{2i}{F_\pi} \int d^3x e^{i\vec{p}\cdot\vec{x}} J_0(x) |\Omega\rangle$$

Multiply out

$$\begin{aligned} \langle \pi(\vec{q}) | \pi(\vec{p}) \rangle &= 2\omega_p (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \\ &= -\frac{2i}{F_\pi} \int d^3x e^{i\vec{p}\cdot\vec{x}} \langle \pi(\vec{q}) | J_0(x) |\Omega\rangle \end{aligned}$$

$$\Rightarrow \langle \pi(\vec{p}) | J_0(x) |\Omega\rangle = i\omega_p F_\pi e^{i\vec{p}\cdot\vec{x}}$$

Or, in Lorentz-covariant form:

$$\langle \pi(\vec{p}) | J_\mu(x) |\Omega\rangle = i p_\mu F_\pi e^{-ipx}$$

→ This has the meaning of a particle of mom p and mass = 0

Goldstone theorem:

Spontaneous breaking of continuous global symmetries implies the existence of massless particles, called Goldstone bosons

The way Goldstone bosons enter the L is severely constrained!

Now consider QCD with 2 flavors

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \quad \text{SU(2) doublet}$$

Hadrons built out of light flavors show remarkable symmetry

$$(P_n) ; \begin{pmatrix} \Delta^{++} \\ \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix} ; \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$$

Neglect quark masses at first

(Good approximation: $m_u \sim 2 \text{ MeV}$, $m_d \sim 5 \text{ MeV}$)

↓

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \underbrace{i\bar{q}\not{D}q}_{= i\bar{q}_L \not{D}q_L + i\bar{q}_R \not{D}q_R}$$

$$q_{LR} \equiv \frac{1}{2}(1 \mp \gamma_5)q, \quad q = \begin{pmatrix} u \\ d \end{pmatrix}$$

L, R components do not talk to each other

The symmetry is

$$SU(2)_L \times SU(2)_R = SU(2)_V \times SU(2)_A$$

Transformations are

$$q \rightarrow e^{i\theta^a \tau^a} e^{i\gamma_5 \beta^a \tau^a}$$

$\downarrow \quad \downarrow$
 $U_V \quad U_A$

$$U_L = U_{V-A}$$

$$U_R = U_{V+A}$$

The full symmetry is $U(2) \times U(2)$

$$SU(2) : \det = 1$$

$$U(2) : |\det| = 1 \rightarrow U(2) = SU(2) \times U(1)$$

$$SO(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$$

$V \quad A \quad e^{id} \quad e^{i\lambda \gamma_5}$

$$\text{Noether currents : } J_\mu^\alpha = \bar{q} \tau^\alpha \gamma^\mu q$$

$$J_{\mu 5}^\alpha = \bar{q} \tau^\alpha \gamma^\mu \gamma_5 q$$

$$J_\mu^V = \bar{q} \gamma^\mu q$$

$$J_\mu^A = \bar{q} \gamma^\mu \gamma_5 q$$

$U(1)_{V,A}$ are broken by quantum anomalies

$U(1)_V \rightarrow$ Noether charge : baryon # B

$B-L$ is conserved in SM (L -lepton #)

Will not deal with anomalies now

As the universe cooled down to $T_c \sim \Lambda_{QCD}$, $SU(2) \times SU(2)$ symmetry was broken by quark condensates

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = V^3 \quad ([V] = M^*)$$

The mechanism of formation of quark condensates is not fully understood in QCD

→ active research at GSI Darmstadt, RHIC@BNL (Brookhaven), ALICE@CERN
to study QCD phase transition

Even without understanding the exact mechanism, can learn much about the dynamics of Goldstone bosons!

$$\rightarrow \mathcal{L} = 1.2\mu \sum | \dot{\Sigma} |^2 + m^2 | \Sigma |^2 - \frac{\lambda}{4} | \Sigma |^4$$

Notation: $\Sigma = \Sigma_{ij} \rightarrow$ adjoint representation of $SU(2)$
 $| \Sigma |^2 = \text{tr}(\Sigma \Sigma^\dagger) = \sum_{ij} \Sigma_{ji}^+$

$SU(2)$ transformations:

$$\Sigma \rightarrow g_L \Sigma g_R^+, \quad \Sigma^+ \rightarrow g_R \Sigma^+ g_L^+$$

$$g_{L,R} = e^{i \theta_{L,R}^a T^a}$$

Potential minimized at $\langle \Sigma \rangle = \frac{V}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

with $v = \frac{2M}{\sqrt{\lambda}}$,

Expect $v \sim V \sim \Lambda_{QCD}$

Just as before \rightarrow linear sigma model

$$\Sigma(x) = \frac{v + \phi(x)}{\sqrt{2}} \exp \left[2i \frac{\pi^a(x) \bar{c}^a}{F_\pi} \right]$$

π transforms as

$$\pi^a \rightarrow \pi^a + \frac{F_\pi}{2} (\theta_L^a - \theta_R^a) - \frac{1}{2} f^{abc} (\theta_L^b + \theta_R^c) \pi^a$$

Unbroken isospin $\rightarrow SU(2)_V$, $\theta_L = \theta_R$

$\rightarrow \pi^a$ transforms under adjoint repr.,

$\rightarrow (\pi^+, \pi^0, \pi^-)$ isotriplet is generated

Broken $SU(2)_A$, $\theta_L = -\theta_R$

$\hookrightarrow \pi^a \rightarrow \pi^a + F_\pi \theta_L^a$ shift transf.

Mass term for π^a is forbidden

Integrate out ϕ ($m, \lambda \rightarrow \infty$, v fixed)

$$\begin{aligned} \hookrightarrow \frac{\sqrt{2}}{v} \Sigma &= U(x) = e^{i \frac{\pi^a x^a}{F_\pi}} \\ &= \exp \left[\frac{i}{F_\pi} \begin{pmatrix} \pi^0 & \sqrt{2} \pi^- \\ \sqrt{2} \pi^+ & -\pi^0 \end{pmatrix} \right] \end{aligned}$$

$$UU^\dagger = \mathbb{1} ; U \rightarrow g_L U g_R^\dagger$$

Non-linear G-model for pion fields

(Chiral Lagrangian) \rightarrow from symmetries

$$\begin{aligned}
\mathcal{L}_X = & \frac{\pi^2}{4} + \text{tr}(D_\mu U)(D^\mu U^+) \\
& + L_1 \text{tr}[(D_\mu U)(D^\mu U^+)]^2 \\
& + L_2 \text{tr}[(D_\mu U)(D_\nu U^+)] \text{tr}[(D^\mu U)(D^\nu U^+)] \\
& + L_3 \text{tr}[(D_\mu U)(D^\mu U^+)(D_\nu U)(D^\nu U^+)] + \dots
\end{aligned}$$

Covariant derivatives \rightarrow to include
EW interaction;

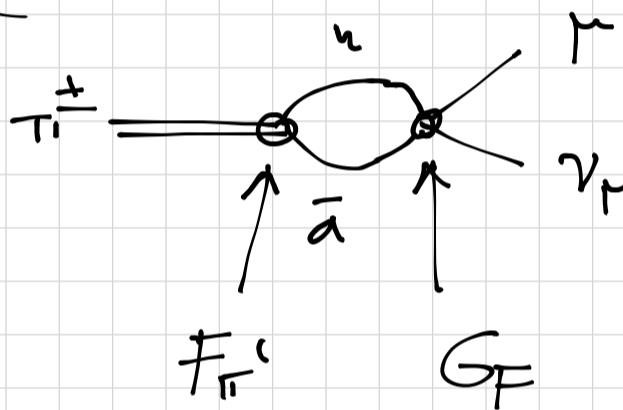
No gluons \rightarrow were integrated out for $T < T_c$
the condensate is colorless

First term: Taylor expand

$$\begin{aligned}
\frac{\pi^2}{4} (\partial_\mu U)(\partial^\mu U^+) = & \frac{1}{2} (\partial_\mu \pi^0)(\partial^\mu \pi^0) \\
& + (\partial_\mu \pi^+)(\partial^\mu \pi^-) \\
& + \frac{1}{\pi^2} \left[-\frac{1}{3} (\pi^0)^2 (\partial_\mu \pi^+) (\partial^\mu \pi^-) \right. \\
& \quad \left. + \text{permutations} \right] \\
& + \frac{1}{\pi^4} \left[\frac{1}{18} (\pi^+ \pi^-)^2 (\partial_\mu \pi^0)(\partial^\mu \pi^0) \right. \\
& \quad \left. + \dots \right] \\
& + \dots
\end{aligned}$$

- L_χ contains higher derivatives
- ⇒ is non-renormalizable
- higher derivatives enter in powers of (E/F_π)
- as long as $(E/F_\pi) \ll 1$ is predictive
- Low-Energy Constants (LEC) can be fitted to observables

The pion decay constant F_π → from pion lifetime



$$\Gamma(\pi^+ \rightarrow \mu^+ \nu_\mu) = \frac{G_F^2 F_\pi^2}{4\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

$$\bar{\tau}_{\pi^+} \approx 2.6 \cdot 10^{-8} \text{ s} \Rightarrow F_\pi \approx 92.4 \text{ MeV}$$

Can make predictions for other processes.

Now, quark masses are non-zero → chiral symmetry is broken not only spontaneously, but also explicitly

$$L_m = \bar{q} M q, \quad M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

Trick: treat masses as (constant) fields

$SU(2)$ transformation

$$M \longrightarrow g_L M g_R^+$$

\Rightarrow Also mass-term is $SU(2)$ invariant

New term in non-linear σ -model

$$\begin{aligned} \mathcal{L}_M &= \frac{V^3}{2} \text{tr}(M U + M U^\dagger) \\ &= V^3 (m_u + m_d) - \frac{V^3}{2 F_\pi^2} \pi^2 + O\left(\frac{\pi^3}{F_\pi^3}\right) \end{aligned}$$

\Rightarrow Gell-Mann - Okubo - Renner relation

$$m_\pi^2 = \frac{V^3}{F_\pi^2} (m_u + m_d)$$

$$m_\pi \approx 140 \text{ MeV}; \quad F_\pi = 92.4 \text{ MeV} \quad V \approx \Lambda_{QCD} \sim 300 \text{ MeV}$$

$$m_u + m_d \approx 7 \text{ MeV}$$

Generalization to $SU(3)$ (u, d, s)

$$\rightarrow U(x) = e^{2i\pi^a T^a / F_\pi}$$

$$= \exp\left[\frac{\sqrt{2}i}{F_\pi} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ \bar{K}^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}\right]$$

Chiral perturbation theory \rightarrow
relations among octet masses
(Gell-Mann - Okubo)

e.g. for mesons

$$\frac{1}{2} \left[\frac{M_{K^+}^2 + M_{K^0}^2}{2} + \frac{M_{K^-}^2 + M_{\bar{K}^0}^2}{2} \right]$$

$$= \frac{3 m_\eta^2 + m_\pi^2}{4}$$

holds within 5%

Similarly for baryons (not related to chiral symmetry, but $SU(3)$ flavor)

Spontaneous breaking of chiral sym. together with its explicit breaking by quark masses explains why the pion is so light

(recall constituent quarks w. mass ~ 300 MeV to explain baryon masses \rightarrow naive expectation would be $M_\pi \sim 600$ MeV)

Pions are realized as pseudo-Goldstone bosons (acquire finite mass due to explicit symmetry breaking)

Pion decay constant is an important constant in particle physics;

Extending to 3 flavors $\rightarrow F_\pi, F_K$

Exact knowledge of F_K/F_π crucial

for tests of unitarity of the
CKM quark mixing matrix