The Muon $g-2$
A sensitive probe for new physics

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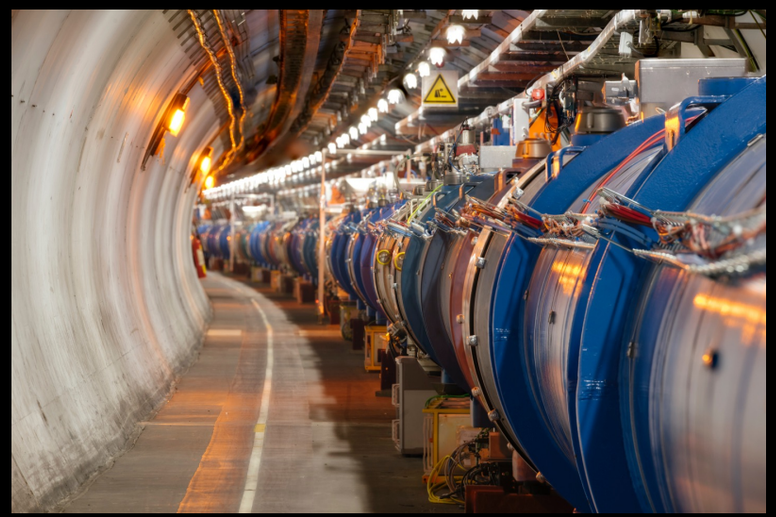
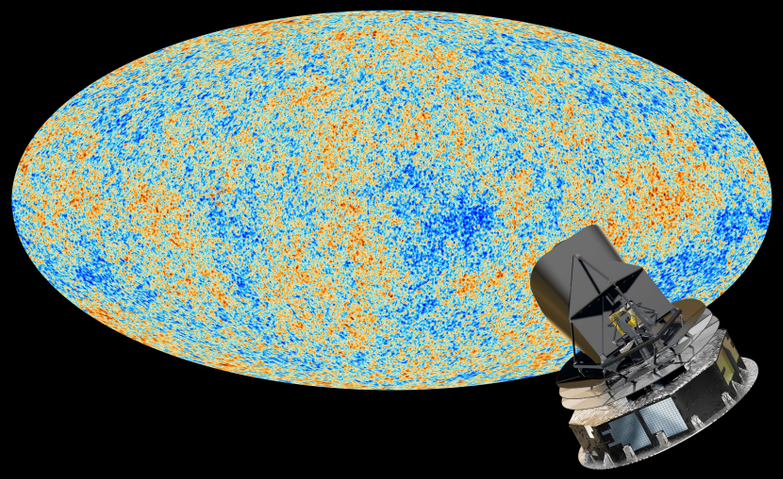
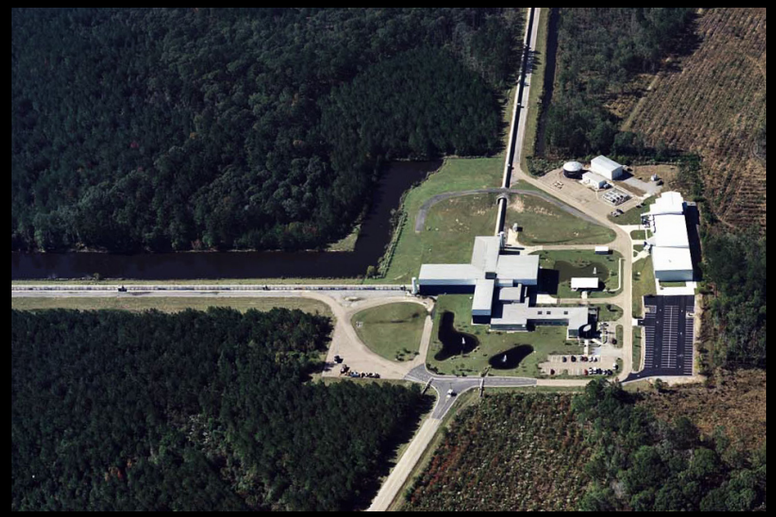
SFB 1044
THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

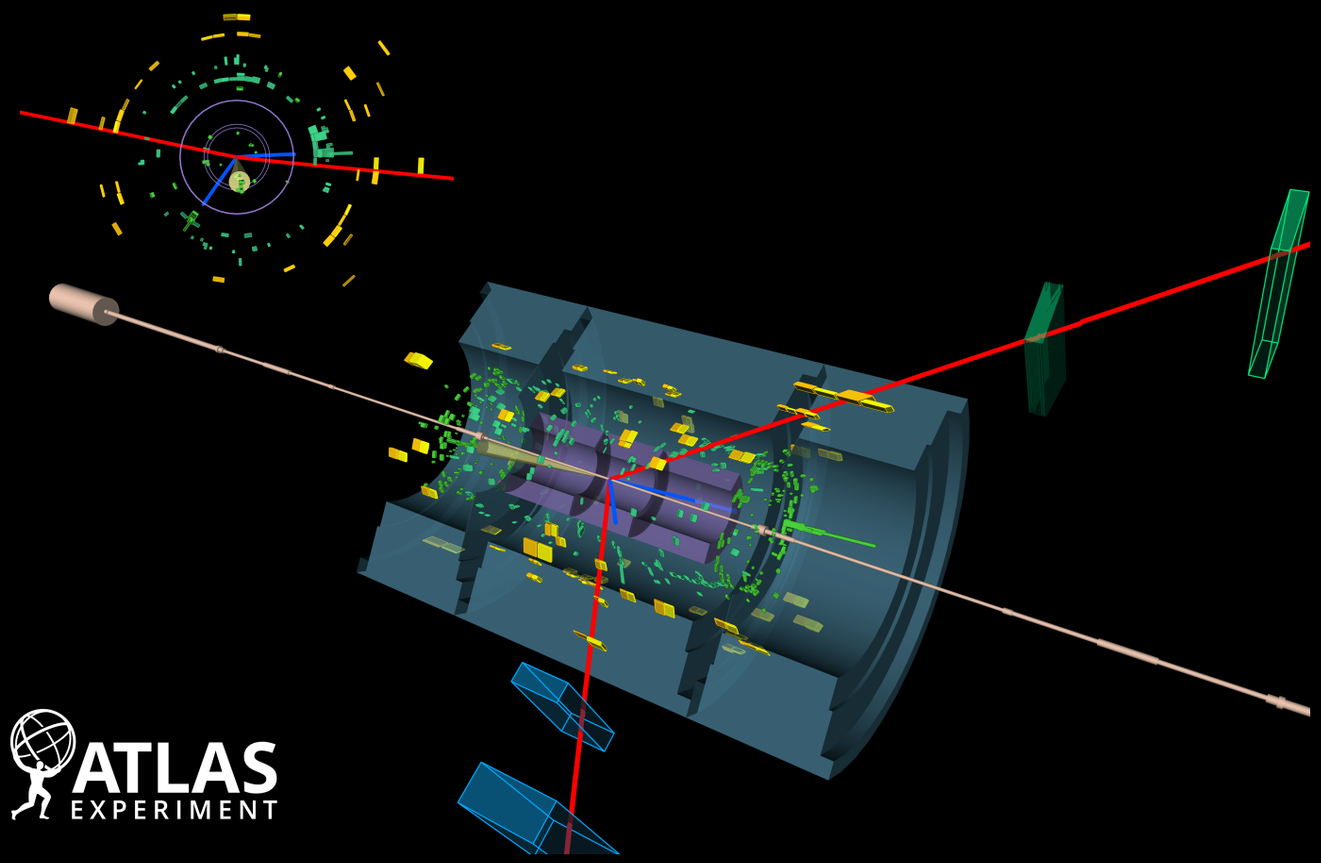
The Quest for New Physics

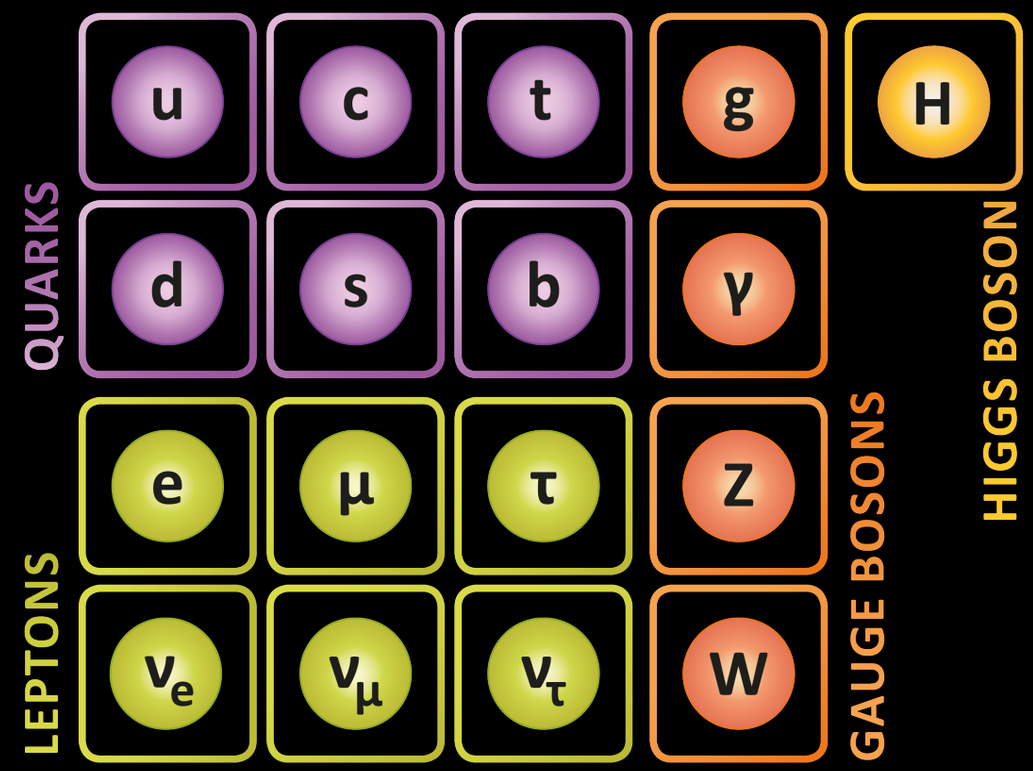






The Quest for New Physics

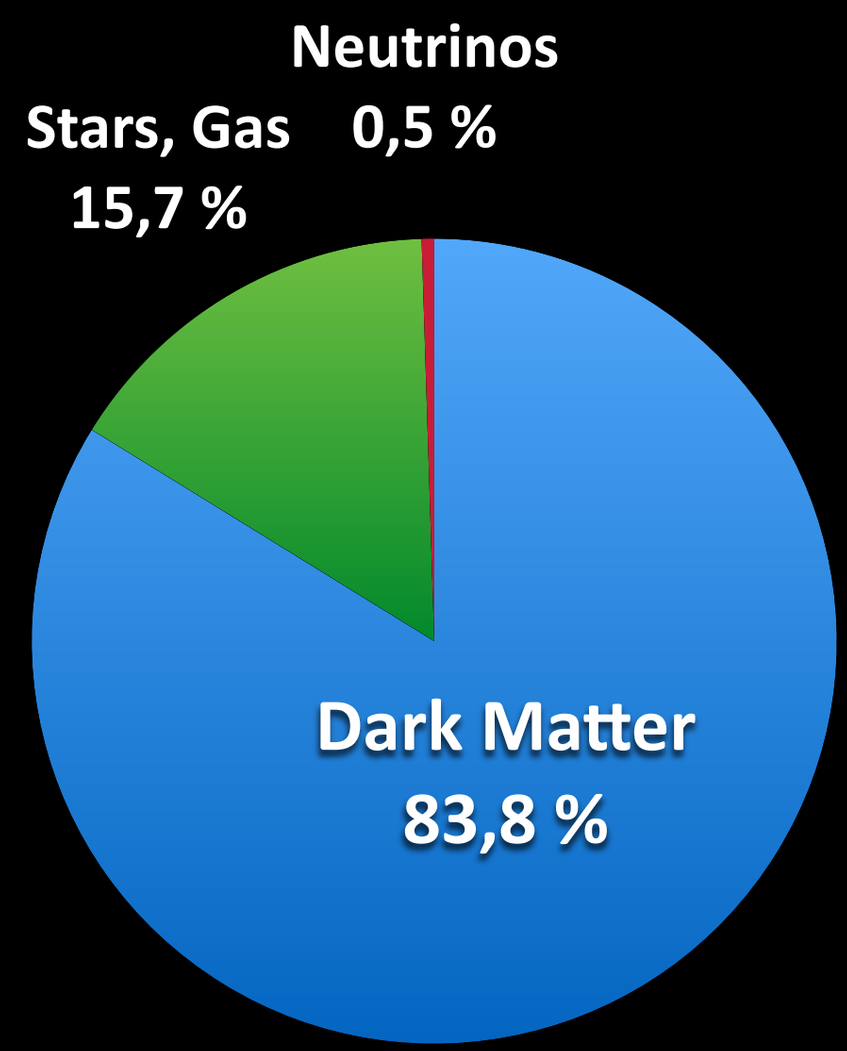




The Quest for New Physics

Standard Model does not explain

- Baryon asymmetry
- Mass and scale hierarchies
- Existence of dark matter



Standard Model does not provide a complete description of Nature

The Quest for New Physics

* Explore the limits of the Standard Model
  - Search for new particles and phenomena at high energies
  - Search for enhancement of rare phenomena
  - Compare precision measurements to SM predictions

* Realise extreme levels of experimental sensitivity, matched by equally precise theoretical calculations

* Control over “hadronic uncertainties” — effects arising from the strong interaction

* Prominent example: anomalous magnetic moment of the muon

Magnetic moment of particles and nuclei

Particle with charge e and mass m:

$$\boldsymbol{\mu} = g\,\frac{e\hbar}{2m}\,\boldsymbol{S}, \qquad \boldsymbol{S} = \frac{\boldsymbol{\sigma}}{2}$$

Pauli equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{x},t) = \left\{\frac{1}{2m}\left[\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A})\right]^2 + e\Phi\right\}\psi(\boldsymbol{x},t)$$

$\Leftrightarrow$

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{x},t) = \left\{\frac{1}{2m}\,(\boldsymbol{p}-e\boldsymbol{A})^2 + e\Phi - \frac{e\hbar}{2m}\,\boldsymbol{\sigma}\cdot\boldsymbol{B}\right\}\psi(\boldsymbol{x},t)$$

* Non-relativistic limit of Dirac equation: $g = 2$

* Experimental measurement: $g_e = 2.0023193\ldots$

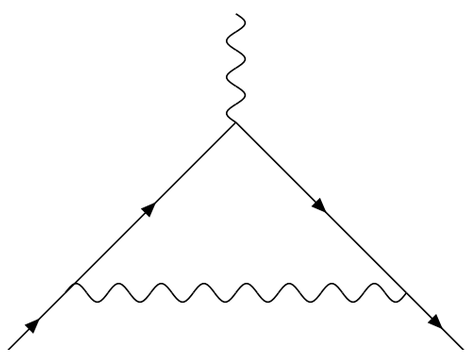
  $g_\mu = 2.0023318\ldots$

Anomalous magnetic moment

* Dirac value of $g = 2$ modified by quantum corrections

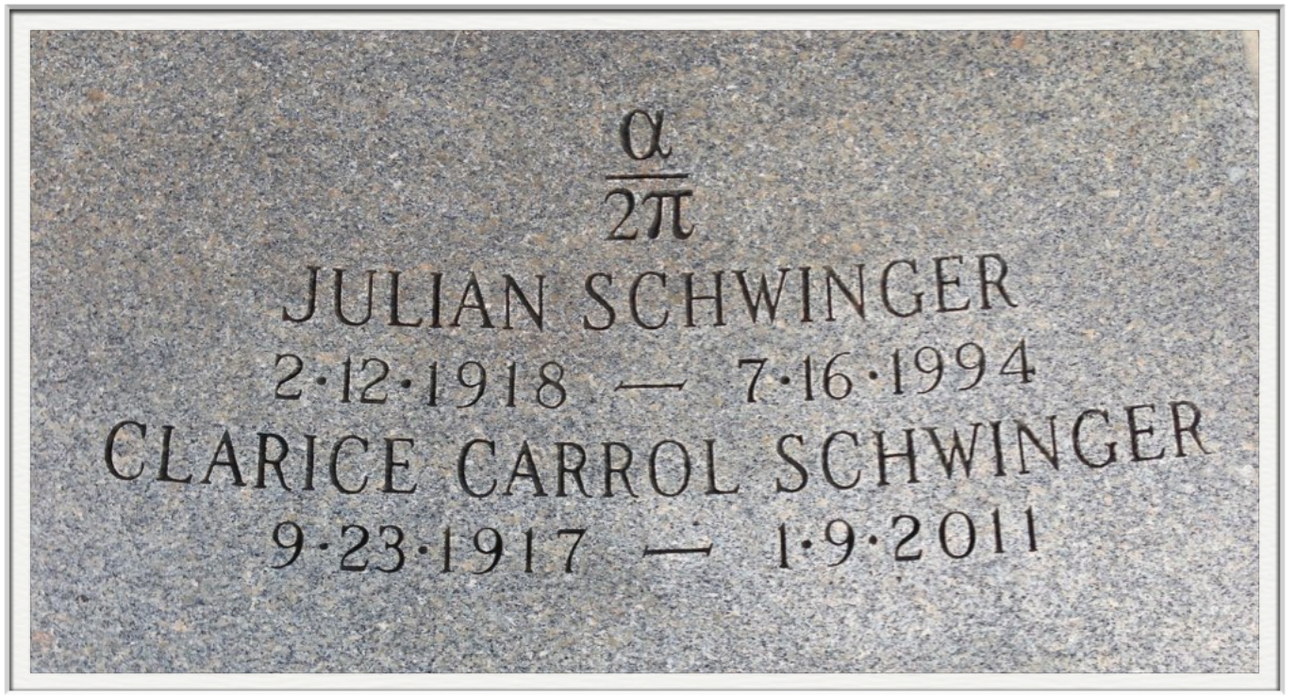
$$g = 2(1 + a) \quad \Rightarrow \quad a = \tfrac{1}{2}(g - 2)$$

* First-order QED correction:



$$a^{(2)} = \frac{\alpha}{2\pi} = 0.001\,161\,40\ldots$$

[J. Schwinger, Phys Rev 73 (1948) 416]

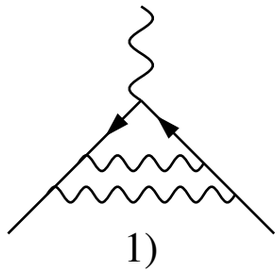


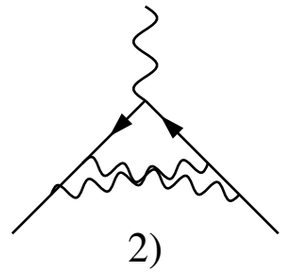
$$a_e^{\text{exp}} = 0.001\,159\,652\,181\,643(764)$$

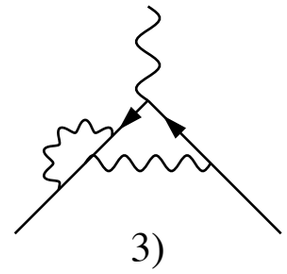
$$a_\mu^{\text{exp}} = 0.001\,165\,920\,9(6)$$

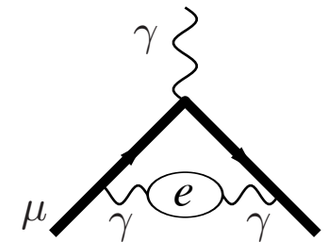
Higher-order corrections

* QED corrections:









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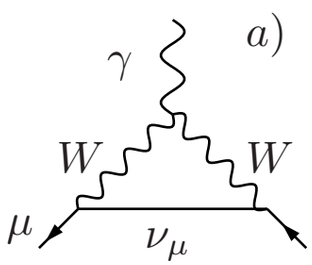
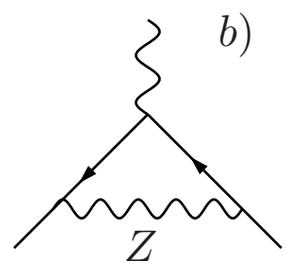
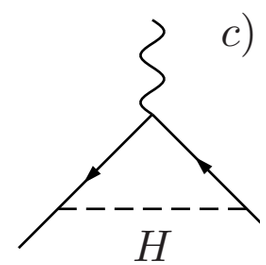
* Weak corrections:







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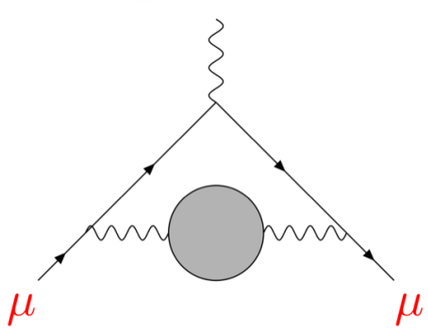
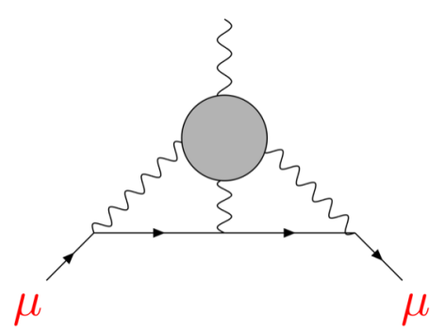
* Strong corrections:





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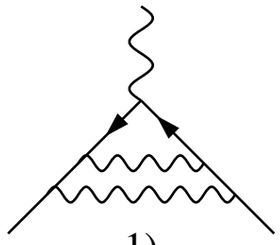
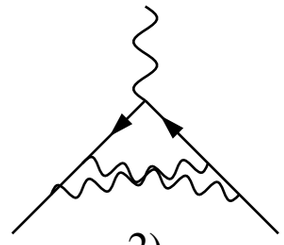
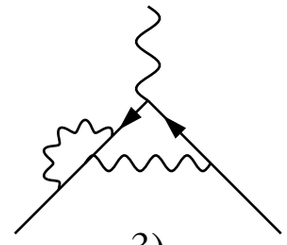
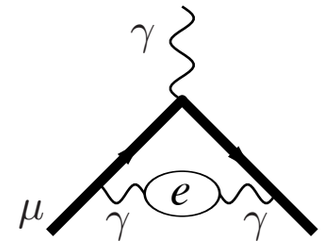
Higher-order corrections

* QED corrections:









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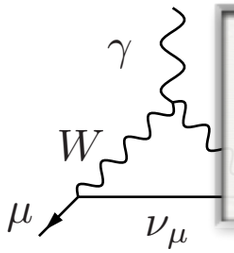
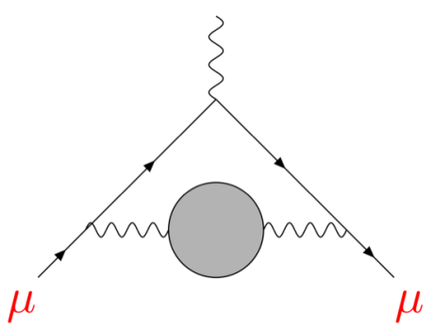
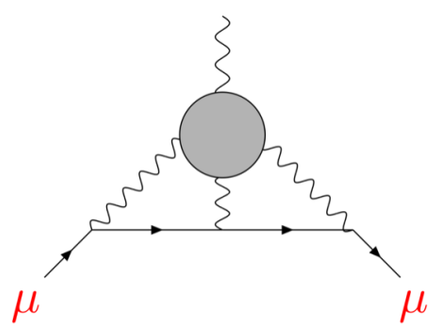


99.994% of a_μ are due to QED!

* Strong corrections:





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Why do we care?

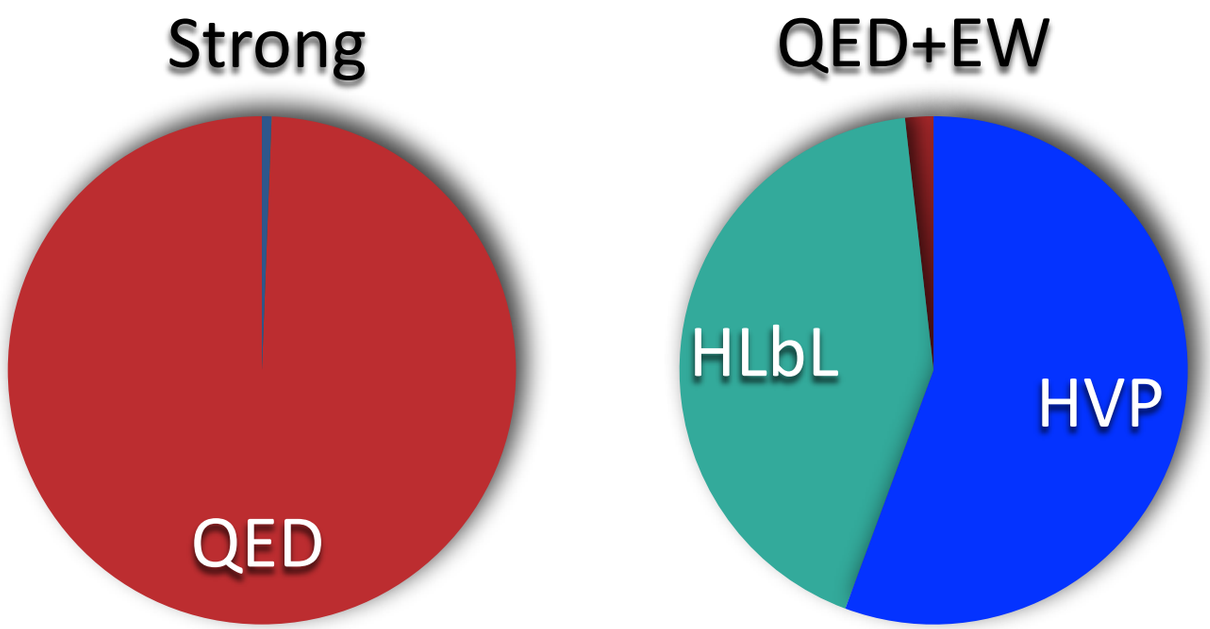
* Standard Model estimate of a_μ deviates from experiment:

$$a_\mu \equiv \tfrac{1}{2}(g-2)_\mu = \begin{cases} 116\,592\,080(54)(33)\cdot 10^{-11} \\ 116\,591\,825(34)(26)(1)\cdot 10^{-11} \end{cases}$$

3.5 σ

* SM estimate dominated by QED; error dominated by QCD

$$a_\mu = a_\mu^{\rm QED} + a_\mu^{\rm weak} + a_\mu^{\rm strong}$$



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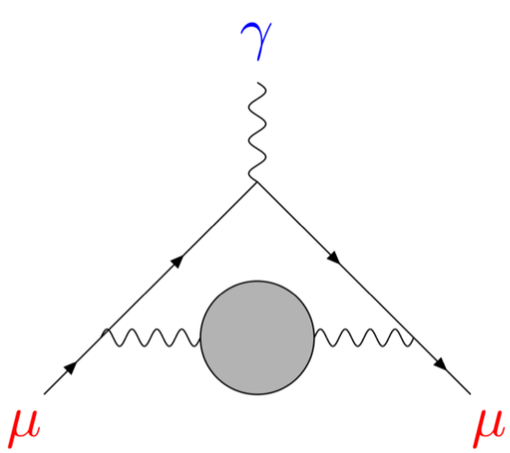
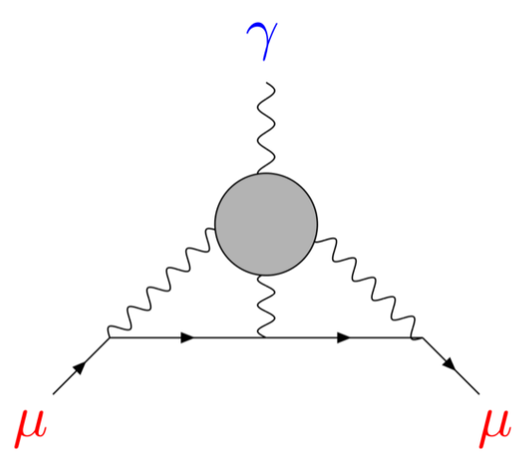
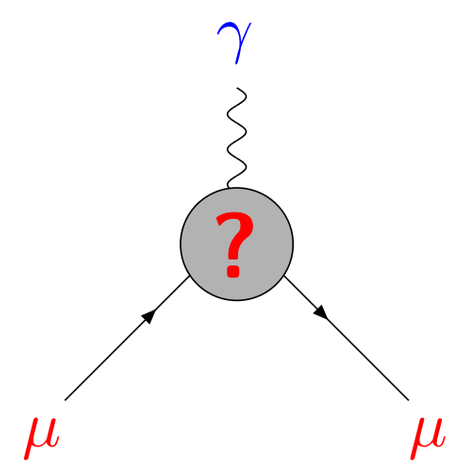
Hadronic vacuum polarisation:



Hadronic light-by-light scattering:



Contribution from "New Physics"?



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$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{strong}} + a_\mu^{\text{NP?}}$$

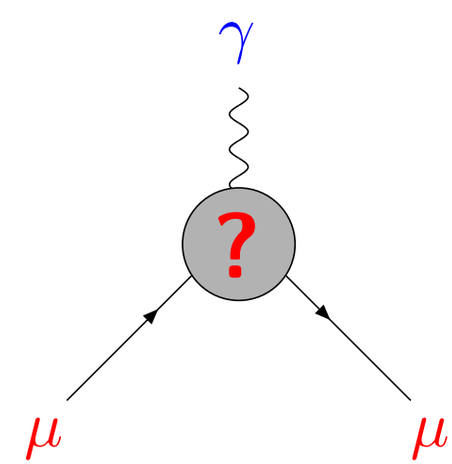
* New physics effects enhanced by

$$\delta a_\ell \propto m_\ell^2 / M_?^2$$

⇒ Muon is more sensitive by a factor

$$(m_\mu/m_e)^2 \approx 4.3\cdot 10^4$$

Contribution from "New Physics"?



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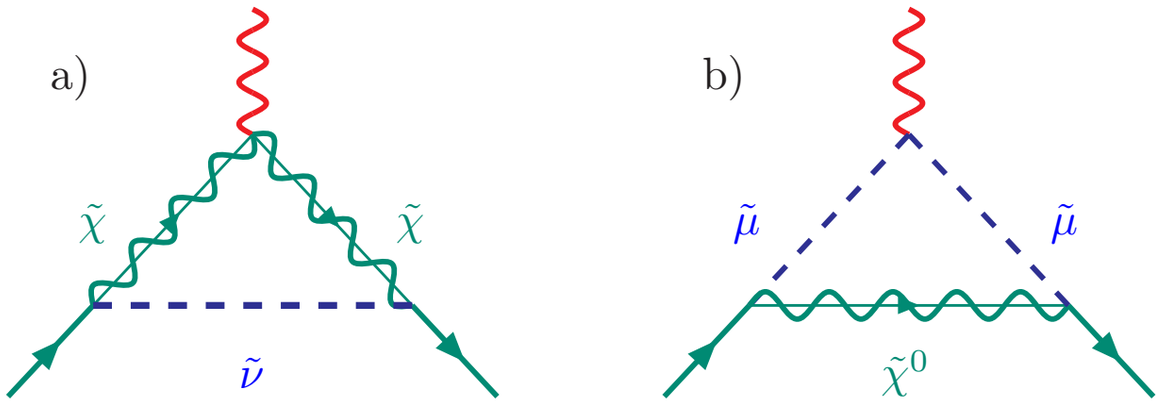
* Candidates for BSM physics:
  - Supersymmetric particles
  - Additional gauge bosons:

$$G_{\rm SM} \longrightarrow G_{\rm SM} \times U(1)^n$$



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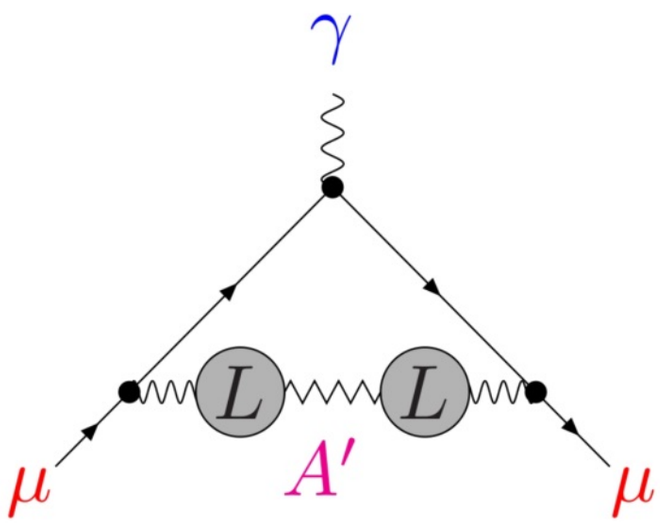
A' — "dark photon"



Outline

Experimental determination of a_μ

QED contribution to a_μ

Hadronic contributions to a_μ

Hadronic contributions to a_μ from lattice QCD

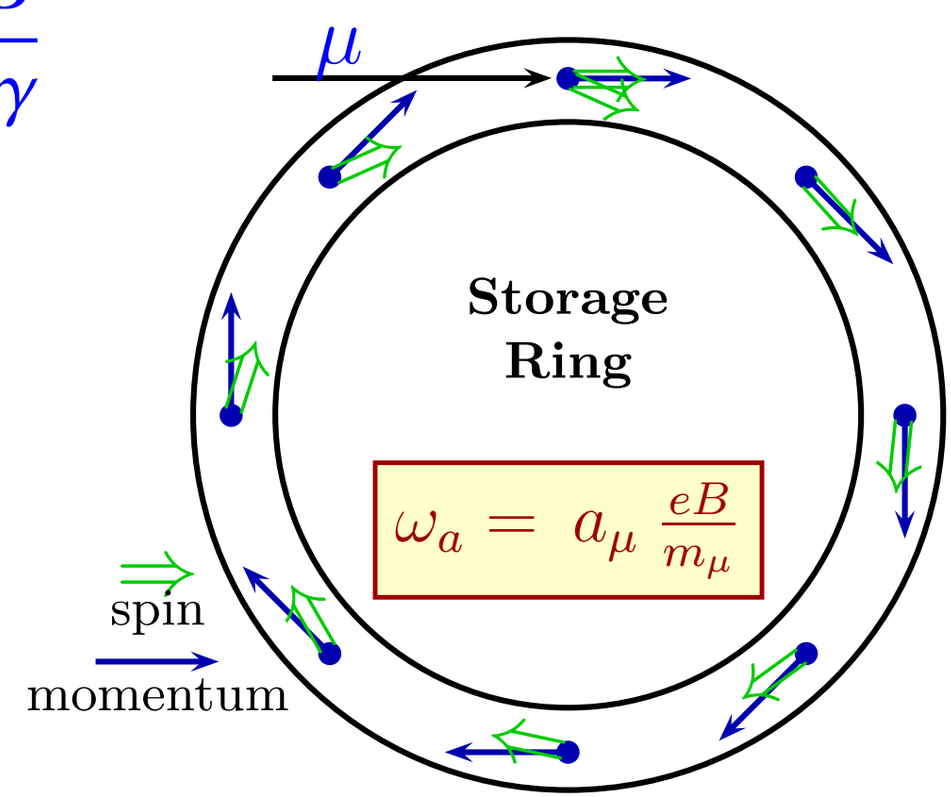
Measuring a_μ at storage rings

* Particle with charge e moving in a magnetic field:
  - Momentum turns with cyclotron frequency ω_C
  - Spin turns with ω_S

$$\omega_C = -\frac{eB}{m\gamma}, \quad \omega_S = -g\,\frac{eB}{2m} - (1-\gamma)\frac{eB}{m\gamma}$$

⇒ Spin turns relative to the momentum with frequency ω_a

$$\omega_a = \omega_S - \omega_C = -\underbrace{\tfrac{1}{2}(g-2)}_{a}\,\frac{eB}{m}$$



actual precession × 2

[Jegerlehner & Nyffeler, Phys Rep 477 (2009) 1]

Measuring a_μ at storage rings

* Storage rings require vertical focussing — apply electric quadrupole field

$$\omega_a = -\frac{e}{m}\left\{a_\mu \boldsymbol{B} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\boldsymbol{\beta}\times\boldsymbol{E}}{c}\right\}$$

0

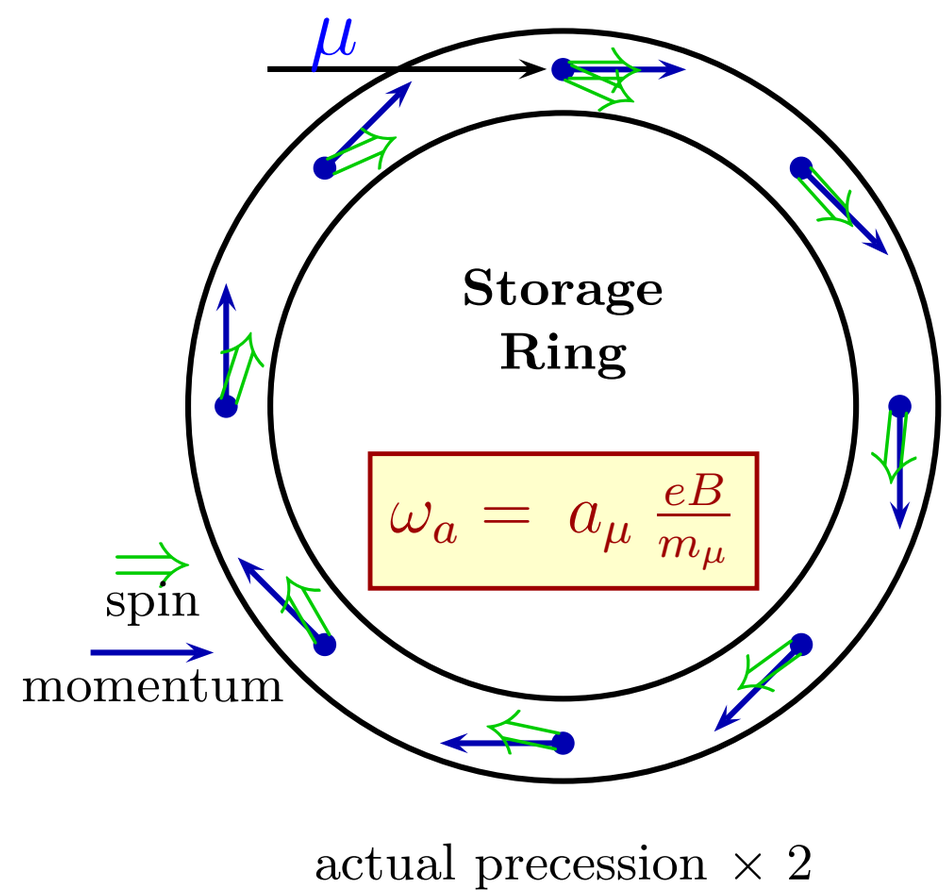
[Bargmann, Michel & Telegdi 1959]

* Tune γ such that term $\propto (\boldsymbol{\beta}\times\boldsymbol{E})$ vanishes

  "magic" γ :

$$\gamma_{\text{magic}} = 29.3 \quad \Leftrightarrow \quad p_{\text{magic}} = 3.09\,\text{GeV}/c$$

* Measure two quantities: $\omega_a,\ \boldsymbol{B}$

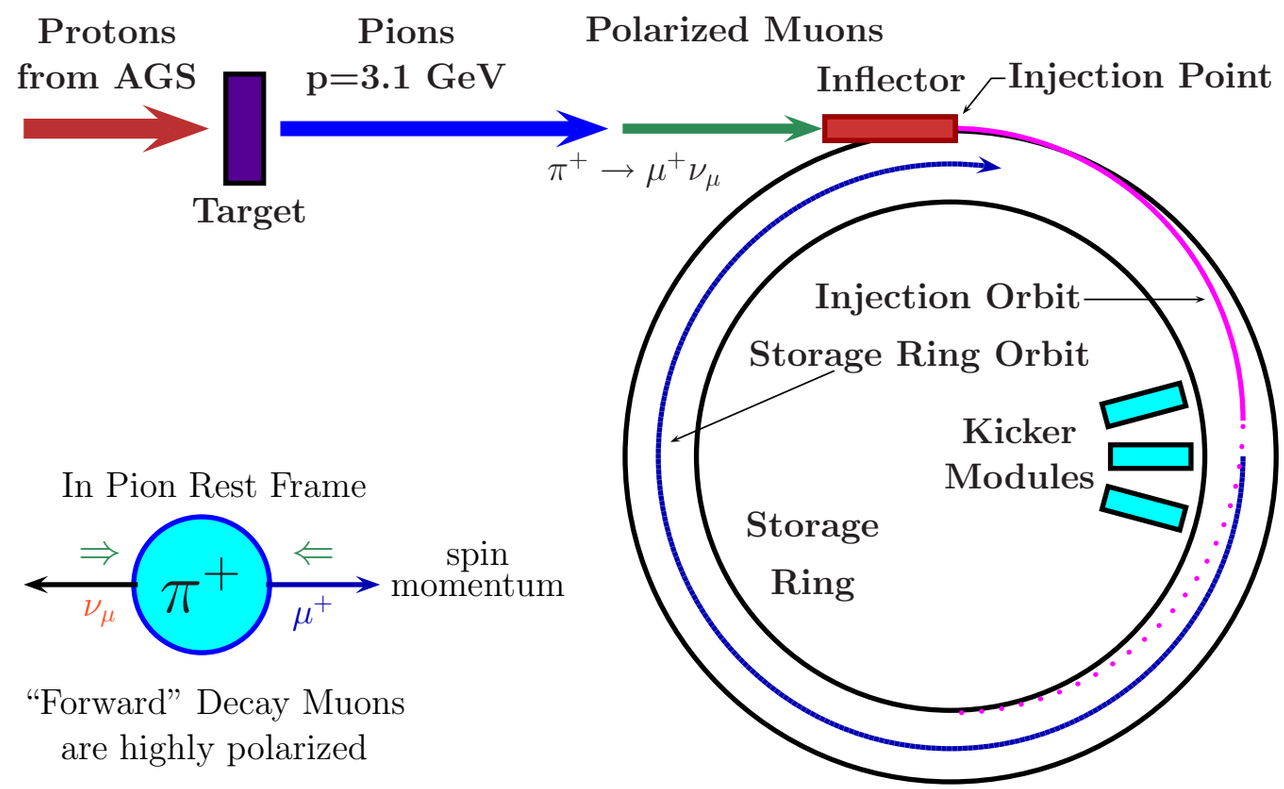


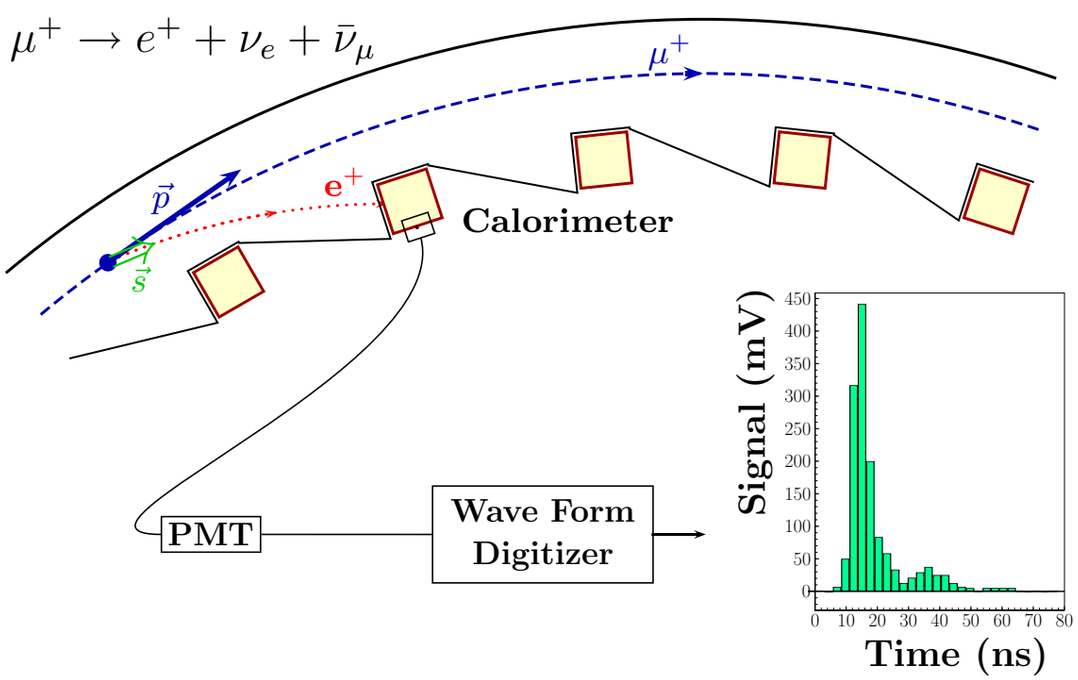
[Jegerlehner & Nyffeler, Phys Rep 477 (2009) 1]

BNL experiment E821

Birth: $\pi^+ \to \mu^+ + \nu_\mu$

Death: $\mu^+ \to e^+ + \bar{\nu}_\mu + \nu_e$





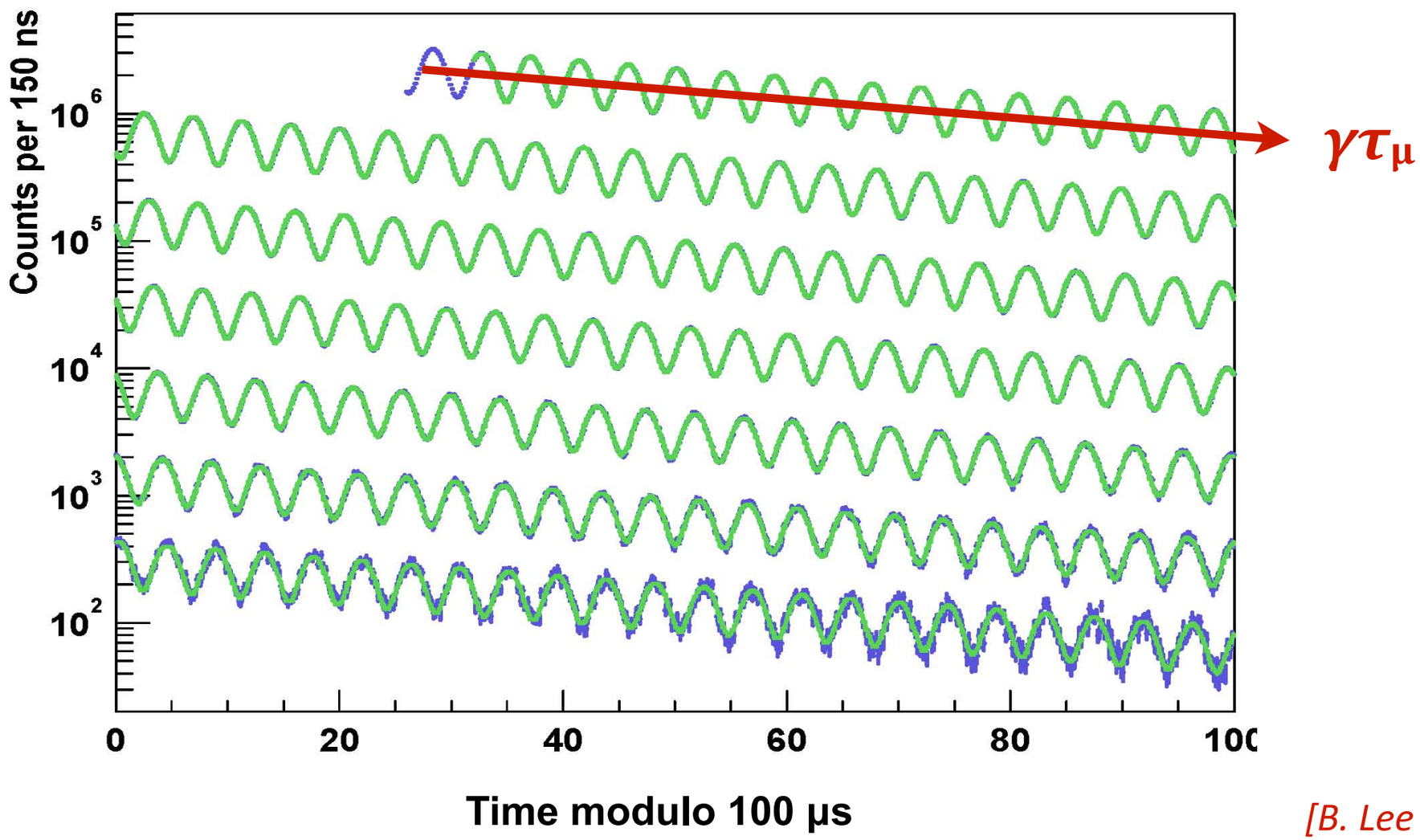
- Highest energy positrons emitted along muon spin axis
- Select positron above threshold energy

[Jegerlehner & Nyffeler, Phys Rep 477 (2009) 1]

BNL experiment E821

* Count rate and wiggle plot:

$$N(t) = N_0(E)\,\exp\left(-\frac{t}{\gamma\tau_\mu}\right)\left\{1 + A(E)\sin\left(\omega_a t + \phi(E)\right)\right\}$$



[B. Lee Roberts]

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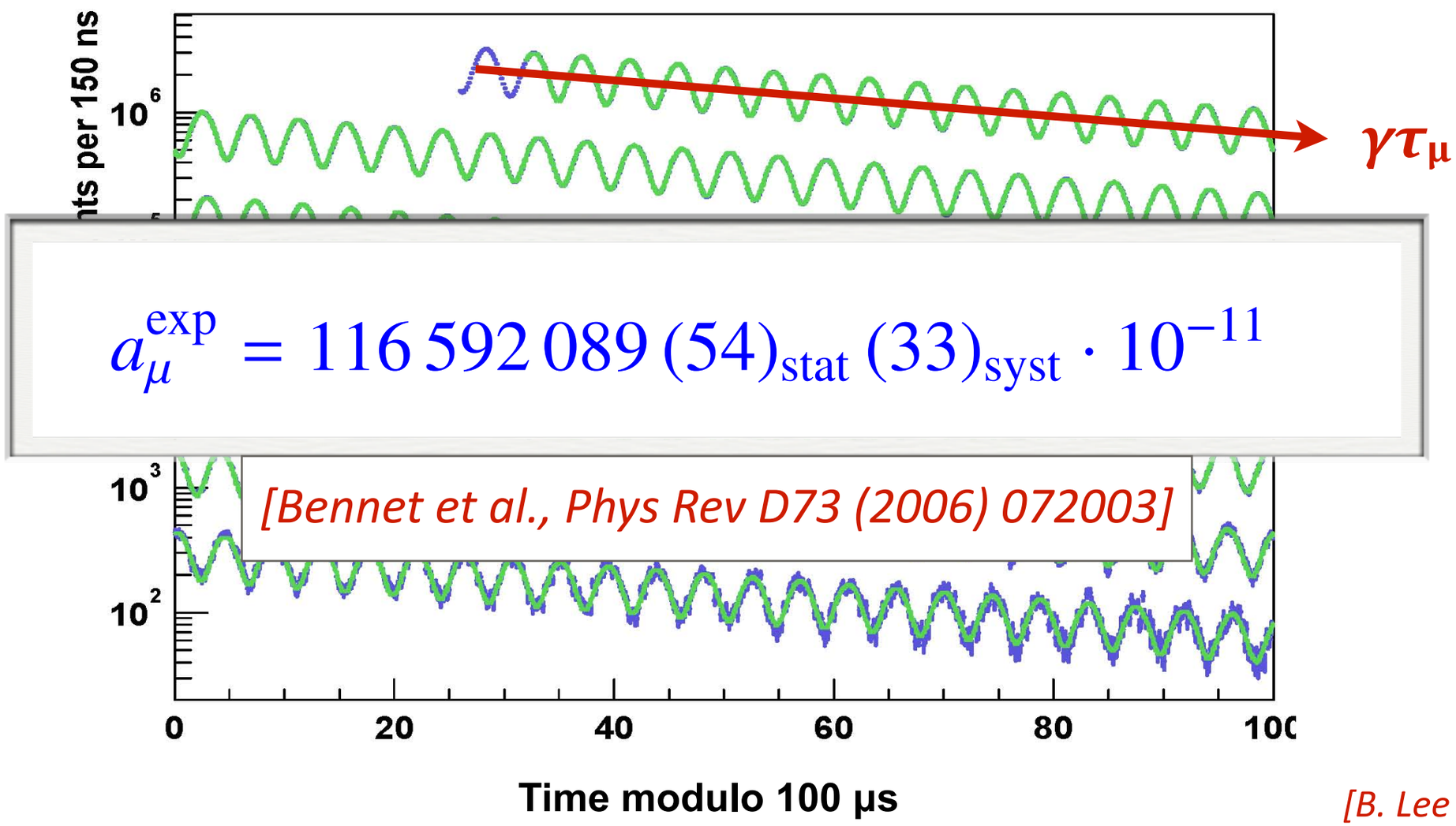


$$a_\mu^{\text{exp}} = 116\,592\,089\,(54)_{\text{stat}}\,(33)_{\text{syst}} \cdot 10^{-11}$$

[Bennet et al., Phys Rev D73 (2006) 072003]

[B. Lee Roberts]

From BNL E821 to Fermilab E989

$$a_\mu^{\text{exp}} = 116\,592\,089\,(54)_{\text{stat}}\,(33)_{\text{syst}} \cdot 10^{-11}$$

* Total precision of $0.54\ \text{ppm}$, dominated by statistics
* Use hotter beam of Fermilab proton booster: $8\ \text{GeV/c}$
* Suppress pion background — longer pion decay channel

  BNL: $80\ \text{m}$ $\longrightarrow$ Fermilab: $2\ \text{km}$

* Aim for $100\ \text{ppb}$ statistical and $100\ \text{ppb}$ systematic error

  $\longrightarrow$ $0.14\ \text{ppm}$ total error

* Transport BNL storage ring to Fermilab

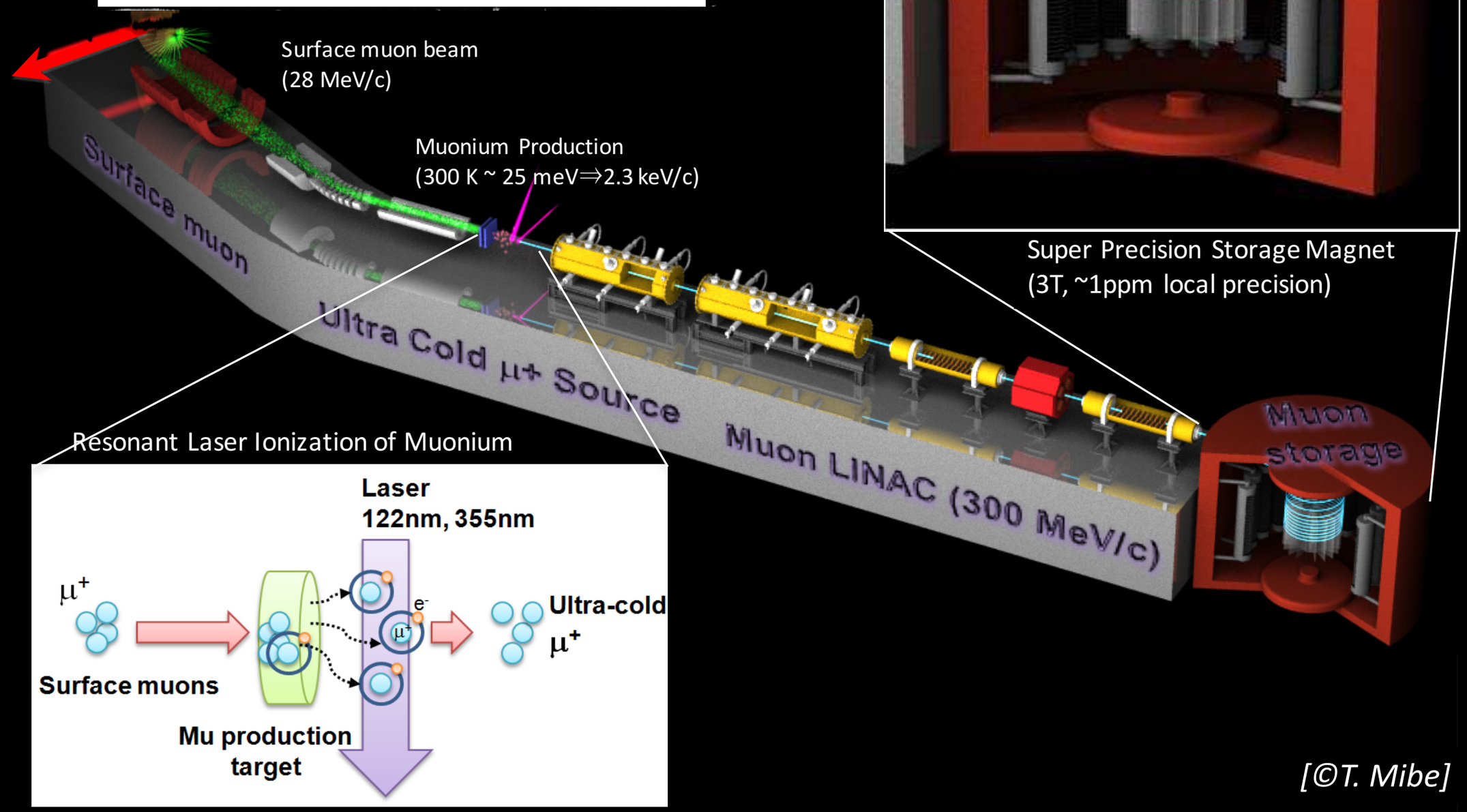

EMMERT
EMMERT

Re-assembly of the storage ring

[©B. Lee Roberts]

E34 experiment at J-PARC

- ultra-cold muon beam
- measure a_μ alongside d_μ
- target precision of **0.1 ppm**

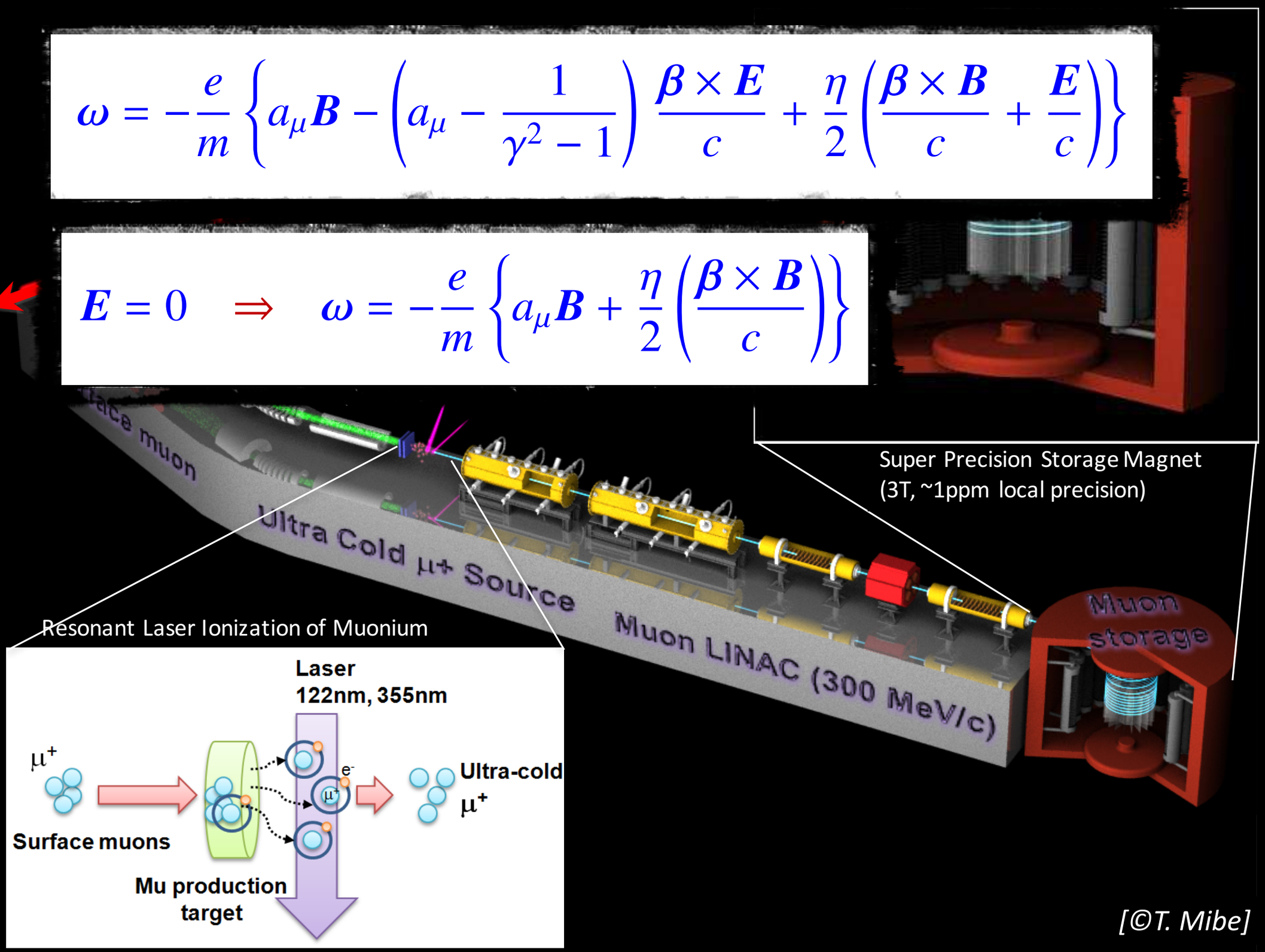


[©T. Mibe]

E34 experiment at J-PARC

$$\boldsymbol{\omega} = -\frac{e}{m}\left\{a_\mu \boldsymbol{B} - \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\frac{\boldsymbol{\beta}\times\boldsymbol{E}}{c} + \frac{\eta}{2}\left(\frac{\boldsymbol{\beta}\times\boldsymbol{B}}{c} + \frac{\boldsymbol{E}}{c}\right)\right\}$$

$$\boldsymbol{E} = 0 \quad \Rightarrow \quad \boldsymbol{\omega} = -\frac{e}{m}\left\{a_\mu \boldsymbol{B} + \frac{\eta}{2}\left(\frac{\boldsymbol{\beta}\times\boldsymbol{B}}{c}\right)\right\}$$

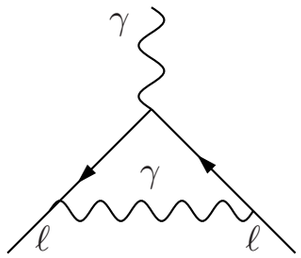


[©T. Mibe]

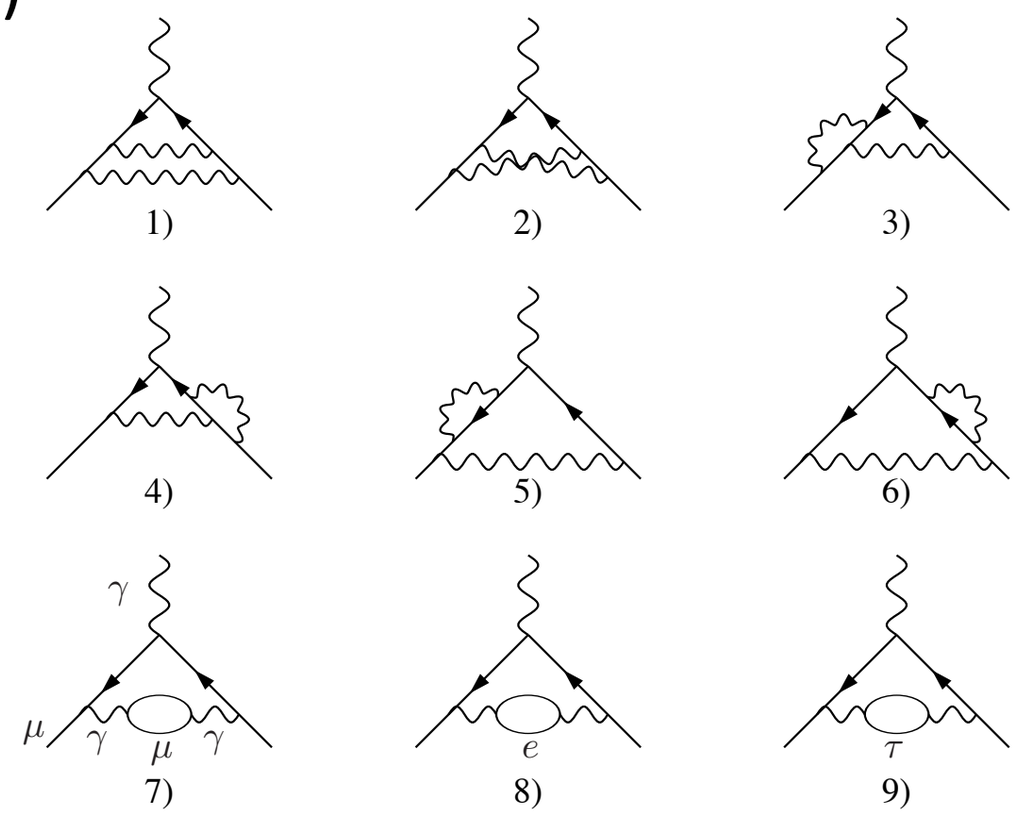
QED contribution to a_μ

* QED contribution has been worked out in perturbation theory to 5th order in α

* Order α (1-loop)



* Order α^2 (2-loop)



QED contribution to a_μ

SM	116 591 776.000	100 %	#diagrams
QED, tot	116 584 718.951	99,9939 %	
2	116 140 973.318	99,6133 %	1
4	413 217.629	0,3544 %	9
6	30 141.902	0,0259 %	72
8	381.008	0,0003 %	891
10	5.094	$4 \cdot 10^{-6}$ %	12672

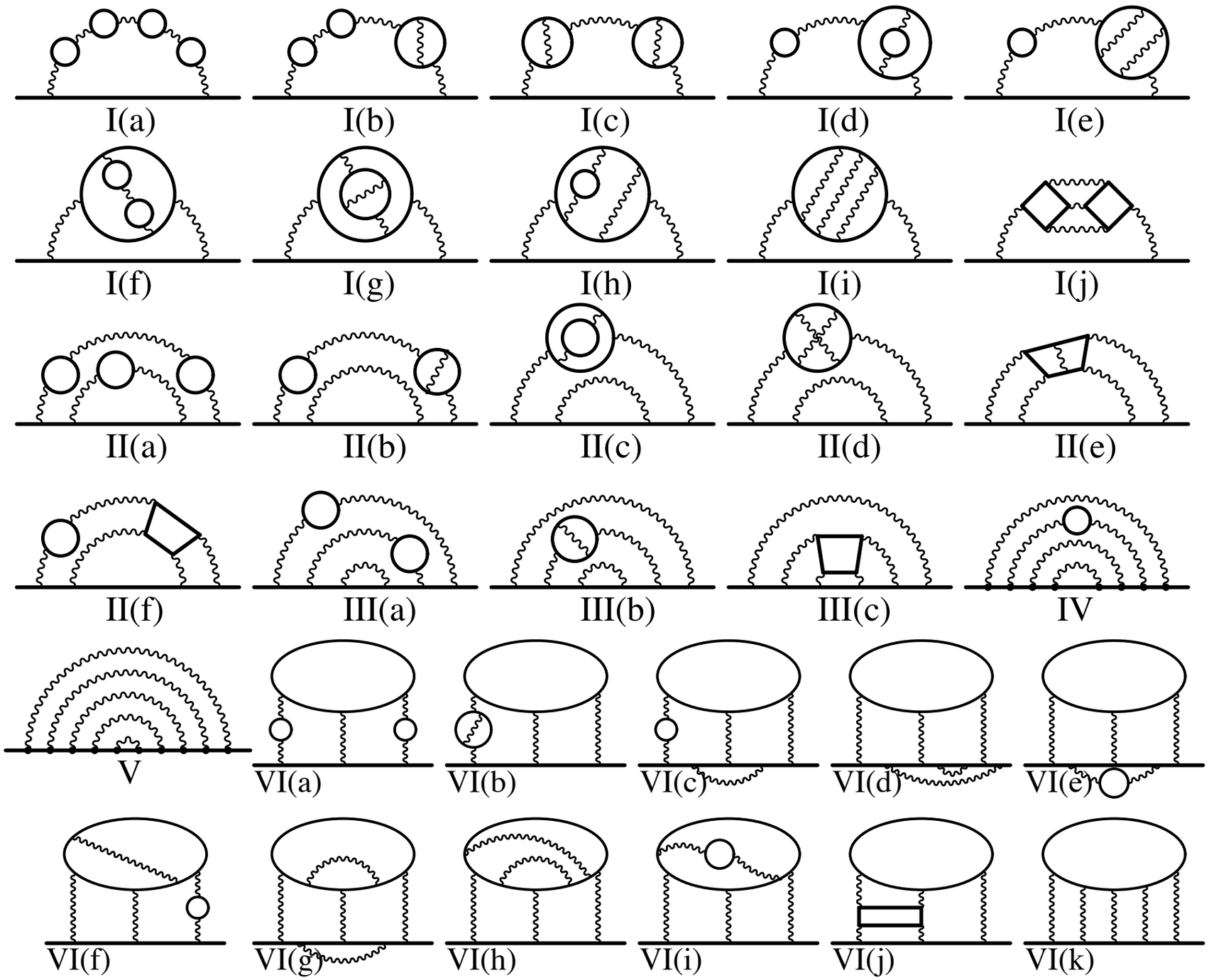
PRL **109,** 111808 (2012) PHYSICAL REVIEW LETTERS week ending 14 SEPTEMBER 2012

Complete Tenth-Order QED Contribution to the Muon $g - 2$

Tatsumi Aoyama,[1,2] Masashi Hayakawa,[3,2] Toichiro Kinoshita,[4,2] and Makiko Nio[2]

[1]*Kobayashi-Maskawa Institute for the Origin of Particles and the Universe (KMI), Nagoya University, Nagoya, 464-8602, Japan*
[2]*Nishina Center, RIKEN, Wako, Japan 351-0198*
[3]*Department of Physics, Nagoya University, Nagoya, Japan 464-8602*
[4]*Laboratory for Elementary Particle Physics, Cornell University, Ithaca, New York 14853, USA*
(Received 24 May 2012; published 13 September 2012)

QED contribution to a_μ



QED contribution to a_μ

Physics Letters B 772 (2017) 232–238

High-precision calculation of the 4-loop contribution to the electron g-2 in QED

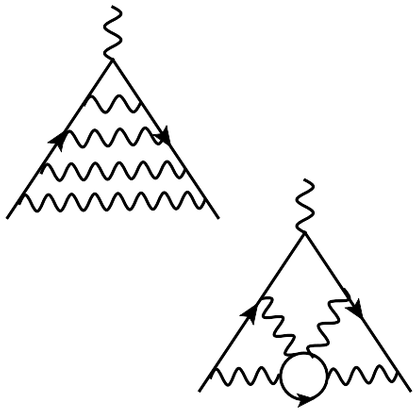
Stefano Laporta

Dipartimento di Fisica, Università di Bologna, Istituto Nazionale Fisica Nucleare, Sezione di Bologna, Via Irnerio 46, I-40126 Bologna, Italy

A B S T R A C T

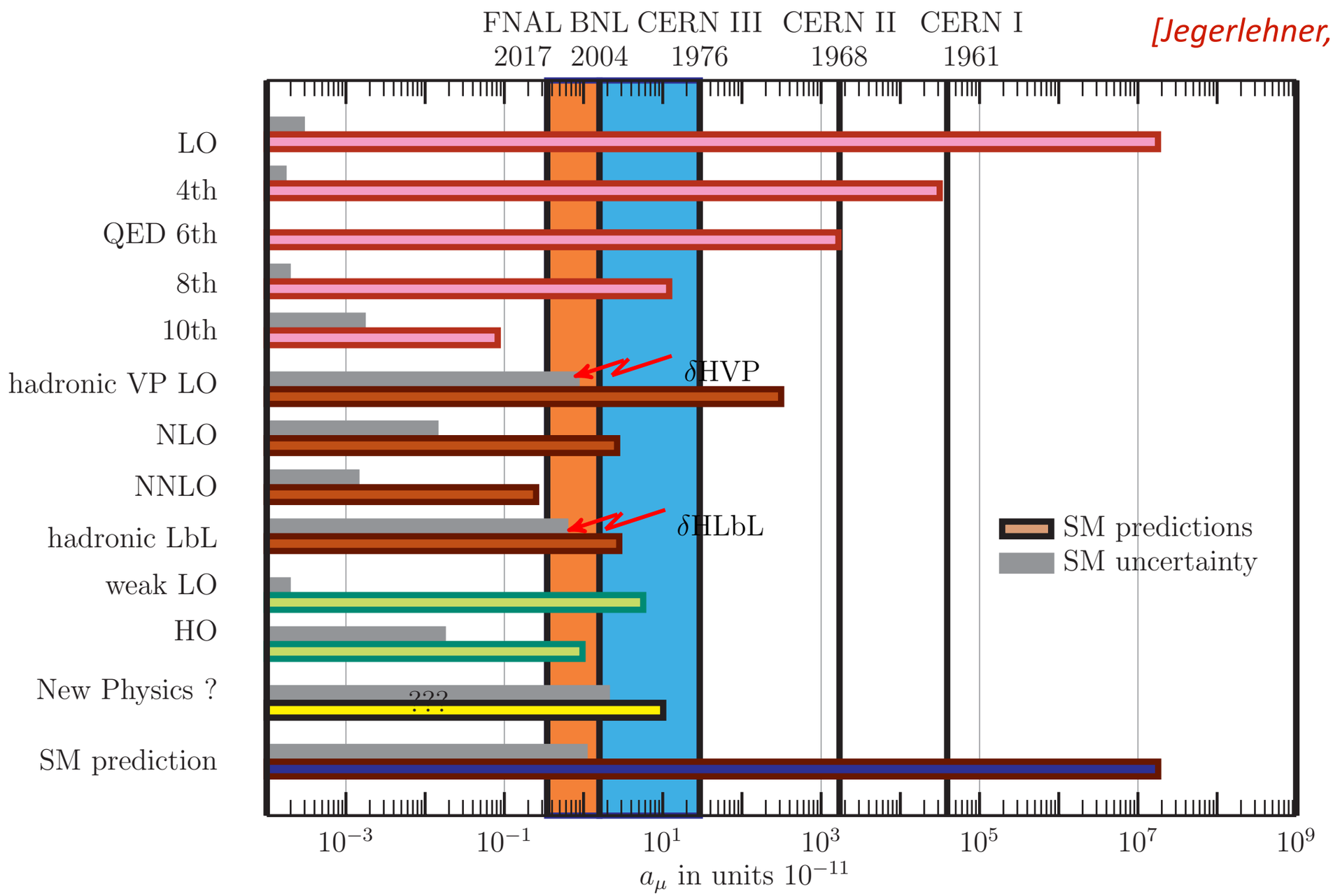
I have evaluated up to 1100 digits of precision the contribution of the 891 4-loop Feynman diagrams contributing to the electron g-2 in QED. The total mass-independent 4-loop contribution is

$$a_e = -1.912245764926445574152647167439830054060873390658725345\ldots\left(\frac{\alpha}{\pi}\right)^4.$$



[Marquard,Smirnov,Smirnov, Steinhauser,Wellmann]	[Laporta'17]	[Aoyama,Hayakawa, [Kinoshita,Nio'12]
-2.161 ± 0.065		-2.1755 ± 0.0020
$-2.176866027739540077443259355895893938670$		
0.077 ± 0.031		0.05596 ± 0.0001
$0.056110899897828364831469274418908842 23$		

Theory confronts experiment

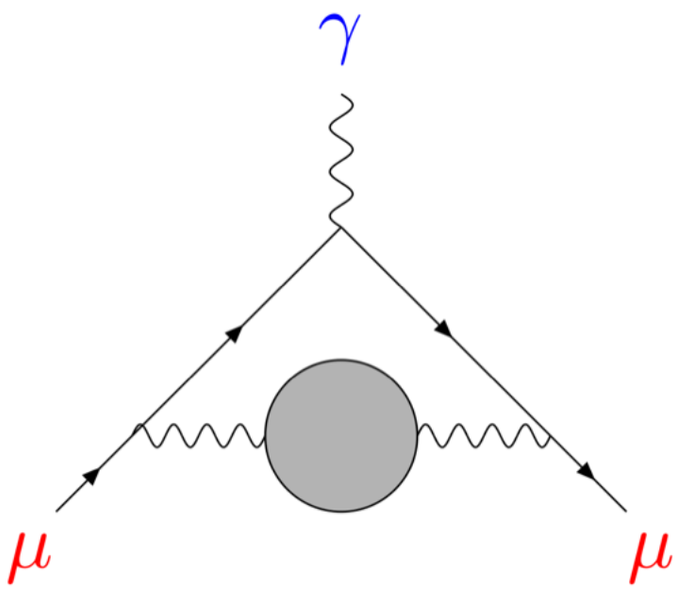


[Jegerlehner, arXiv:1705.00263]

* Experimental sensitivity of E989 exceeds total theory uncertainty by far!

Hadronic contributions to a_μ

Hadronic vacuum polarisation:



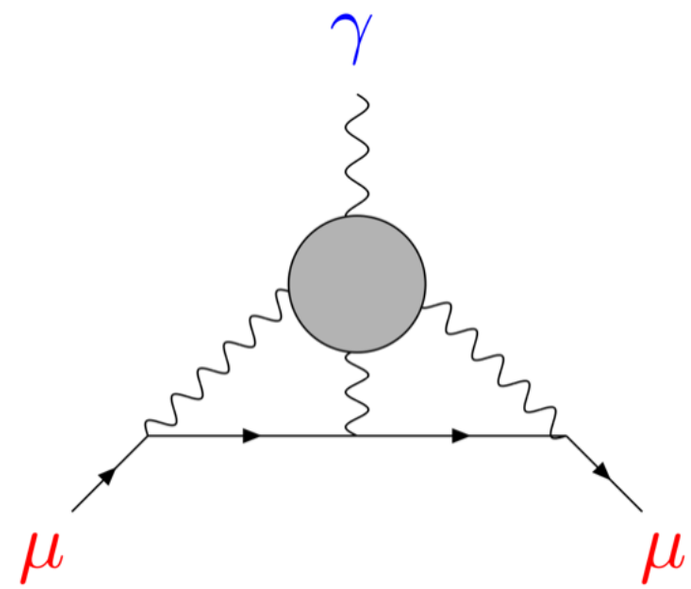
Dispersion theory:

$$a_\mu^{\rm hvp} = (6933 \pm 25) \cdot 10^{-11}$$

(combined e^+e^- data)

[Keshavarzi et al., arXiv:1802.02995]

Hadronic light-by-light scattering:



Model estimates:

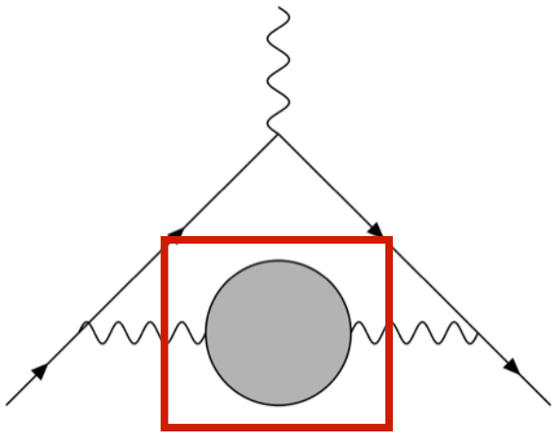
$$a_\mu^{\rm hlbl} = (105 \pm 26) \cdot 10^{-11}$$

"Glasgow consensus"

[Prades, de Rafael, Vainshtein 2009]

Hadronic vacuum polarisation

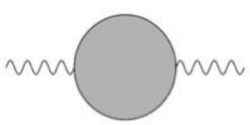
* Hadronic electromagnetic current:



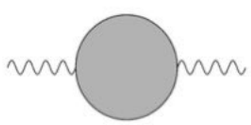
$$J^\mu(x) = \tfrac{2}{3}\bar{u}\gamma^\mu u - \tfrac{1}{3}\bar{d}\gamma^\mu d - \tfrac{1}{3}\bar{s}\gamma^\mu s + \tfrac{2}{3}\bar{c}\gamma^\mu c + \ldots$$

$$(q^\mu q^\nu - q^2 g^{\mu\nu})\,\Pi(q^2) = ie^2 \int d^4x\, \mathrm{e}^{iq\cdot x}\, \langle 0\,|T\, J^\mu(x) J^\nu(0)|\,0\rangle$$

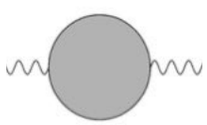
* **Optical theorem:**



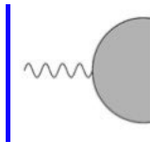
$$= \int \frac{ds}{\pi(s-q^2)}\,\mathrm{Im}$$



$$2\,\mathrm{Im}$$

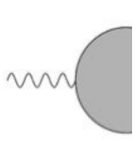


$$= \sum_{\mathrm{had}} \int d\Phi \,\Big|$$



$$\Big|^2$$

$$\Big|$$



$$\Big|^2 \propto \sigma(e^+e^- \to \mathrm{hadrons})$$

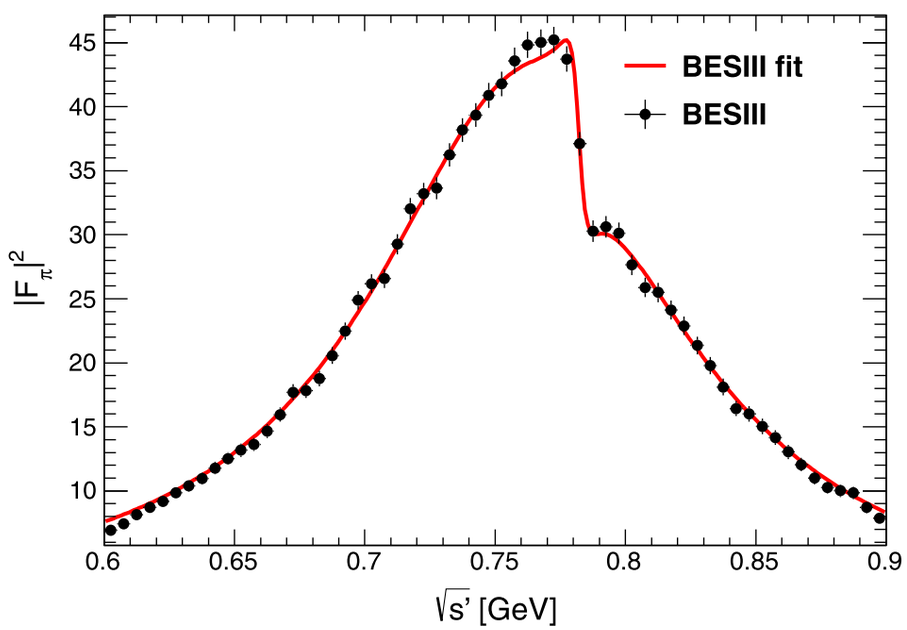
$$a_\mu^{\mathrm{hvp}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} ds\, \frac{R_{\mathrm{had}}(s)\,\hat{K}(s)}{s^2}, \quad R_{\mathrm{had}}(s) = \sigma(e^+e^- \to \mathrm{hadrons}) \Big/ \frac{4\pi\,\alpha(s)}{(3s)}$$

HVP contribution from dispersion relations

* Knowledge of $R_{\rm had}(s)$ required down to pion threshold

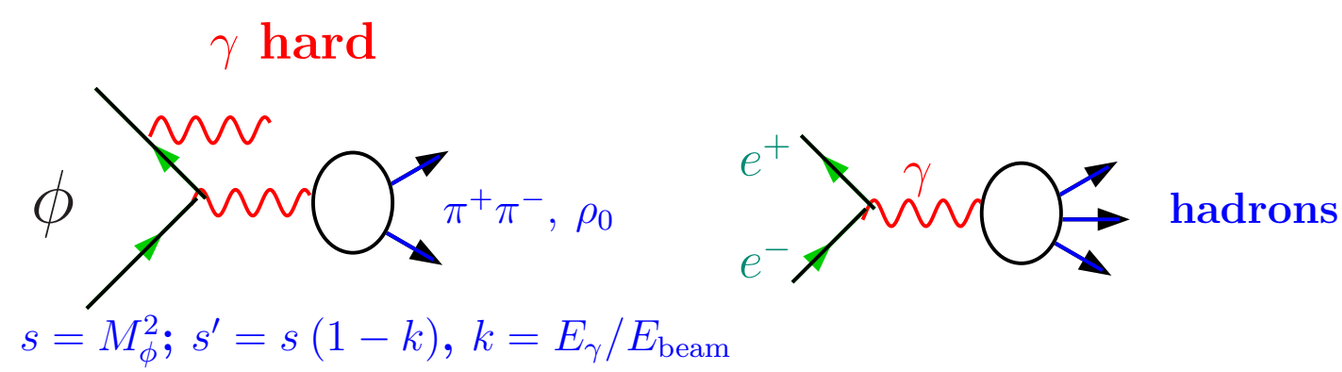
$$a_\mu^{\rm hvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left\{ \int_{m_{\pi^0}^2}^{E_{\rm cut}^2} ds\, \frac{R_{\rm had}^{\rm data}(s)\,\hat{K}(s)}{s^2} + \int_{E_{\rm cut}^2}^{\infty} ds\, \frac{R_{\rm had}^{\rm pQCD}(s)\,\hat{K}(s)}{s^2} \right\}$$

⇒ Use experimental data for cross section ratio $R_{\rm had}(s)$



[BESIII Collaboration, 2016]

Initial state radiation (ISR) vs. beam energy tuning:



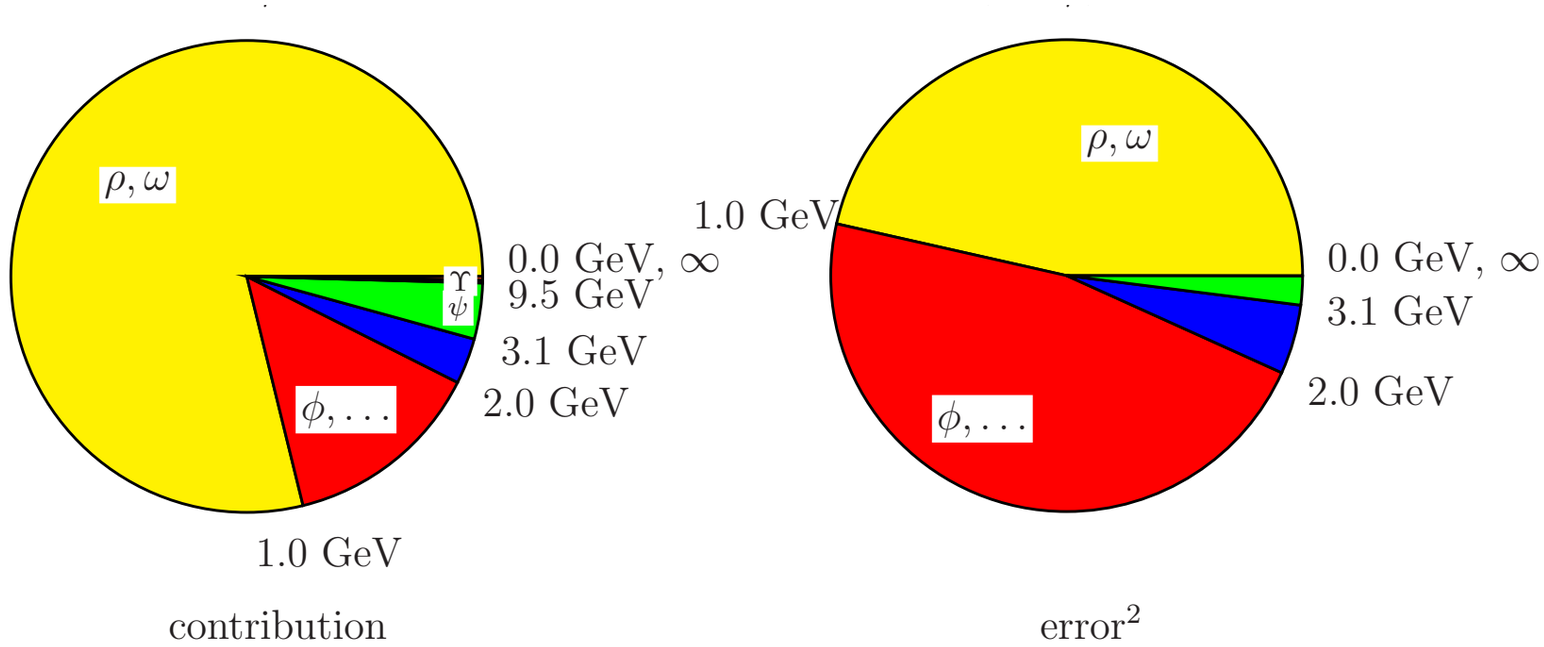
[Jegerlehner, arXiv:1705.00263]

HVP contribution from dispersion relations

* Knowledge of $R_{\rm had}(s)$ required down to pion threshold

$$a_\mu^{\rm hvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left\{ \int_{m_{\pi^0}^2}^{E_{\rm cut}^2} ds\, \frac{R_{\rm had}^{\rm data}(s)\,\hat{K}(s)}{s^2} + \int_{E_{\rm cut}^2}^{\infty} ds\, \frac{R_{\rm had}^{\rm pQCD}(s)\,\hat{K}(s)}{s^2} \right\}$$

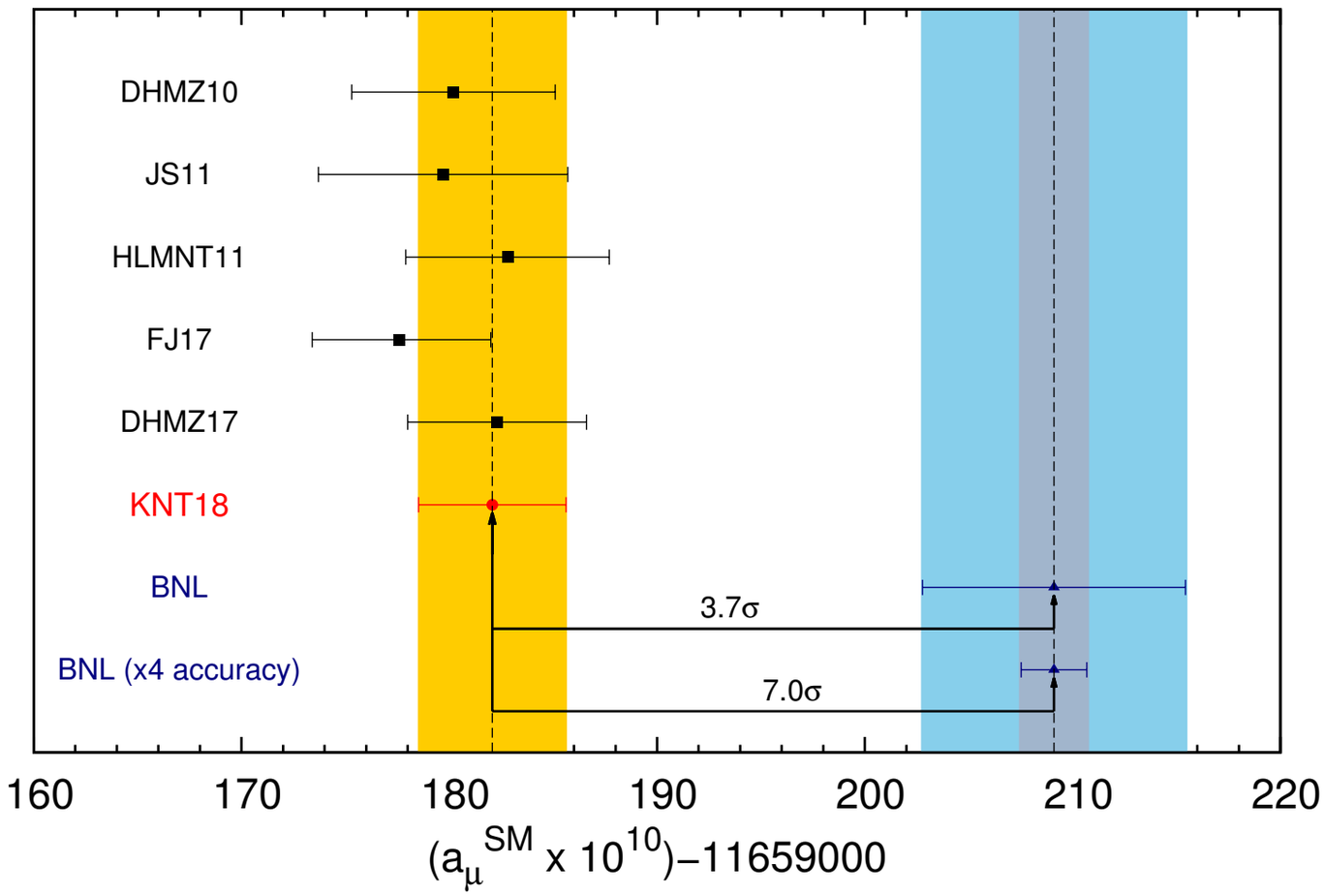
⇒ Use experimental data for hadronic cross section $R_{\rm had}(s)$



* Low-energy region dominates

[Jegerlehner, arXiv:1705.00263]

HVP contribution from dispersion relations

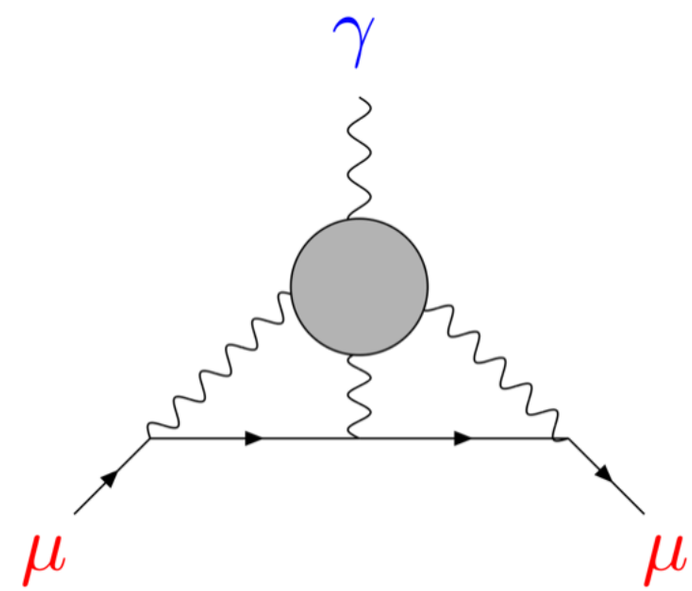


[Keshavarzi et al., arXiv:1802.02995]

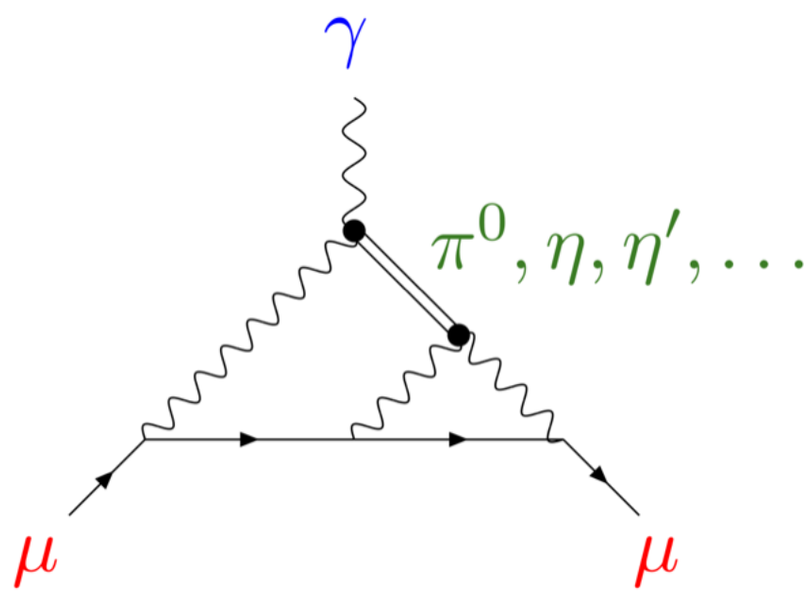
* Tension of ≈ 3.7 standard deviations between SM and experiment
* Overall precision of HVP estimate: $\approx 0.4\%$
* Theory estimate subject to experimental uncertainties
* Disagreement over individual hadronic channels

Hadronic Light-by-Light scattering

* No simple dispersive framework



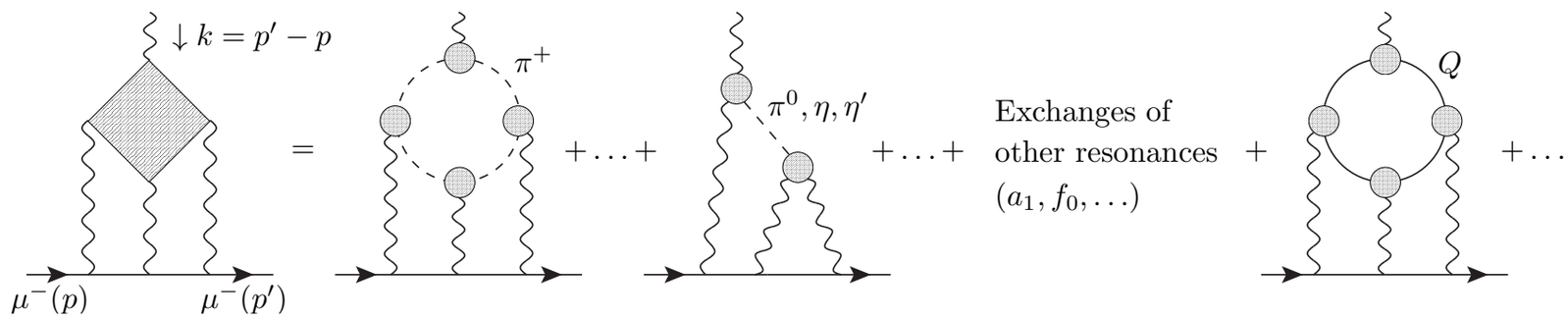
* Identify dominant sub-processes, e.g.



* Individual contributions estimated using model calculations
* Dispersive formalism set up for various subprocesses
  [Colangelo et al., Pauk & Vanderhaeghen, 2014 ff]
* Other approaches: functional methods, lattice QCD

Hadronic Light-by-Light scattering

* Dominant hadronic contributions to a_μ^{hlbl} *[Nyffeler, arXiv:1710.09742]*



Contribution:	pion-loop (dressed)	pseudoscalar exchanges	quark-loop (dressed)
Chiral counting:	p^4	p^6	
N_c counting:	1	N_c	

"Glasgow consensus"

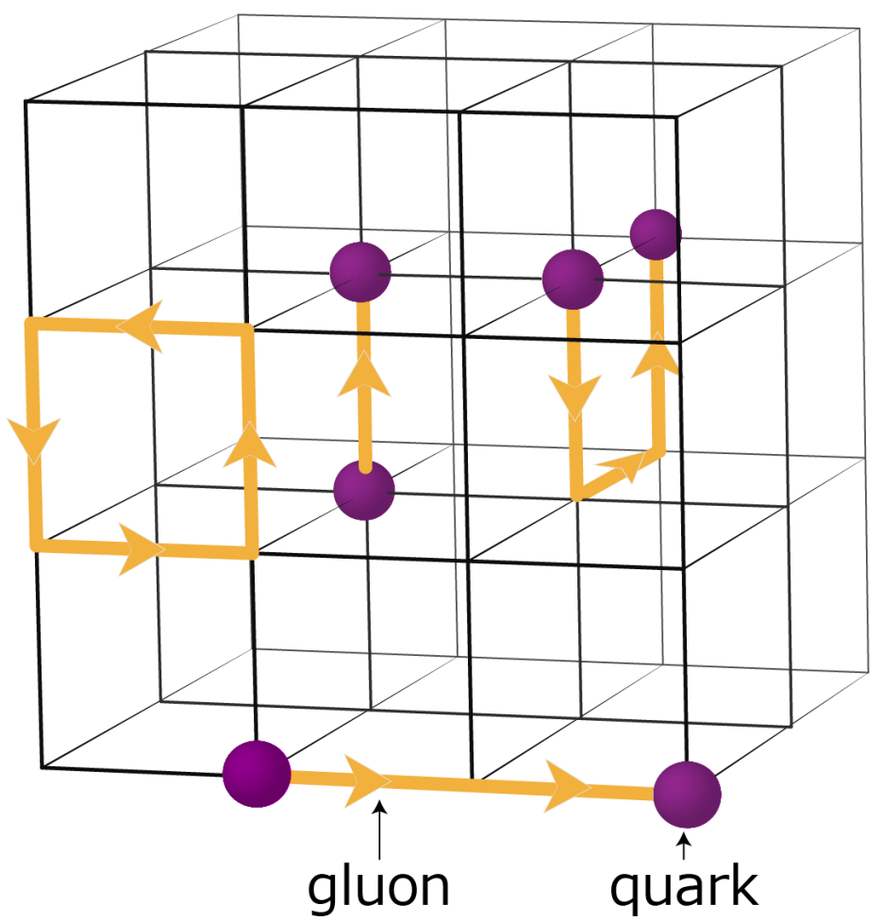
Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

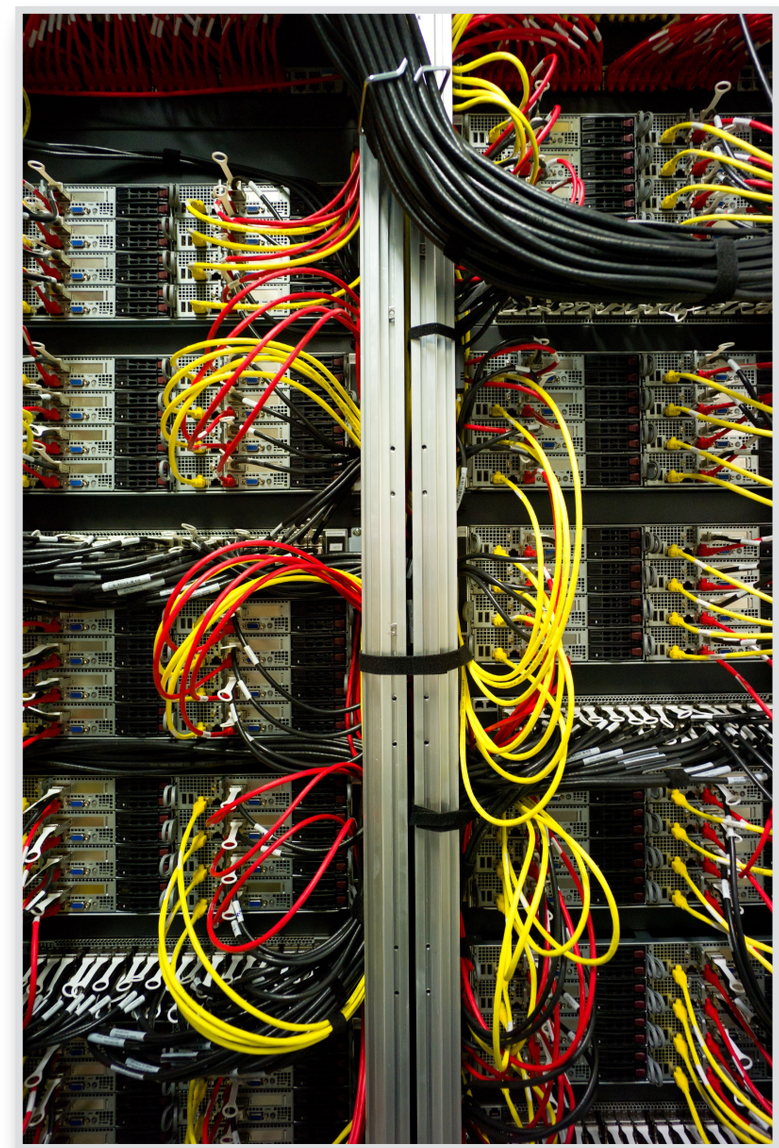
BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

The muon g – 2 in lattice QCD

Hadronic contributions in precision observables

* Accuracy of Standard Model tests limited by hadronic contributions
* Employ “ab initio” approach: Lattice QCD





“Clover” @ Mainz

Outline

Basic concepts of Lattice QCD

Hadronic vacuum polarisation from lattice QCD

Hadronic light-by-light scattering

Summary & Outlook

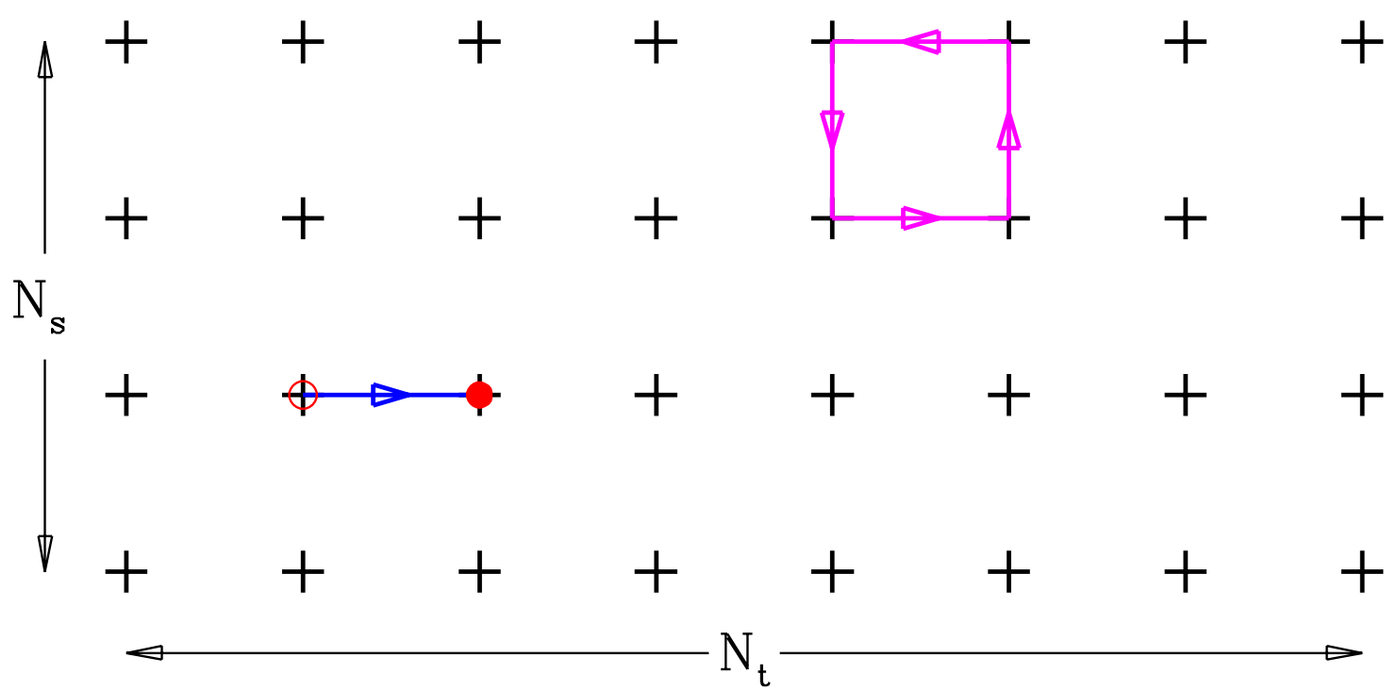
Beyond Perturbation Theory: Lattice QCD

"Lattice QCD is the non-perturbative treatment of the gauge theory of strong interactions, based on ***regularised, Euclidean functional integrals****."*

Minkowski space-time, continuum $\longrightarrow$ Euclidean space-time, discretised

Lattice spacing: $a, \quad x_\mu = n_\mu a, \quad a^{-1} = \Lambda_{\rm UV}$

Finite volume: $L^3 \cdot T, \quad N_s = L/a, \quad N_t = T/a$



(anti)quarks: $\bar{\psi}(x), \psi(x)$

$\longrightarrow$ lattice sites

Gluons: $U_\mu(x) \equiv \mathrm{e}^{aA_\mu(x)} \in \mathrm{SU}(3)$

$\longrightarrow$ links

Field tensor:

$U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu(x+\hat{\nu})^\dagger U_\nu(x)^\dagger$

$\longrightarrow$ "plaquettes"

Beyond Perturbation Theory: Lattice QCD

* Formulate a lattice action in terms of quark fields, link variables and plaquettes:

$$S^{\text{lat}}[U,\bar{\psi},\psi] = S_{\text{G}}^{\text{lat}}[U] + S_{\text{F}}^{\text{lat}}[U,\bar{\psi},\psi]$$

* Reproduce Euclidean continuum action as $a \to 0$

$$S = -\frac{1}{2g_0^2}\int d^4x\,\text{Tr}\,F_{\mu\nu}(x)^2 + \int d^4x \sum_{f=u,d,s,\ldots} \bar{\psi}_f(x)\left(\not{D} + m_f\right)\psi(x)$$

* Discretisation is not unique! Widely used fermionic actions:

Staggered	Wilson	Domain wall	Overlap
HISQ	O(a) improved Wilson	Fixed point	
	twisted-mass Wilson	Chirally improved	

Beyond Perturbation Theory: Lattice QCD

Lattice formulation …

… preserves gauge invariance

… defines observables without reference to perturbation theory

… allows for stochastic evaluation of observables

$$\langle \Omega \rangle = \frac{1}{Z} \int D[U]\, D[\bar{\psi}, \psi]\; \Omega\; \mathrm{e}^{-S_{\mathrm{G}}[U] - S_{\mathrm{F}}[U, \bar{\psi}, \psi]}$$

$$= \frac{1}{Z} \int D[U]\; \Omega \prod_{f=u,d,s,\ldots} \det\left(\not{D} + m_f\right) \mathrm{e}^{-S_{\mathrm{G}}[U]}$$

$$= \frac{1}{Z} \int \prod_{x,\mu} dU_\mu(x)\; \Omega \prod_{f=u,d,s,\ldots} \det\left(\not{D}^{\mathrm{lat}} + m_f\right) \mathrm{e}^{-S_{\mathrm{G}}^{\mathrm{lat}}[U]}$$

Monte Carlo Simulation

1. Generate set of N_c configurations of gauge fields $\{U_\mu(x)\}_i,\ i = 1,\ldots N_c$ with probability distribution

$$W = \prod_{f=u,d,s,\ldots} \det\left(\not{D}^{\rm lat} + m_f\right) \mathrm{e}^{-S_{\rm G}^{\rm lat}[U]}$$

Define an algorithm based on a Markov process

Generate sequence: $\{U\}_1 \longrightarrow \{U\}_2 \longrightarrow \ldots \longrightarrow \{U\}_{N_c}$

Transition probability given by W "Importance sampling"

Strong growth of numerical cost near physical m_u, m_d

Pion mass, i.e. lightest mass in pseudoscalar channel:

$$\approx 500\,\mathrm{MeV} \longrightarrow \approx 130\,\mathrm{MeV}$$

(2001) ($\gtrsim$ 2015)

Monte Carlo Simulation

1. Generate set of N_c configurations of gauge fields $\{U_\mu(x)\}_i,\ i = 1,\ldots N_c$ with probability distribution

$$W = \prod_{f=u,d,s,\ldots} \det\left(\not{D}^{\rm lat} + m_f\right) \mathrm{e}^{-S_{\rm G}^{\rm lat}[U]}$$

   Define an algorithm based on a Markov process

   Generate sequence: $\{U\}_1 \longrightarrow \{U\}_2 \longrightarrow \ldots \longrightarrow \{U\}_{N_c}$

   Transition probability given by W "Importance sampling"

2. Evaluate observable on each configuration:

$$\overline{\Omega} = \frac{1}{N_c}\sum_{i=1}^{N_c} \Omega_i, \quad \langle\Omega\rangle = \lim_{N_c\to\infty} \overline{\Omega},$$

   statistical error $\propto 1/\sqrt{N_c}$

3. Repeat 1. and 2. for different lattice spacings, quark masses and volumes

Correlation functions

* Spectral information contained in correlation functions

$$\sum_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{x})} \left\langle O_{\mathrm{had}}(y)\, O^{\dagger}_{\mathrm{had}}(x) \right\rangle = \sum_{n} w_n(\boldsymbol{p})\, \mathrm{e}^{-E_n(\boldsymbol{p})(y_0-x_0)}$$

$$\xrightarrow{(y_0-x_0)\to\infty} w_1(\boldsymbol{p})\, \mathrm{e}^{-E_1(\boldsymbol{p})(y_0-x_0)}$$

* Ground state dominates at large distances

* $O_{\mathrm{had}}(x)$: interpolating operator

Pion: $O_\pi = \bar{u}\gamma_5 d,\ \bar{u}\gamma_0\gamma_5 d$

ρ-meson: $O_\rho = \bar{u}\gamma_k d$

Nucleon: $O_N = \epsilon_{abc}\left(u^a\, C\gamma_5\, d^b\right) u^c$

$\longrightarrow$ projects on all states with the same quantum numbers

Correlation functions

* Spectral information contained in correlation functions

$$\sum_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{x})} \left\langle O_{\mathrm{had}}(y)\, O^{\dagger}_{\mathrm{had}}(x) \right\rangle = \sum_{n} w_n(\boldsymbol{p})\, \mathrm{e}^{-E_n(\boldsymbol{p})(y_0-x_0)}$$

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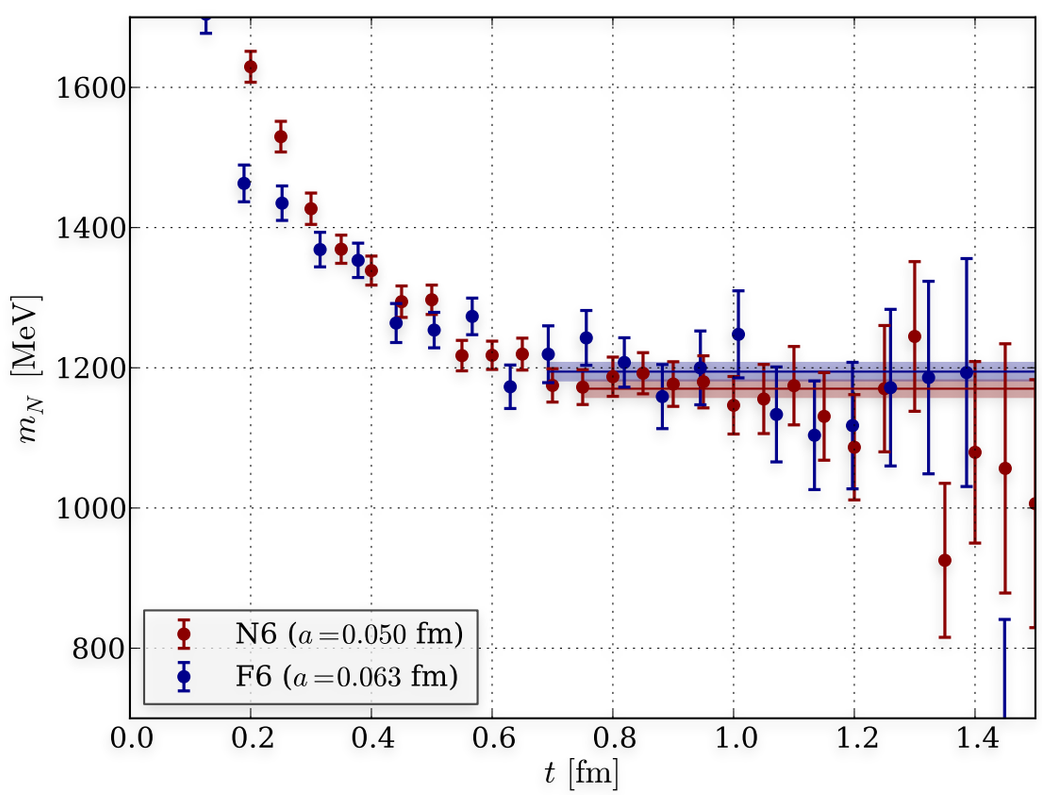


Nucleon "effective mass"

Strong growth of statistical noise as $t \to \infty$

[Capitani et al., arXiv:1504.04628]

Systematic effects

* Lattice artefacts: $$\left\langle \frac{m_{\rm N}}{f_\pi} \right\rangle^{\rm lat} = \left\langle \frac{m_{\rm N}}{f_\pi} \right\rangle^{\rm cont} + O(a^p), \quad p \geq 1$$

  $\longrightarrow$ extrapolate to continuum limit from $a \approx 0.05 - 0.12\,\mathrm{fm}$

* Finite volume effects
  - Empirically: $m_\pi L \geq 4$ sufficient for many purposes
  - Provide information on scattering phase shifts

* Unphysical quark masses
  - Chiral extrapolation to physical values of m_u, m_d

* Inefficient sampling of SU(3) group manifold
  - Simulations become trapped in topological sectors as $a \to 0$
  - Use open boundary conditions in time direction *[Lüscher & Schaefer, 2012]*

Global analyses of Lattice QCD results

FLAG Report:

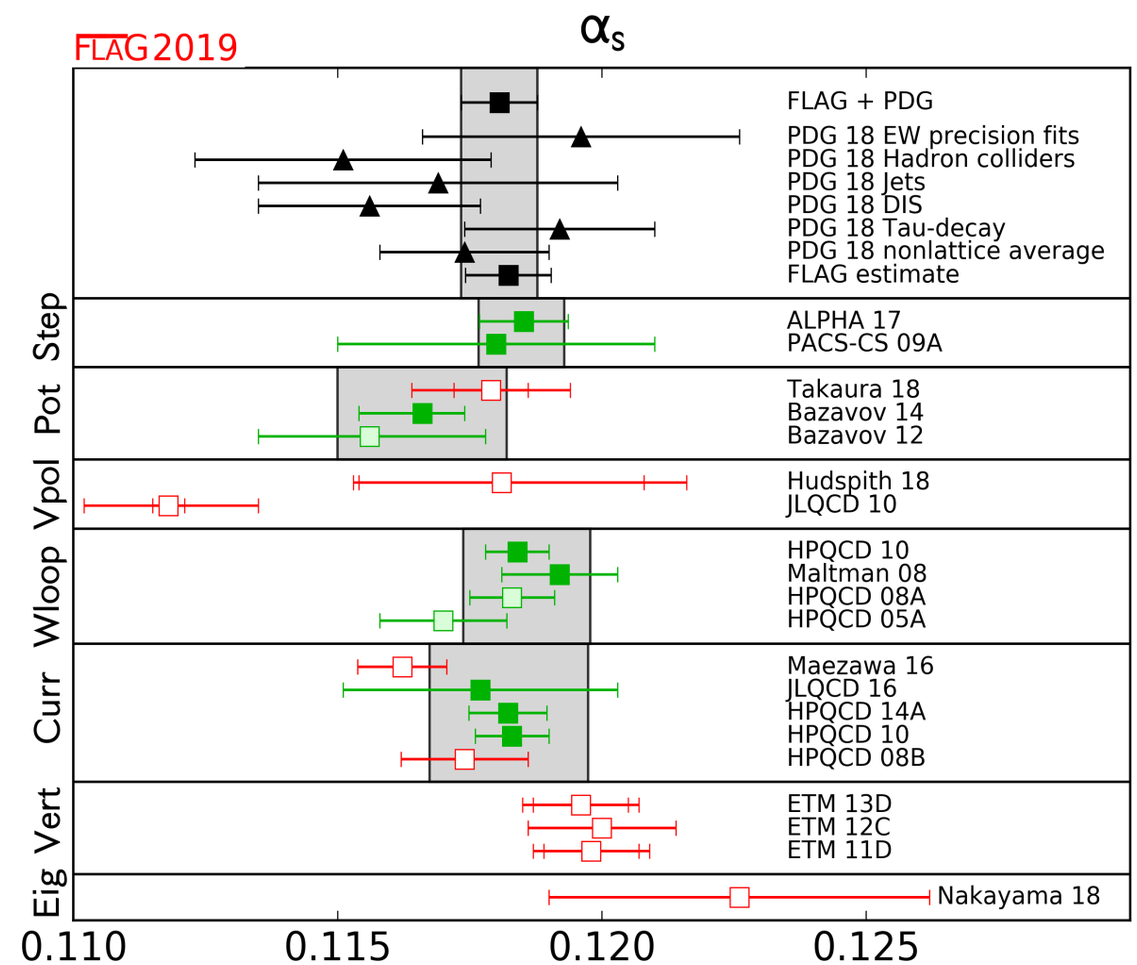
* Performs PDG-style global analyses and averages
  - SM parameters (quark masses, α_s)
  - Flavour physics (K, D, B mesons)
  - Low-energy constants
  - Nucleon matrix elements

* Example: strong coupling

$$\alpha^{(5)}_{\overline{\rm MS}}(M_Z^2) = \begin{cases} 0.11823(81) & \text{FLAG 2019} \\ 0.1174(16) & \text{PDG 2018} \end{cases}$$

* Combination yields

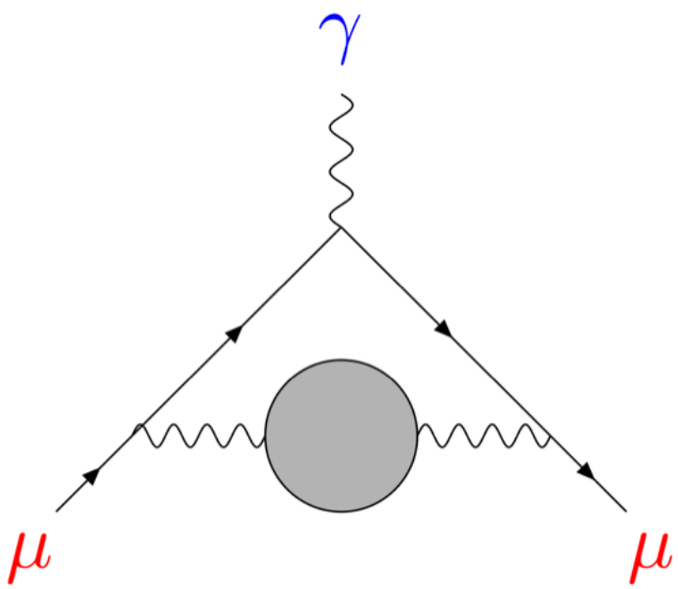
$$\alpha^{(5)}_{\overline{\rm MS}}(M_Z^2) = 0.11806(72)$$



[S. Aoki et al., arXiv:1902.08191]

Reminder: hadronic contributions to a_μ

Hadronic vacuum polarisation:



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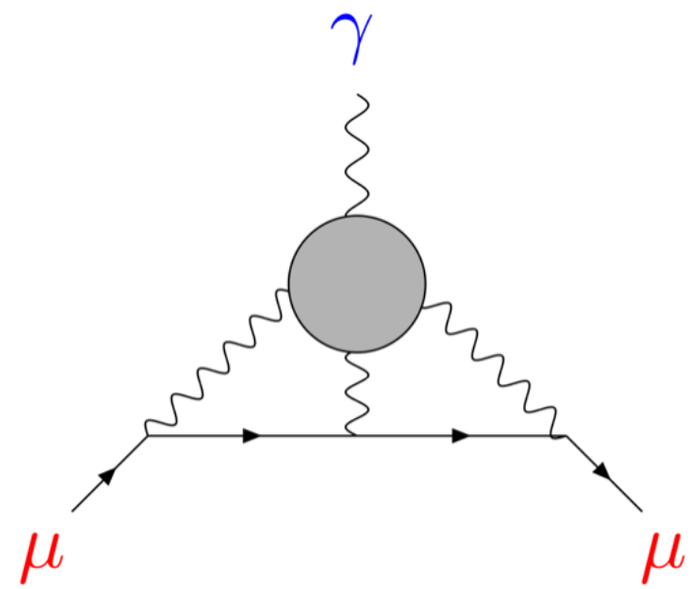

Model estimates:

$$a_\mu^{\rm hlbl} = (105 \pm 26) \cdot 10^{-11}$$

"Glasgow consensus"

[Prades, de Rafael, Vainshtein 2009]

The muon g – 2 in lattice QCD

Motivation for first-principles approach:

* No reliance on experimental data
  — except for simple hadronic quantities to fix bare parameters

* No model dependence
  — except for chiral extrapolation and constraining the IR regime

* Can lattice QCD deliver estimates with sufficient accuracy in the coming years?

$$\delta a_\mu^{\rm hvp}/a_\mu^{\rm hvp} < 0.5\%, \qquad \delta a_\mu^{\rm hlbl}/a_\mu^{\rm hlbl} \lesssim 10\%$$

The muon g – 2 in lattice QCD

ELSEVIER

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Review

Lattice QCD and the anomalous magnetic moment of the muon

Harvey B. Meyer, Hartmut Wittig

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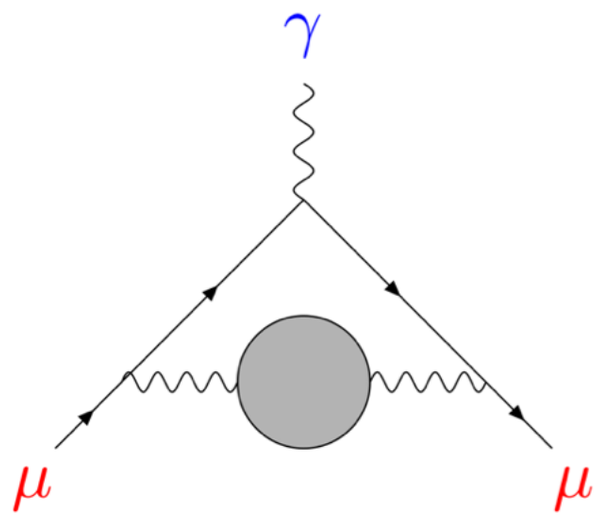
arXiv:1807.09370

The Mainz $(g-2)_\mu$ Lattice QCD project

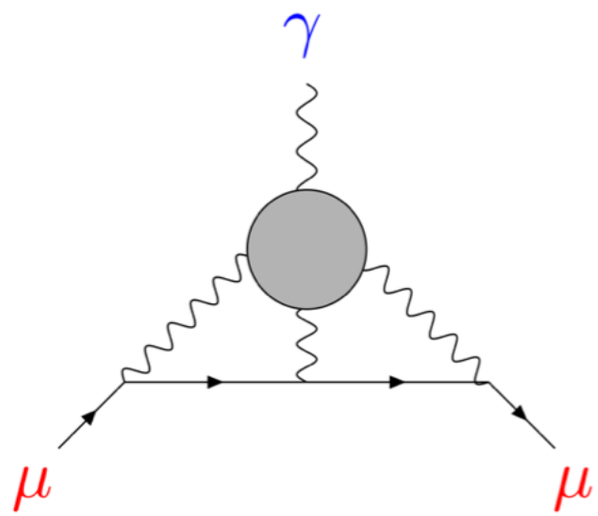
Collaborators:

M. Cè, A. Gérardin, O. Gryniuk, G. von Hippel, H.B. Meyer, K. Miura, A. Nyffeler, K. Ottnad, V. Pascalutsa, A. Risch, T. San José Perez, J. Wilhelm, HW

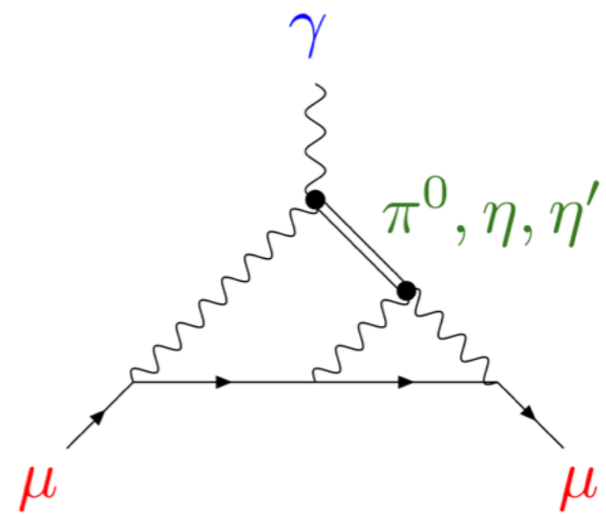
N. Asmussen, J. Green, G. Herdoíza, B. Hörz



- Direct determinations of LO $a_\mu^{\rm hvp}$
- Running of α and $\sin^2\theta_{\rm W}$



- Exact QED kernel
- Forward scattering amplitude



- Transition form factor for $\pi^0 \to \gamma^*\gamma^*$

Lattice QCD approach to HVP

* Convolution integral over Euclidean momenta: *[Lautrup & de Rafael; Blum]*

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\, f(Q^2)\hat{\Pi}(Q^2), \quad \hat{\Pi}(Q^2) = 4\pi^2\left(\Pi(Q^2) - \Pi(0)\right)$$

* Vacuum polarisation tensor:

$$\Pi_{\mu\nu}(Q) = i\int d^4x\, {\rm e}^{iQ\cdot(x-y)}\left\langle J_\mu(x)J_\nu(y)\right\rangle \equiv (Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)\Pi(Q^2)$$

* Electromagnetic current:

$$J_\mu = \tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}\gamma_\mu d - \tfrac{1}{3}\bar{s}\gamma_\mu s + \ldots$$

* Weight function $f(Q^2)$ strongly peaked near muon mass

Lattice QCD approach to HVP

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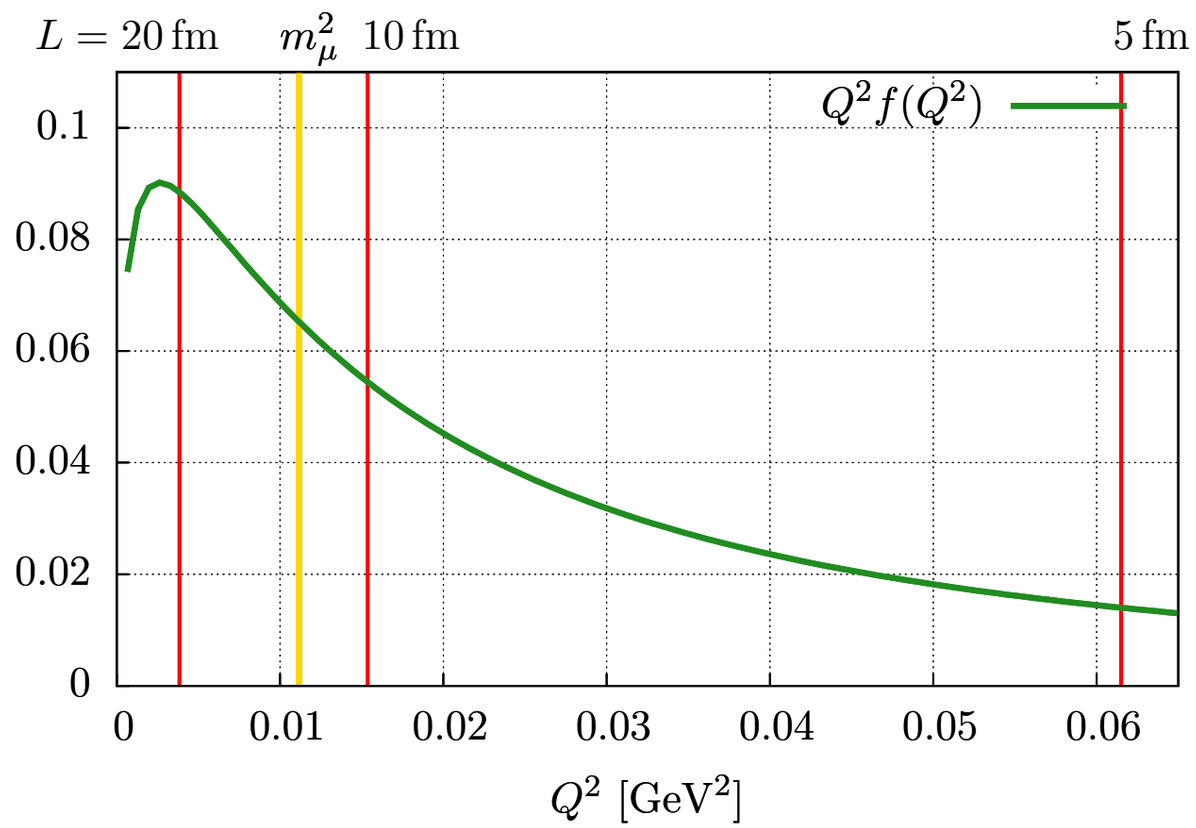


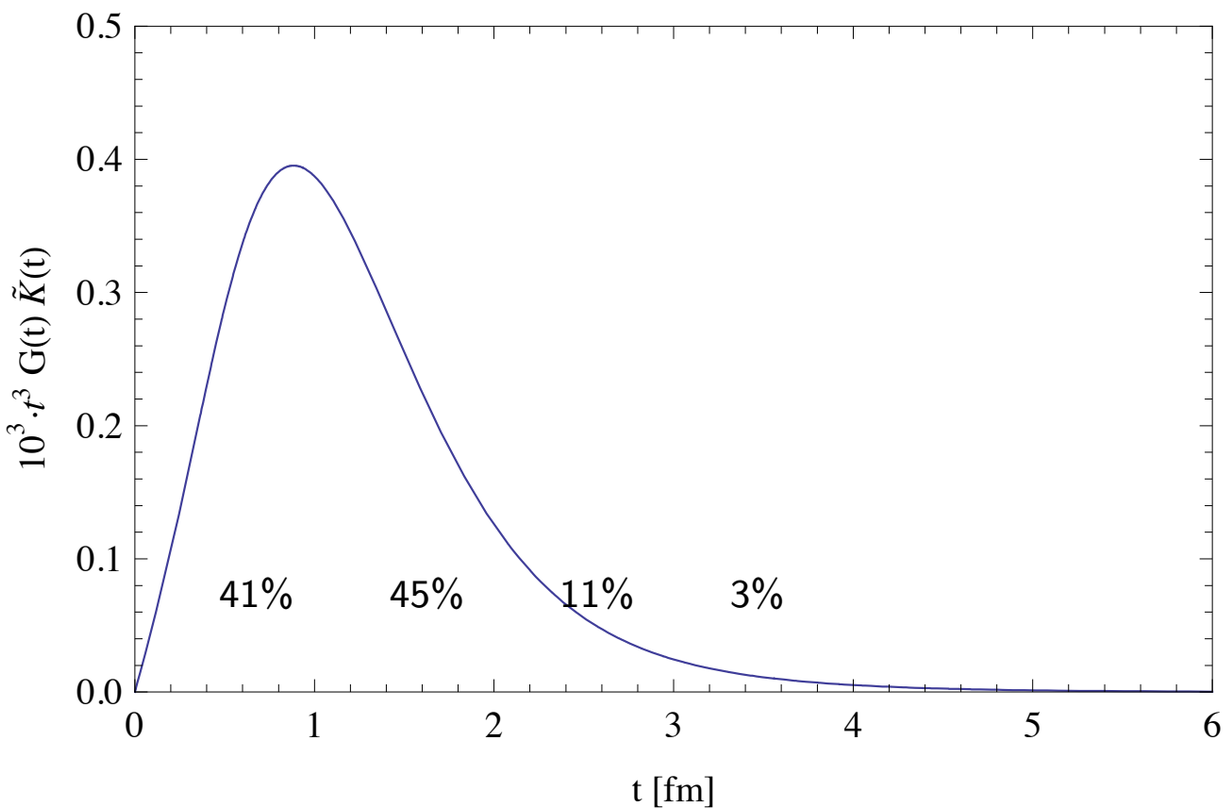
$$Q^2_{\rm min} = (2\pi/L)^2, \qquad m_\pi^{\rm phys} L \approx 4 \quad \Rightarrow \quad L \approx 6\,\mathrm{fm}$$

Lattice QCD approach to HVP

* **Time-momentum representation:** *[Bernecker & Meyer]*

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{K}(x_0)\, G(x_0), \quad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0)\rangle$$

$$\tilde{K}(x_0) = 4\pi^2 \int_0^\infty dQ^2\, f(Q^2) \left[x_0^2 - \frac{4}{Q^2}\sin^2\left(\tfrac{1}{2}Qx_0\right)\right]$$



- Significant contribution from tail of $G(x_0)$
- Exponentially increasing noise-to-signal ratio:

$$R_{\rm NS} \propto \exp\{(m_{\rm V} - m_\pi)x_0\}$$

Lattice QCD approach to HVP

* **Time-momentum representation:** *[Bernecker & Meyer]*

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{K}(x_0)\, G(x_0), \quad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0)\rangle$$

$$\tilde{K}(x_0) = 4\pi^2 \int_0^\infty dQ^2\, f(Q^2) \left[x_0^2 - \frac{4}{Q^2} \sin^2\left(\tfrac{1}{2} Q x_0\right)\right]$$

* Control long-distance behaviour of $G(x_0)$ — large statistical noise

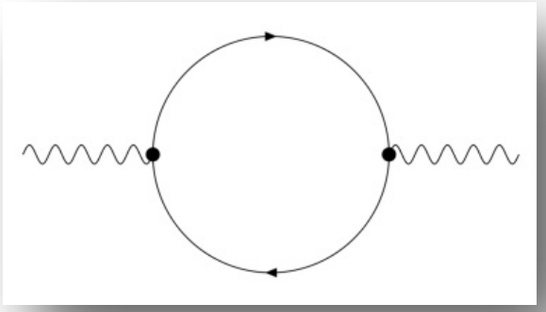
$$G(x_0) = \begin{cases} G(x_0)_{\rm data}, & x_0 \le x_{0,\rm cut} \\ G(x_0)_{\rm ext}, & x_0 > x_{0,\rm cut} \end{cases}$$

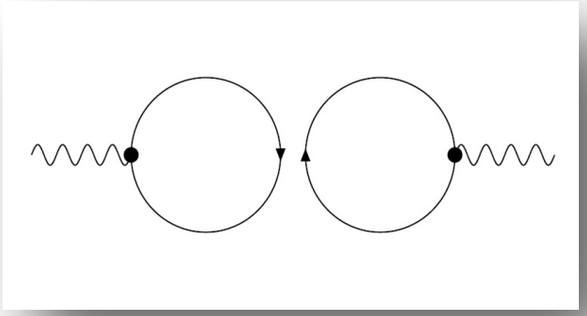
* $G(x_0)$ dominated by two-pion state for $x_0 \to \infty$

Lattice QCD approach to HVP

Challenges:

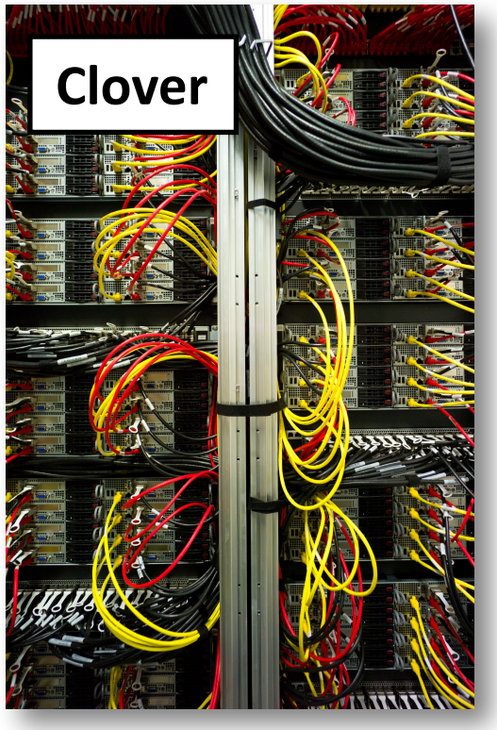
* Statistical accuracy at the sub-percent level required
* Control infrared regime of vector correlator: $G(x_0)$ at large x_0
* Perform comprehensive study of finite-volume effects
* Include quark-disconnected diagrams





* Include isospin breaking: $m_u \neq m_d$, QED corrections

Simulations and Machines



Gauge ensembles

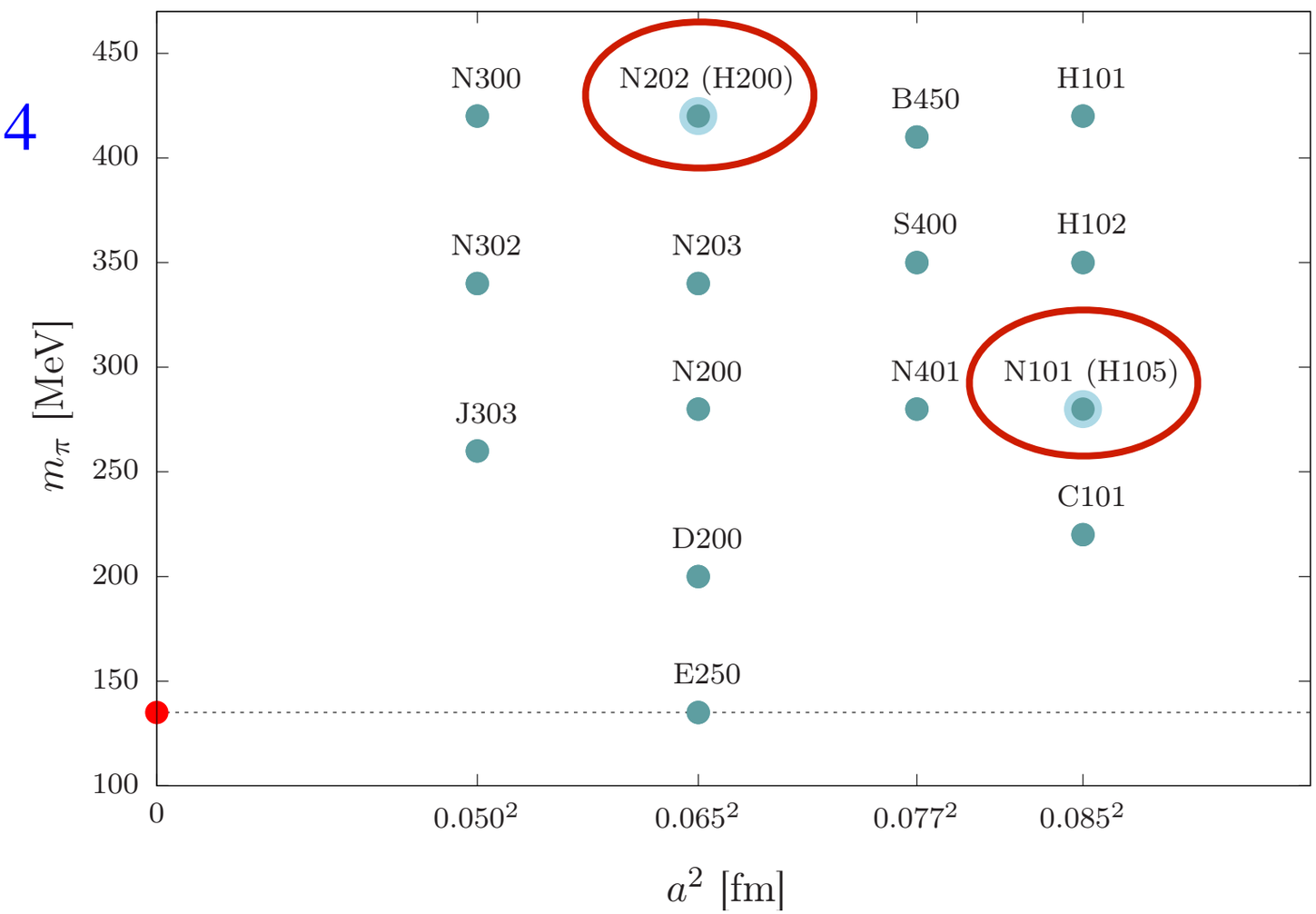
* $N_f = 2 + 1$ flavours of $O(a)$ improved Wilson quarks

* Four values of the lattice spacing: a = 0.085, 0.077, 0.065, 0.050 fm

* Pion masses and volumes:

  $m_\pi^{\min} \approx 135\,\mathrm{MeV}, \quad m_\pi L > 4$

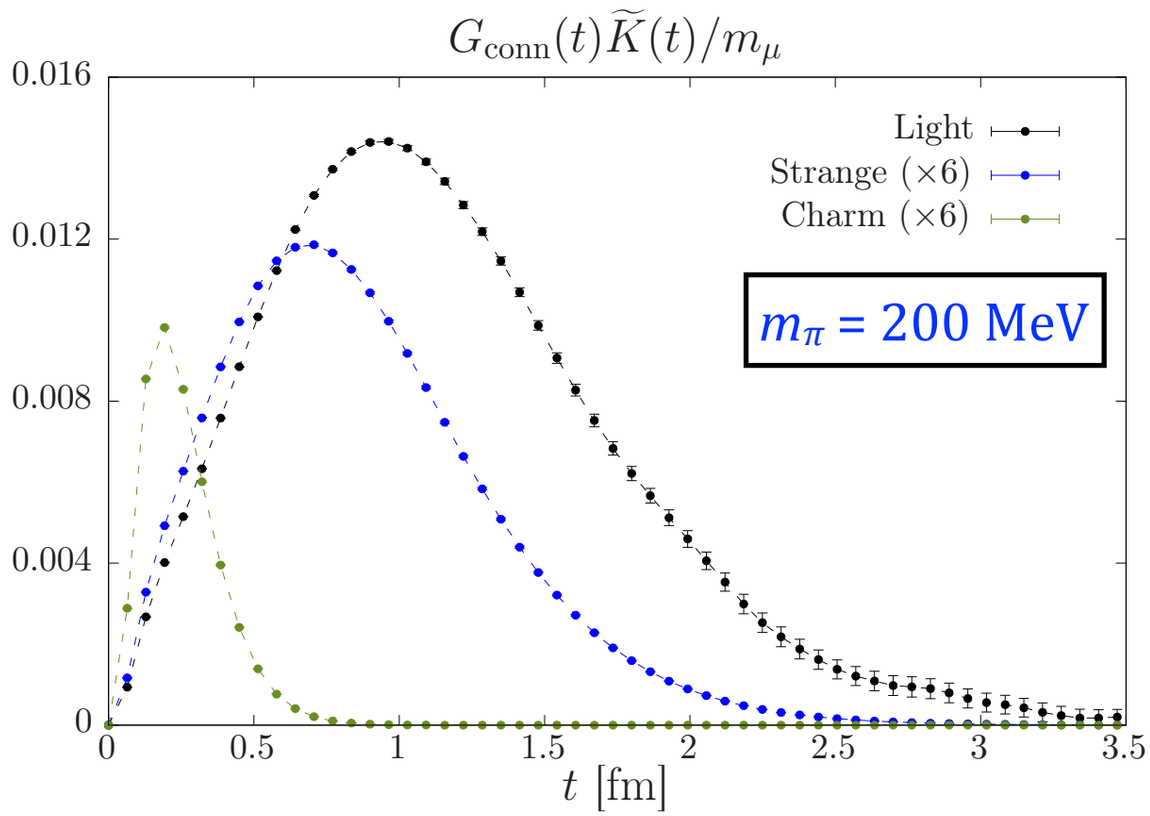
* Check of finite-volume effects

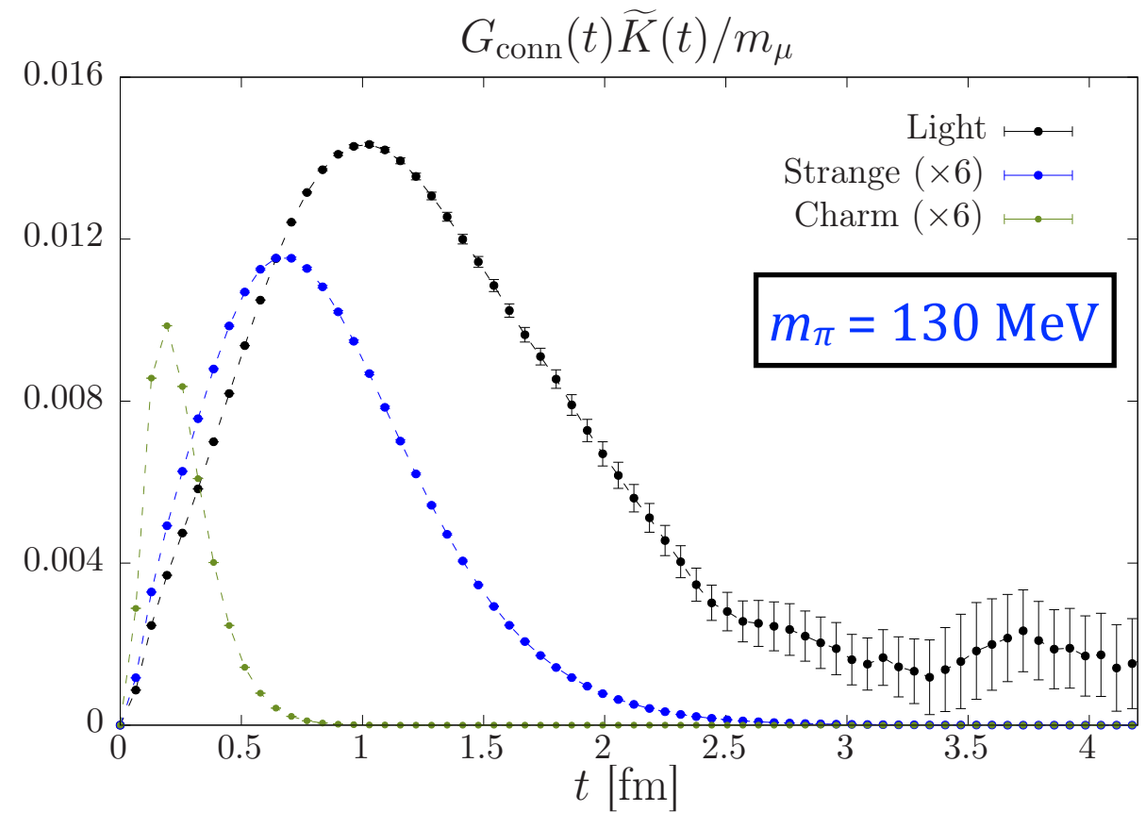


Controlling the infrared regime

* TMR integrand and its long-distance behaviour:

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{K}(x_0)\, G(x_0), \qquad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle$$





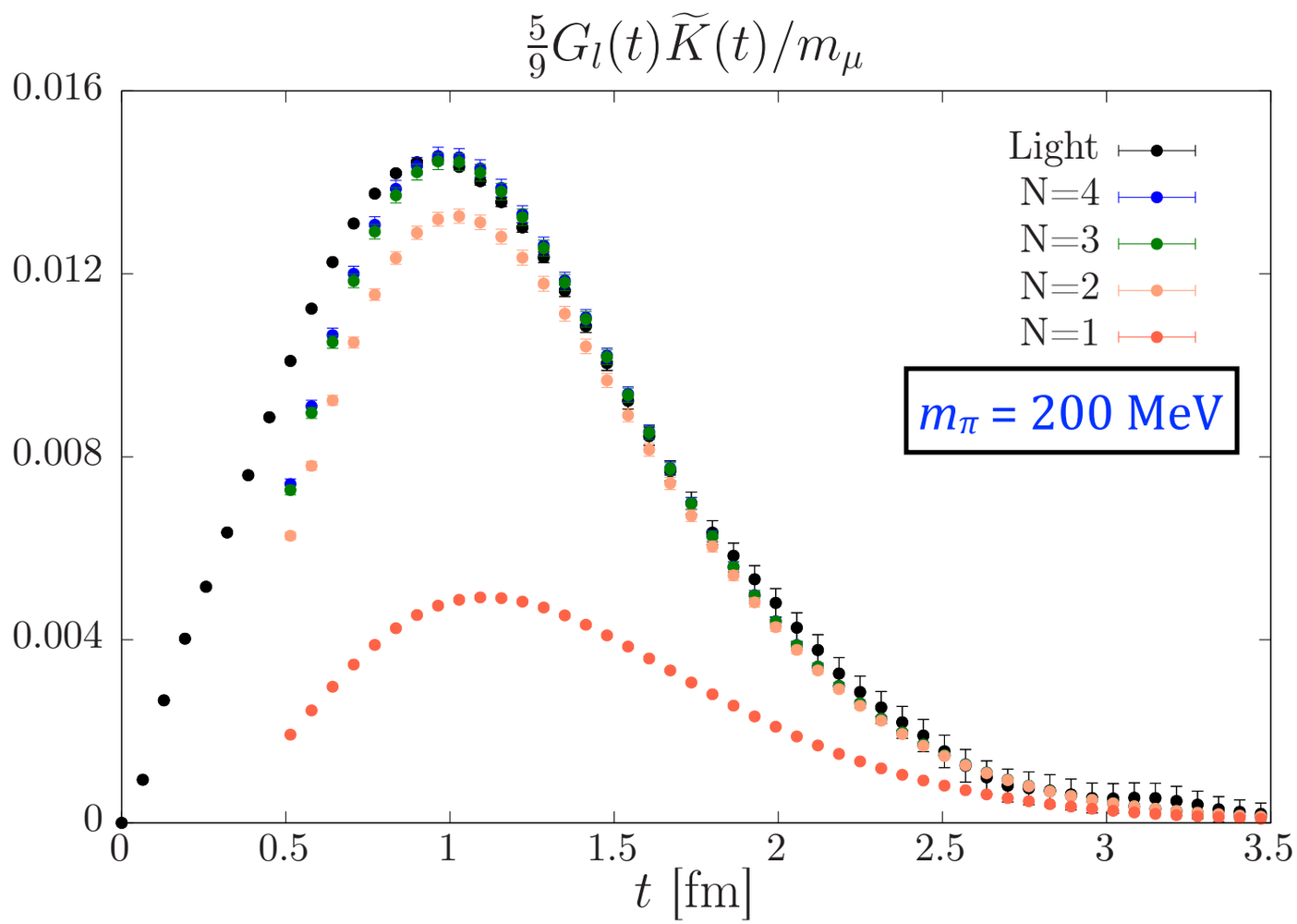
* Large-x_0 regime still statistics-limited for $x_0 \gtrsim 2.5$ fm

[Gérardin et al., arXiv:1904.03120]

Isovector channel

* Saturation of the isovector correlator by low-lying states

$$G(x_0) = \sum_{n=1}^{\infty} A_n \, \mathrm{e}^{-\omega_n x_0} \qquad \text{as } x_0 \to \infty \qquad \omega_n = 2\sqrt{m_\pi^2 + k^2}$$



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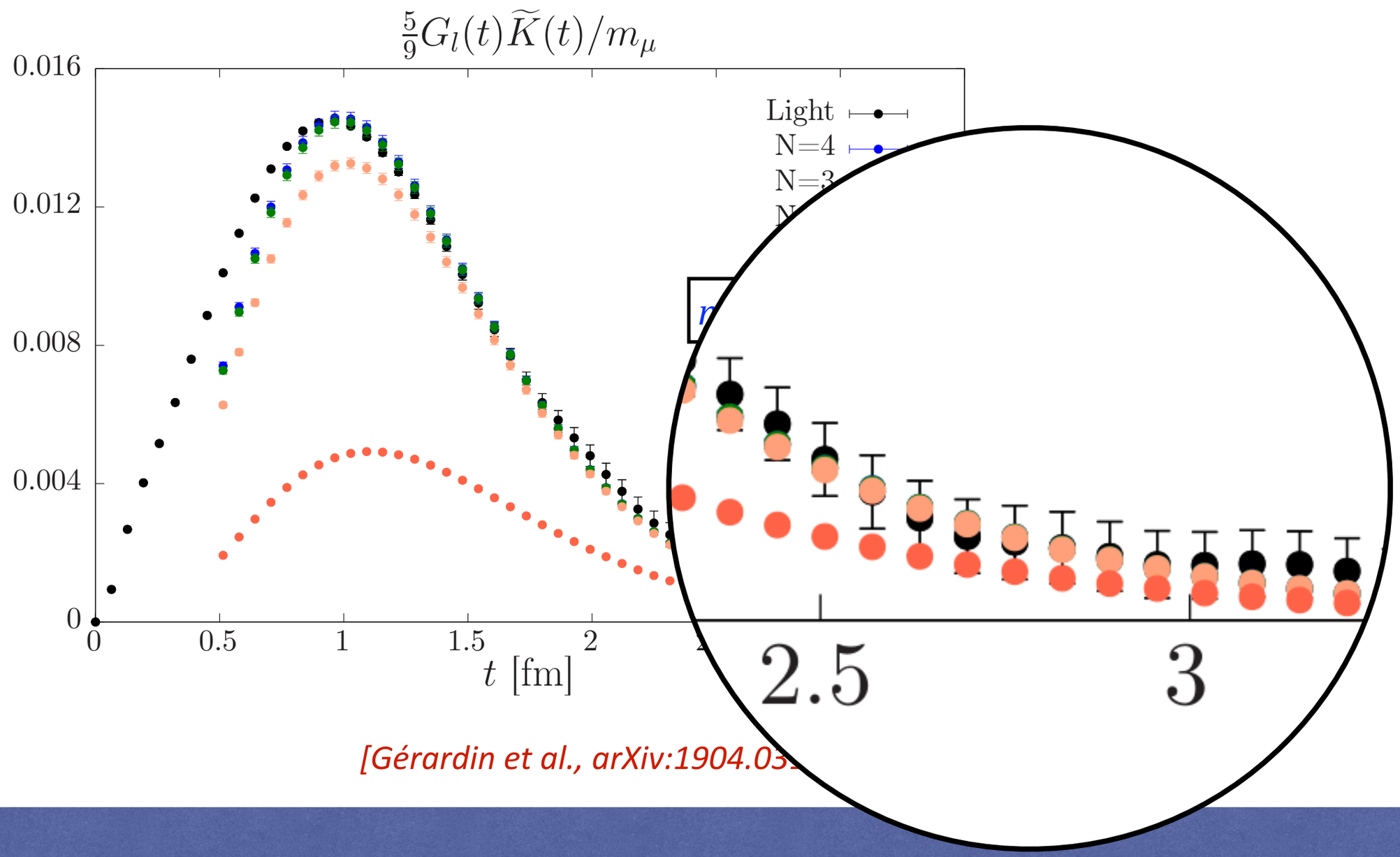


[Gérardin et al., arXiv:1904.03

Finite-volume effects

* Finite-volume correction

$$a_\mu^{\rm hvp}(\infty) - a_\mu^{\rm hvp}(L) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{K}(x_0)\,[G(x_0,\infty) - G(x_0,L)]$$

* Finite volume:

$$G^{\rho\rho}(x_0,L) \overset{x_0\to\infty}{=} \sum_n |A_n|^2\, {\rm e}^{-\omega_n x_0}, \qquad |A_n|^2 = \frac{2k^5}{3\pi\omega^2}\frac{|F_\pi(\omega)|^2}{\left\{k\phi'(k) + k\delta_1'(k)\right\}}$$

* Iso-vector correlator in infinite volume

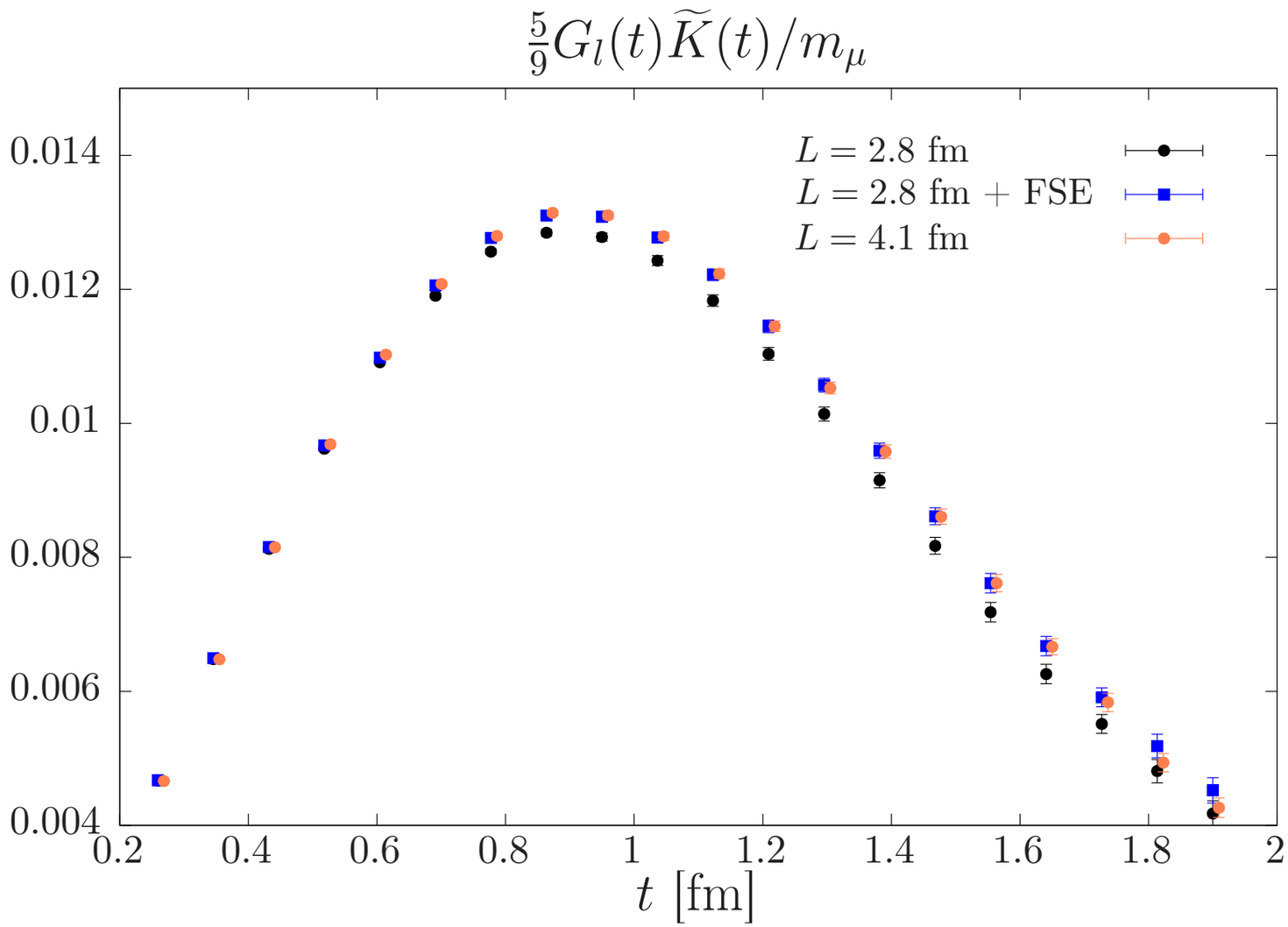
$$G^{\rho\rho}(x_0,\infty) = \int_0^\infty d\omega\, \omega^2 \rho(\omega^2) {\rm e}^{-\omega|x_0|}, \quad \rho(\omega^2) = \frac{1}{48\pi^2}\left(1 - \frac{4m_\pi^2}{\omega^2}\right)^{3/2} |F_\pi(\omega)|^2$$

* Timeline pion form factor: $F_\pi(\omega)$

[Gérardin et al., arXiv:1904.03120]

Finite-volume effects

* Dedicated study for $m_\pi = 280\,\mathrm{MeV}$ (H105, N101: $m_\pi L = 3.8,\ 5.8$)



[Gérardin et al., arXiv:1904.03120]

Finite-volume effects

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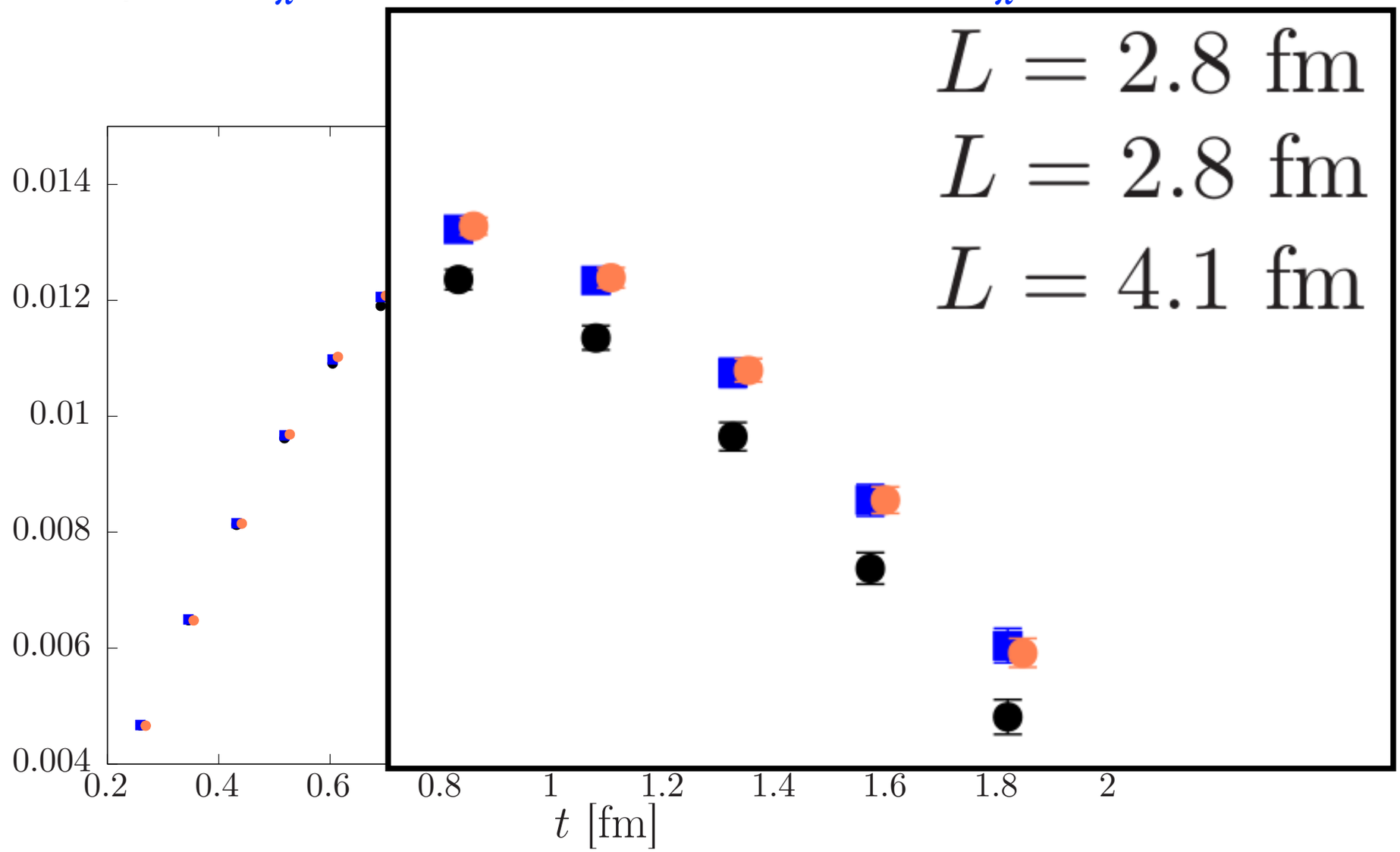
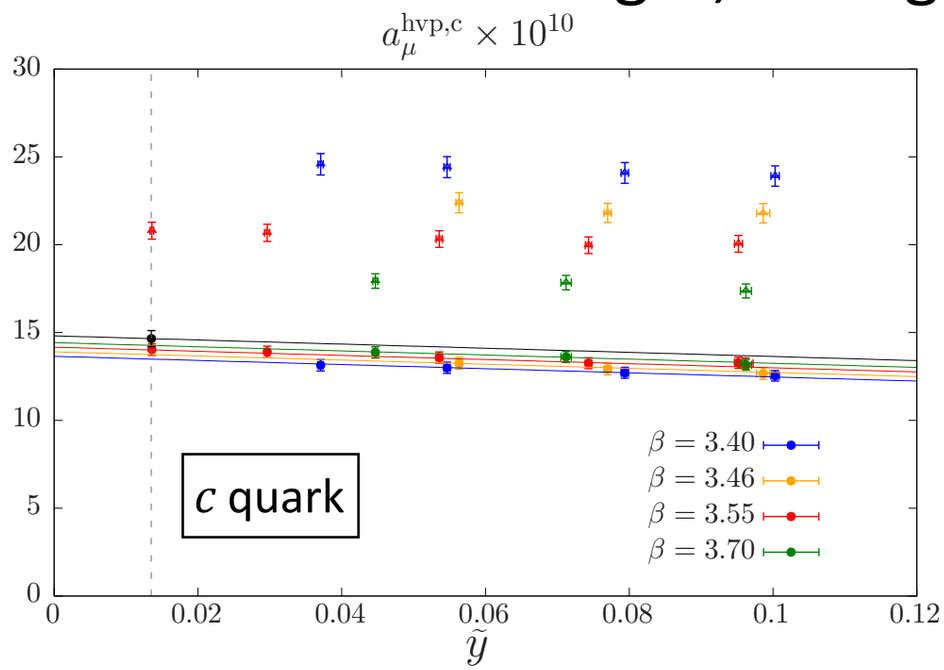


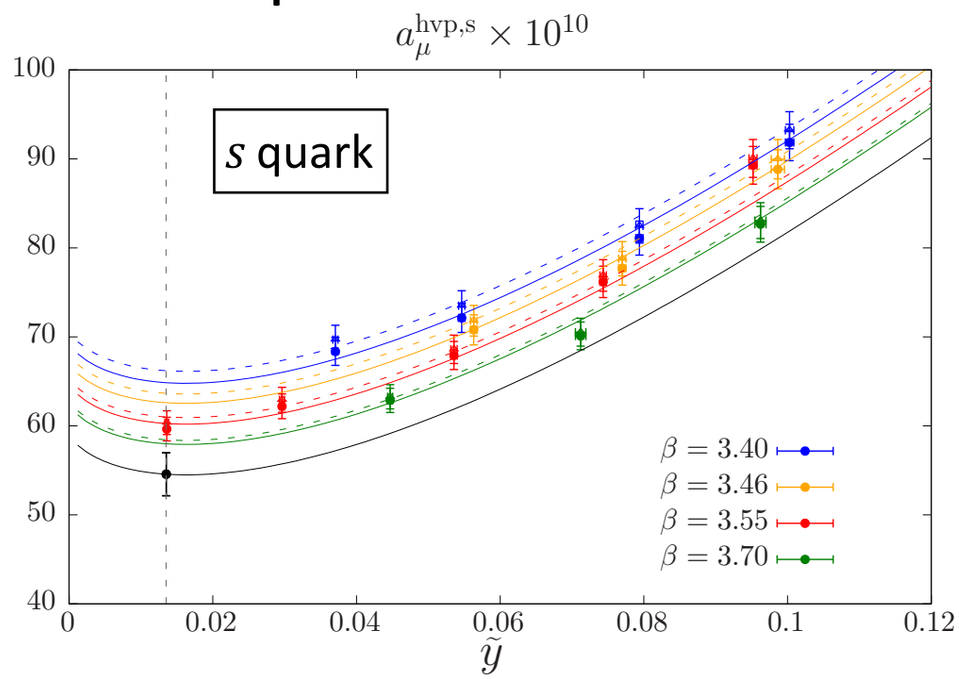
* Finite-size effects well described by parameterisation of $F_\pi(\omega)$

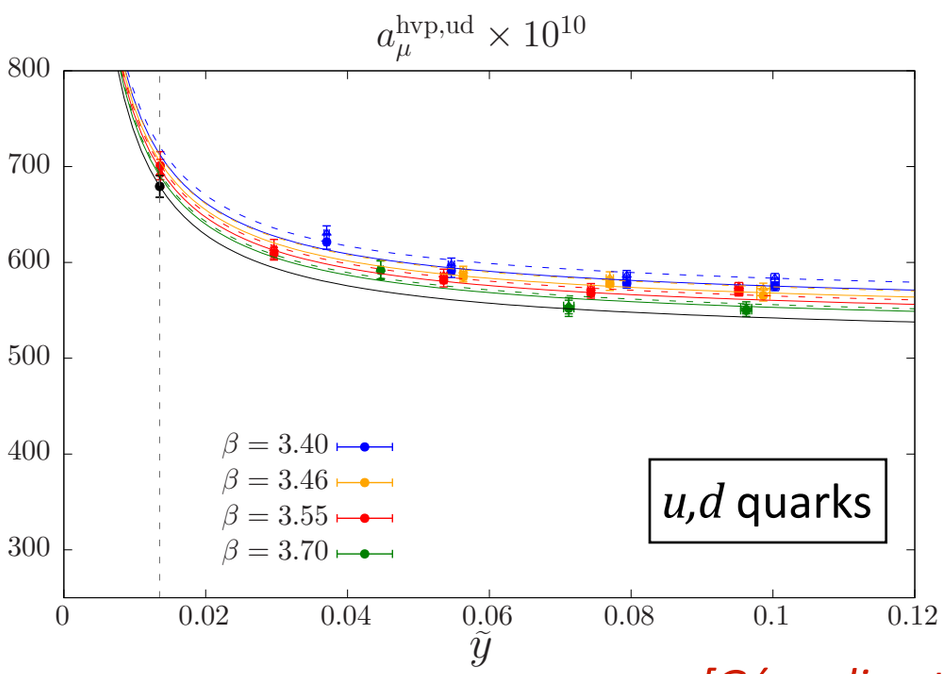
[Gérardin et al., arXiv:1904.03120]

Extrapolation to the physical point

* Contributions from light, strange and charm quarks







$$(a_\mu^{\mathrm{hvp}})^c = 14.66(45)(6) \times 10^{-10}$$

$$(a_\mu^{\mathrm{hvp}})^s = 54.5(2.4)(0.6) \times 10^{-10}$$

$$(a_\mu^{\mathrm{hvp}})^{ud} = 674(12)(5) \times 10^{-10}$$

Error limited by lattice scale

[Gérardin et al., arXiv:1904.03120]

Extrapolation to the physical point

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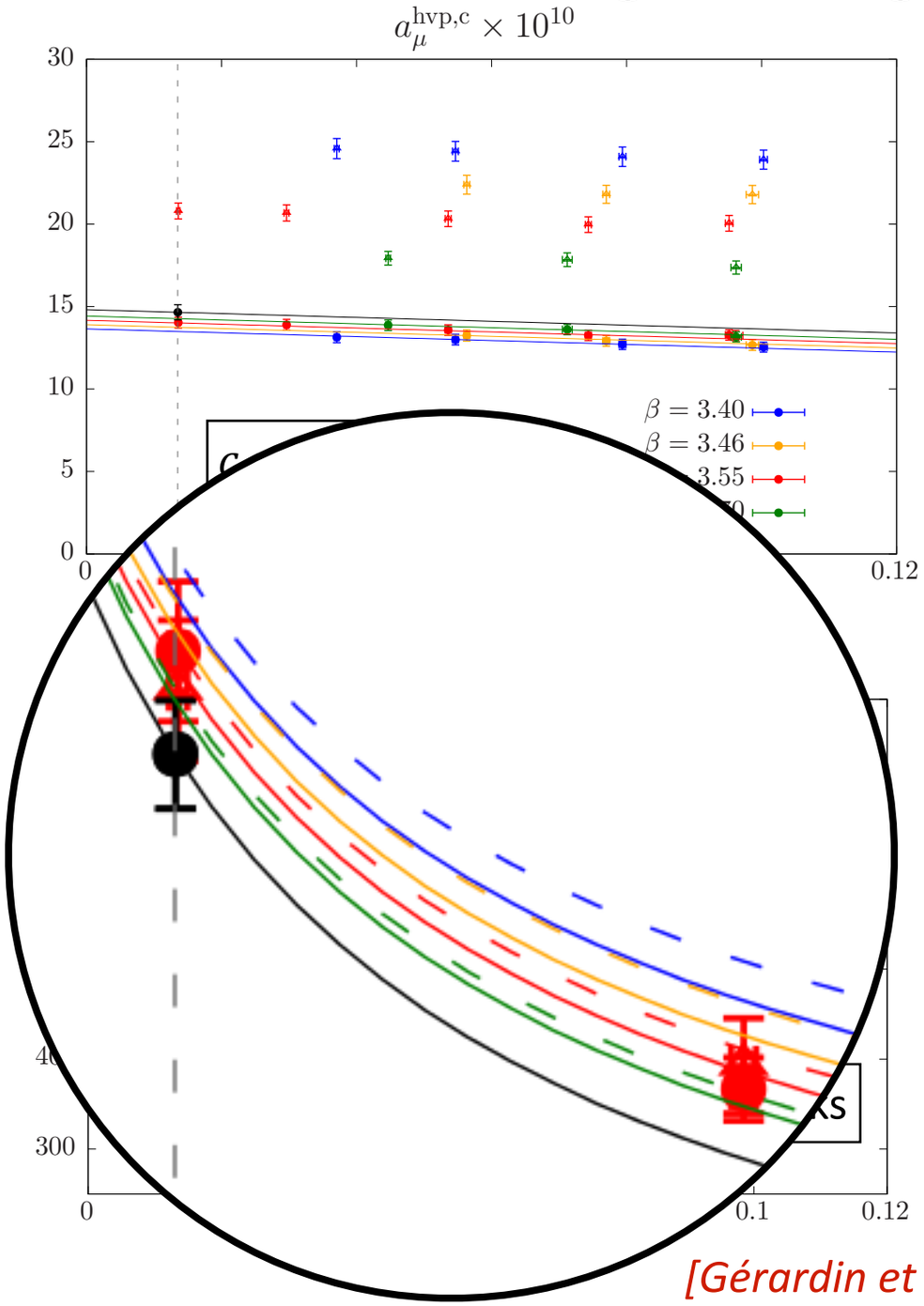

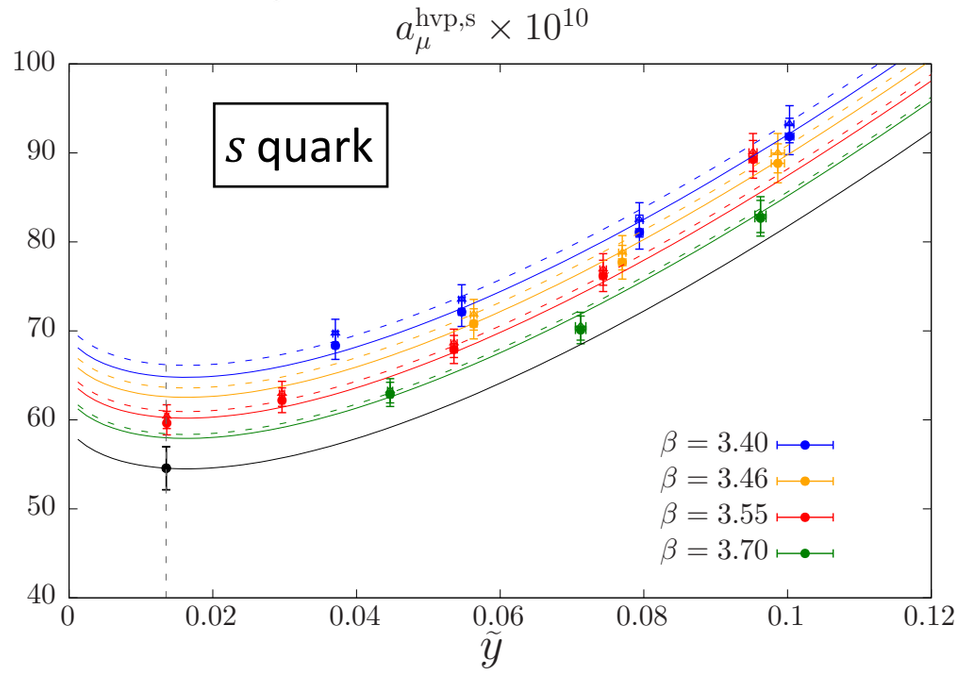

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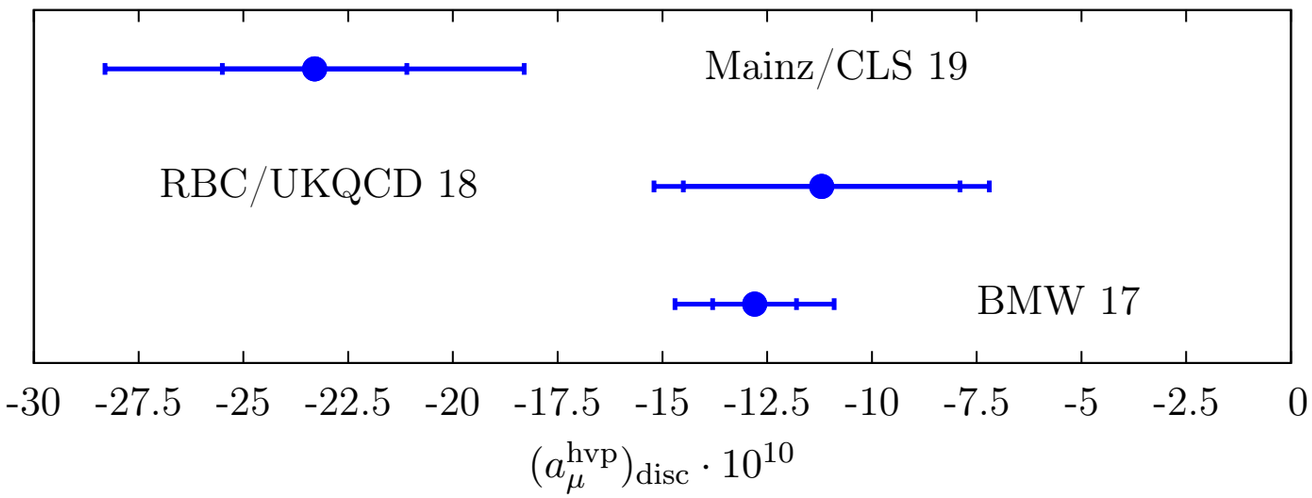
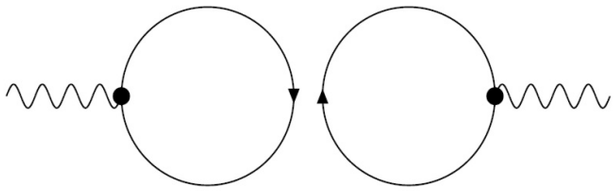
[Gérardin et al., arXiv:1904.03120]

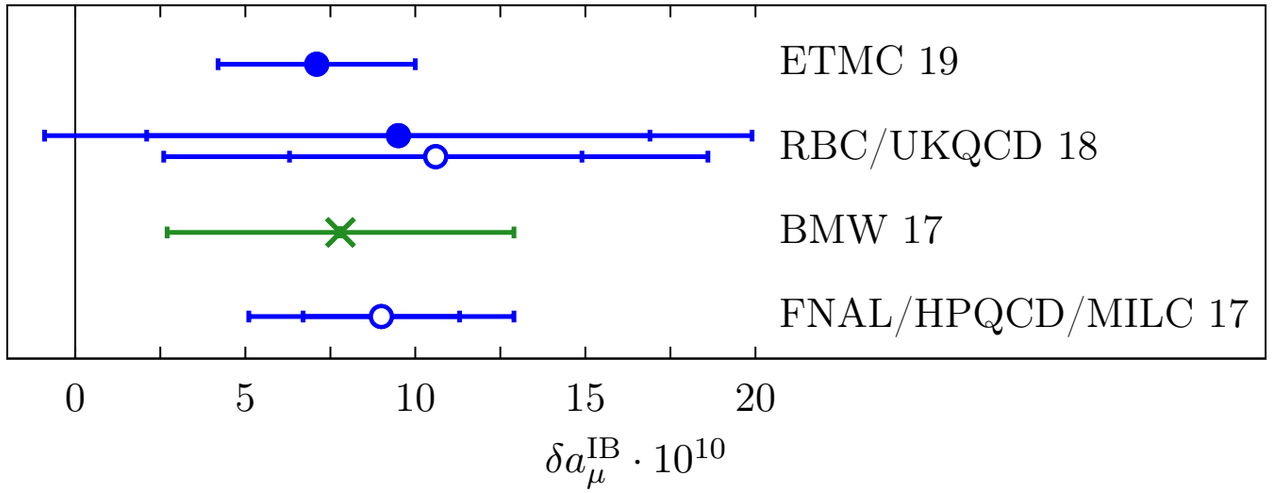
Towards sub-percent accuracy

* Quark-disconnected diagrams





* Isospin breaking corrections



$m_u/m_d = 0.46(2)(2)$

$q_u = 2/3, \quad q_d = -1/3$

Isospin breaking effects

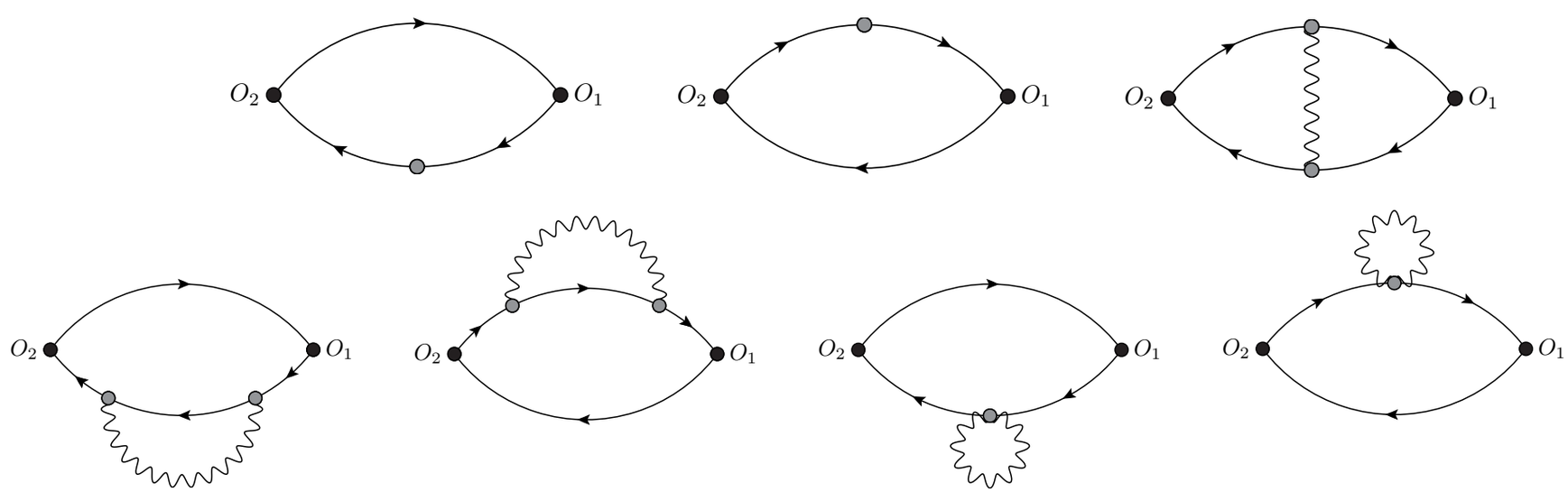
* Simulate QCD+QED for $m_u \neq m_d$ *[Boyle, Gülpers et al., arXiv:1706.05293]*

$$\langle \Omega \rangle = \frac{1}{Z} \int D[A]\, D[U]\, D[\bar{\psi}, \psi]\; \Omega\; \mathrm{e}^{-S_\gamma[A] - S_{\mathrm{G}}[U] - S_{\mathrm{F}}[A,U,\bar{\psi},\psi]}$$

Generate $U(1)$ gauge field stochastically and multiply $\mathrm{SU}(3)$ link variables:

$$U_\mu(x) \longrightarrow \mathrm{e}^{ieA_\mu(x)} U_\mu(x) \qquad \text{(electro-quenched)}$$

* Expand correlator in powers of $(m_d - m_u)$ and $\alpha = e^2/4\pi$



[Divitiis et al., arXiv:1110.6294, arXiv:1303.4896; Giusti et al., arXiv:1704.06561; Risch, HW, arXiv:1811.00895]

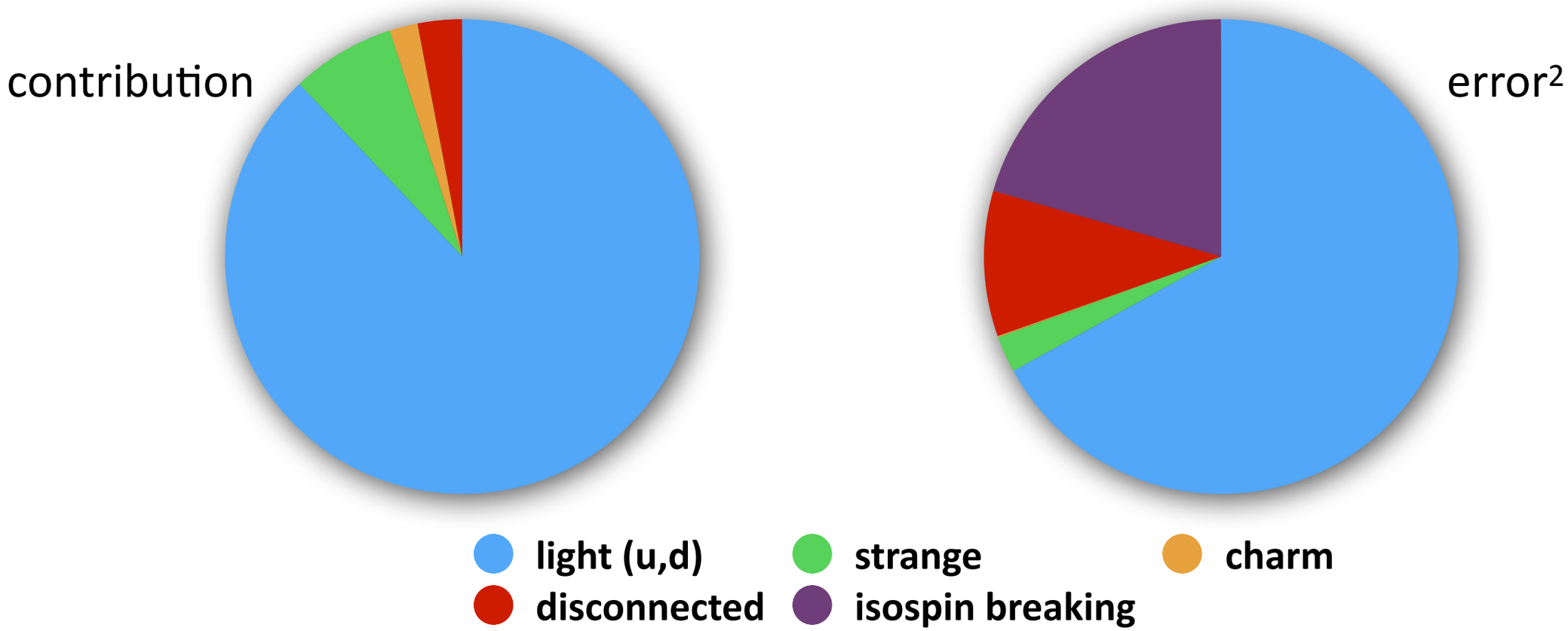
Results and comparison

* Final result at the physical point

$$a_\mu^{\rm hvp} = (720 \pm 12.4_{\rm stat} \pm 9.9_{\rm syst}) \times 10^{-10}$$

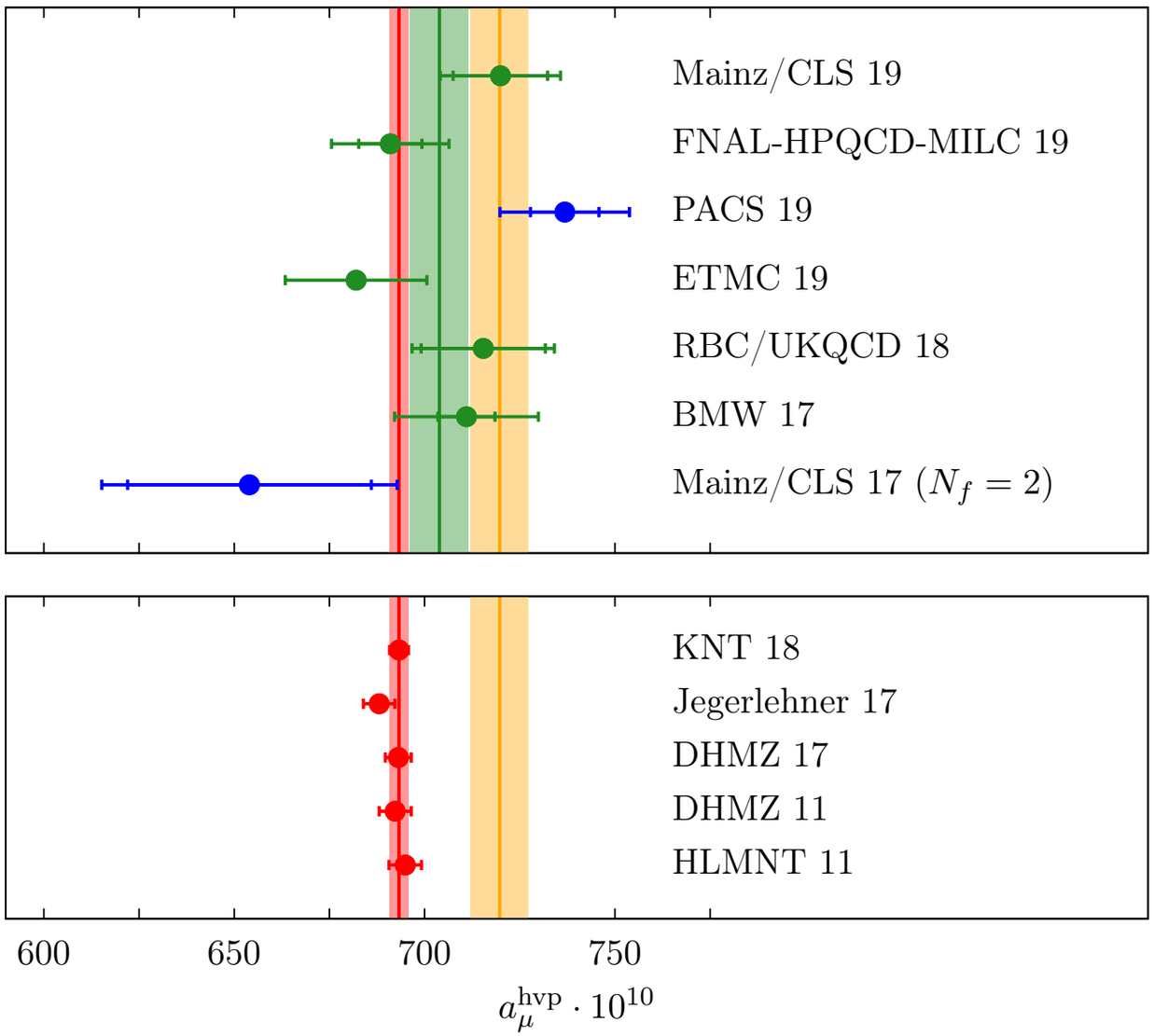
(total uncertainty: 2.2%)

* Individual contributions:



Results and comparison

* **Lattice QCD** vs. **dispersion theory**:



Our result:

$$a_\mu^{\rm hvp} = (720.0 \pm 15.8) \cdot 10^{-10}$$

Dispersion theory:

$$(a_\mu^{\rm hvp})_{\rm disp} = (693.3 \pm 2.5) \cdot 10^{-10}$$

Global lattice average:

$$(a_\mu^{\rm hvp})_{\rm lat} = (703.9 \pm 7.7) \cdot 10^{-10}$$

"No new physics":

$$(a_\mu^{\rm hvp})_{\rm NNP} = (a_\mu^{\rm hvp})_{\rm disp} + (a_\mu^{\rm exp} - a_\mu^{\rm SM})$$

Lattice QCD approaches to HLbL

* Matrix element of e.m. current between muon initial and final states:

$$\left\langle \mu(\boldsymbol{p}', s') \left| J_\mu(0) \right| \mu(\boldsymbol{p}, s) \right\rangle = -e\, \overline{u}(\boldsymbol{p}', s') \left(F_1(Q^2)\gamma_\mu + \frac{F_2(Q^2)}{2m} \sigma_{\mu\nu} Q_\nu \right) u(\boldsymbol{p}, s)$$

$$a_\mu^{\text{hlbl}} = F_2(0)$$

RBC/UKQCD:

* QCD + QED simulations *[Hayakawa et al. 2005; Blum et al. 2015]*
* QCD + stochastic QED *[Blum et al. 2016, 2017]*

Mainz group:

* Exact QED kernel in position space *[Asmussen et al. 2015, 2016, and in prep.]*
* Transition form factors of sub-processes *[Gérardin, Meyer, Nyffeler 2016]*
* Forward scattering amplitude *[Green et al. 2015, 2017]*

QCD + Stochastic QED

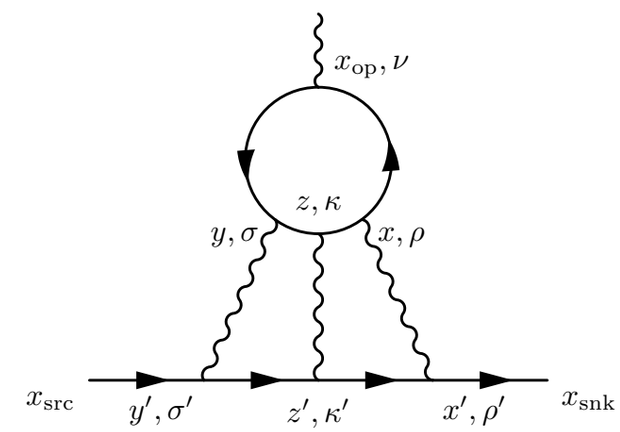
* Stochastic treatment of QED contribution:
  ⇒ insertion of three exact Feynman gauge photon propagators

$$G_{\mu\nu}(x,y) = \frac{1}{VT}\delta_{\mu\nu}\sum_{k,\,|\vec{k}|\neq 0}\frac{e^{ik\cdot(x-y)}}{\hat{k}^2}$$

[Blum et al., Phys Rev D93 (2016) 014503]

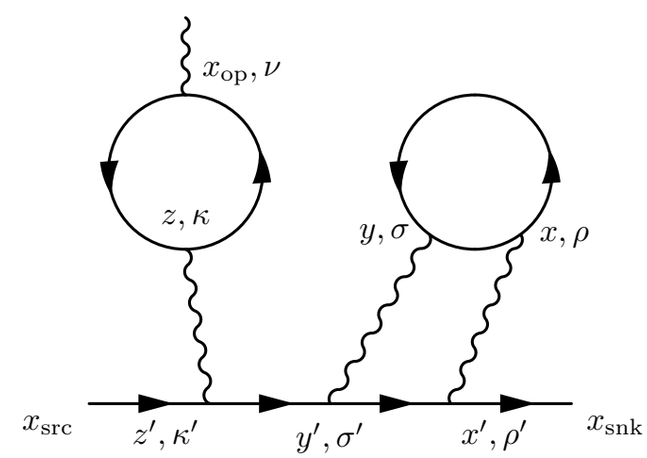
* Connected contribution:

$$(a_\mu^{\rm hlbl})_{\rm con} = (116.0 \pm 9.6)\cdot 10^{-11}$$



* Leading disconnected contribution:

$$(a_\mu^{\rm hlbl})_{\rm disc} = (-62.5 \pm 8.0)\cdot 10^{-11}$$



* Compute sub-leading disconnected diagrams
* Study general systematic effects

[Blum et al., Phys Rev Lett 118 (2017) 022005]

Exact QED kernel in position space

* Determine QED part perturbatively in the continuum in infinite volume

⇒ no power-law volume effects

$$a_\mu^{\rm hlbl} = F_2(0) = \frac{me^6}{3}\int d^4y \int d^4x\, \overline{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\, i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y)$$

* QCD four-point function: $i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\, z_\rho \left\langle J_\mu(x) J_\nu(y) J_\sigma(z) J_\lambda(0)\right\rangle$

* QED kernel function: $\overline{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ *[Asmussen, Green, Meyer, Nyffeler, in prep.]*
  - Infra-red finite; can be computed semi-analytically
  - Admits a tensor decomposition in terms of six weight functions which depend on $x^2, \quad y^2, \quad x\cdot y$

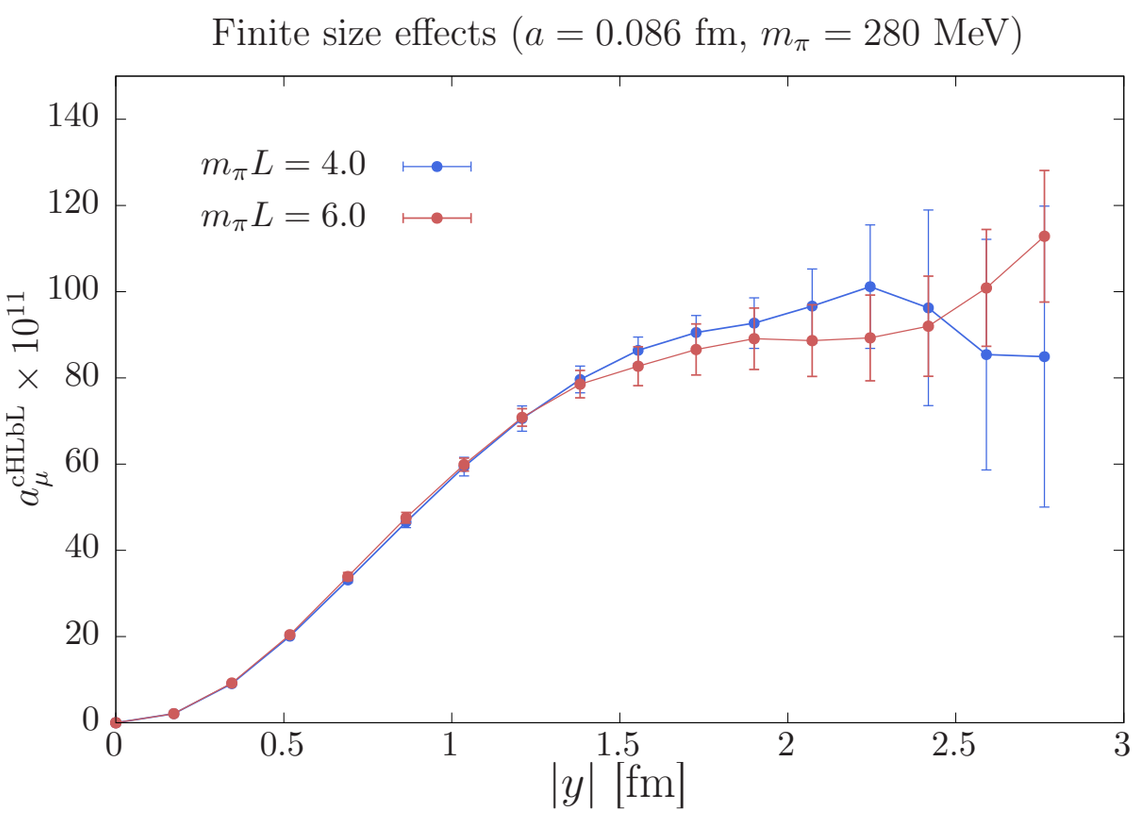
  ⇒ 3D integration instead of $\int d^4x \int d^4y$

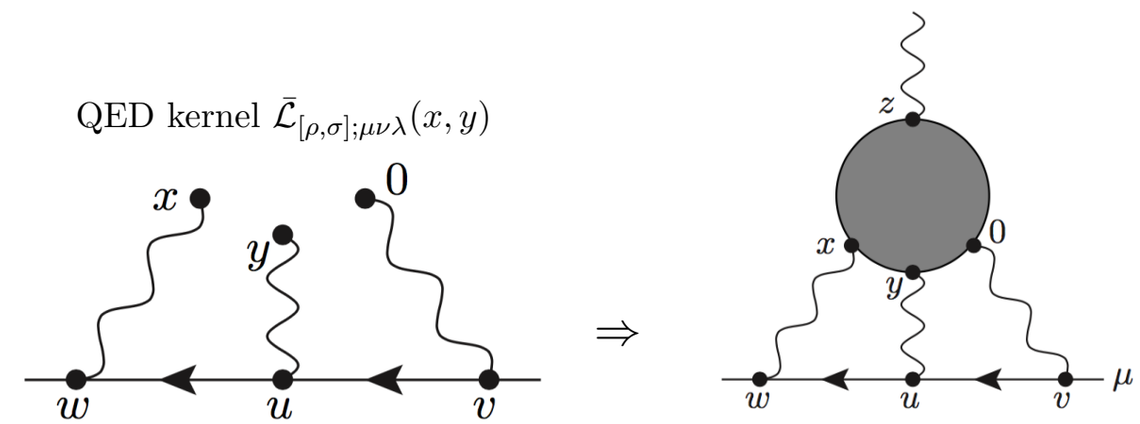
* Weight functions computed and stored on disk

Preliminary results

* Accumulated connected contribution

$$a_\mu^{\text{chlbl}} = \frac{me^6}{3}\, 2\pi^2 \int_0^{y_{\max}} |y|^3\, d|y| \int d^4x\, \overline{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\, i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y)$$



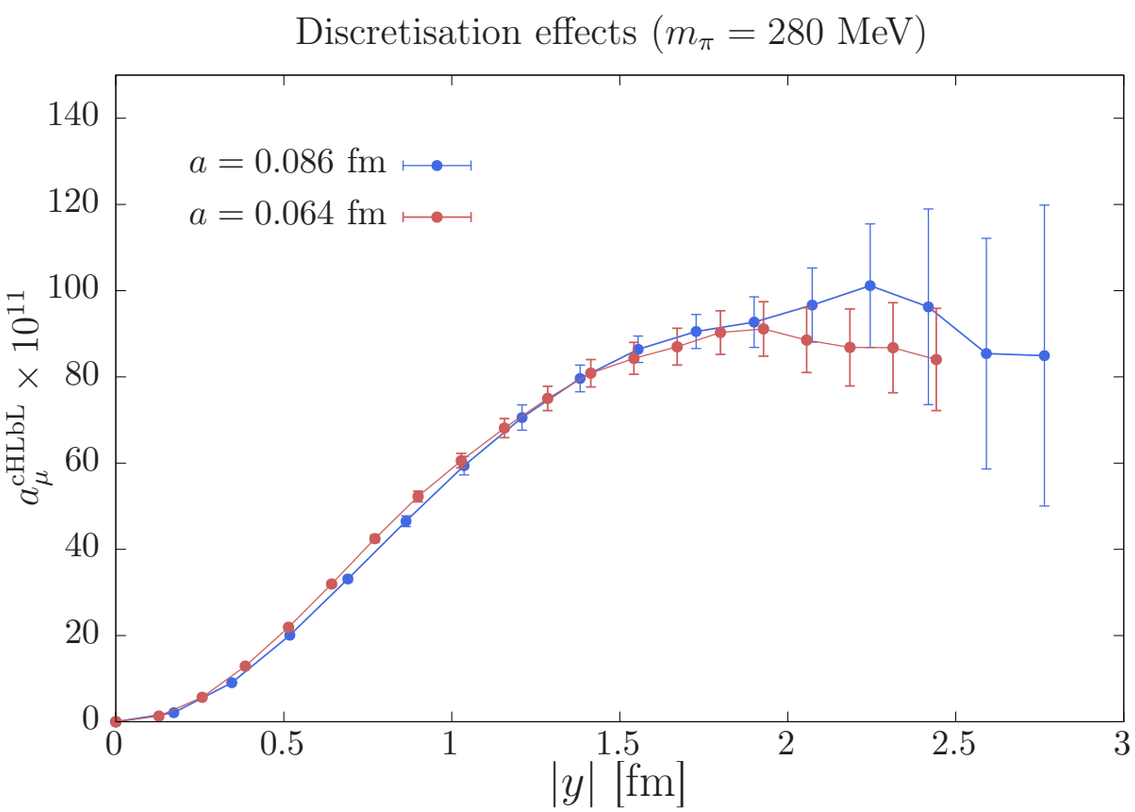


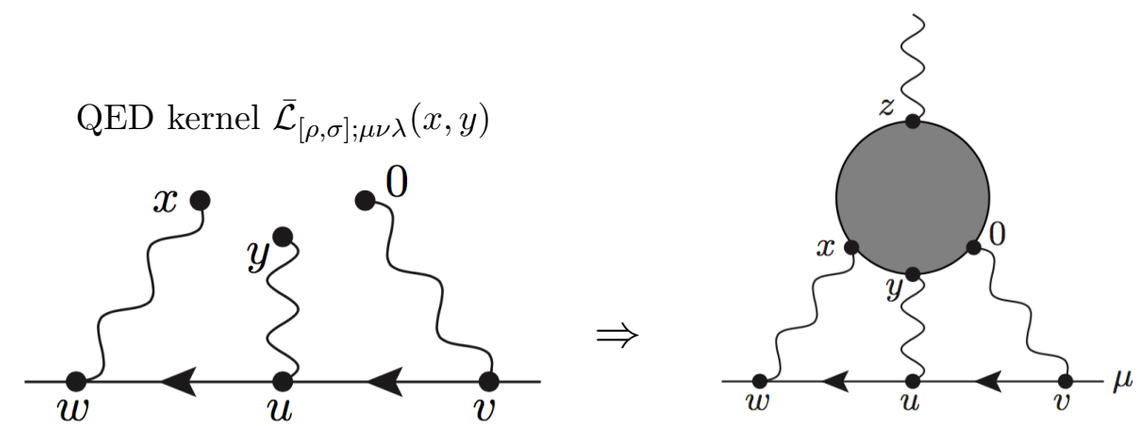
[Asmussen, Gérardin, Nyffeler, Meyer, arXiv:1811.08320]

Preliminary results

* Accumulated connected contribution

$$a_\mu^{\text{chlbl}} = \frac{me^6}{3}\, 2\pi^2 \int_0^{y_{\max}} |y|^3\, d|y| \int d^4x\, \overline{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\, i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y)$$



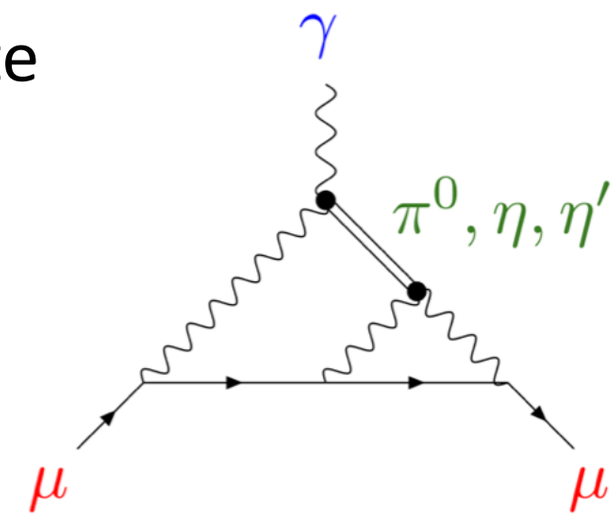


[Asmussen, Gérardin, Nyffeler, Meyer, arXiv:1811.08320]

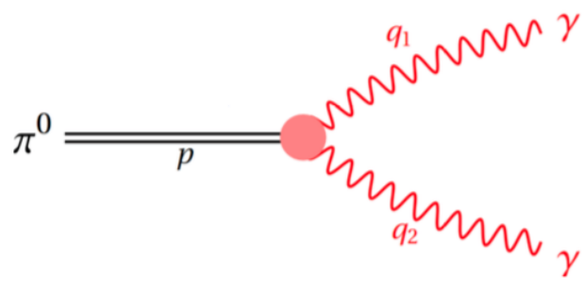
* Controlled discretisation and finite-volume effects

Transition form factor $\pi^0 \to \gamma^*\gamma^*$

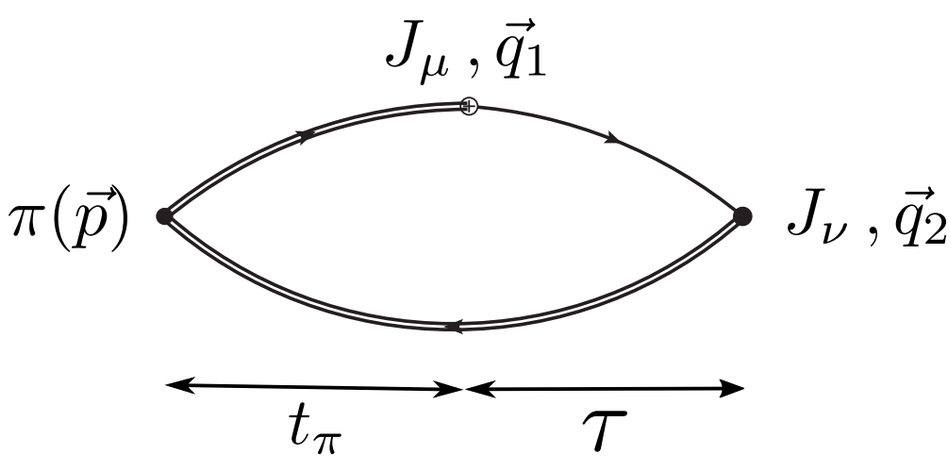
* Pseudoscalar meson exchange expected to dominate LbL scattering *[Nyffeler, arXiv:1602.03737]*



* Compute transition form factor between π^0 and two off-shell photons:



$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2; q_1^2, q_2^2) \equiv M_{\mu\nu}$$



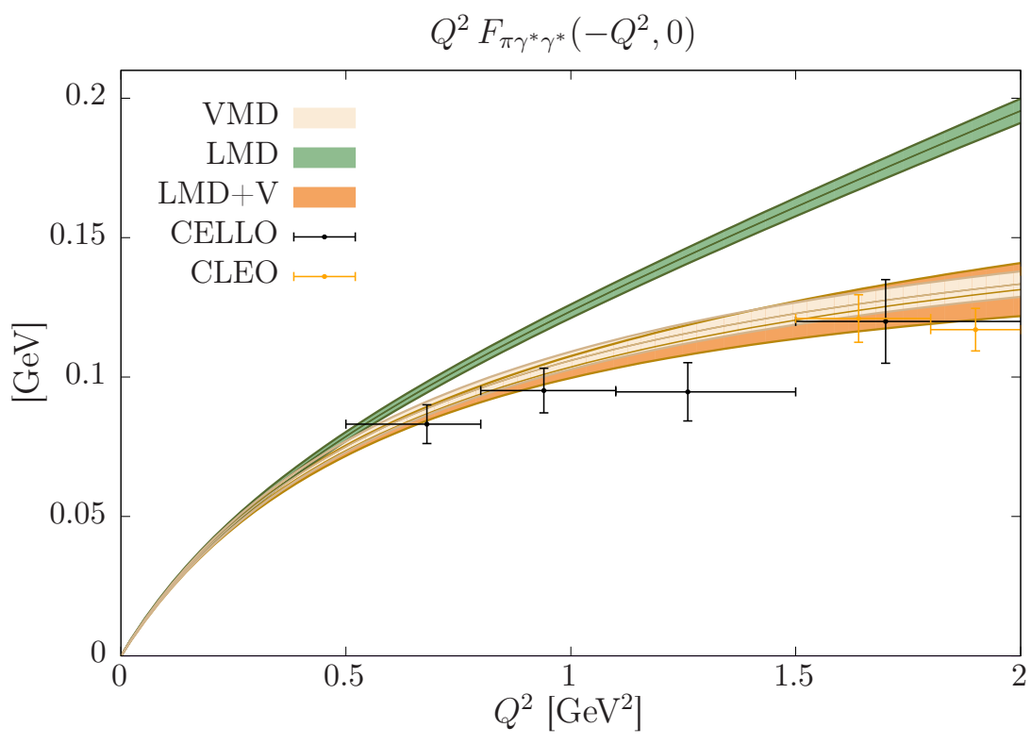
$$M_{\mu\nu} \sim C^{(3)}_{\mu\nu}(\tau, t_\pi; \vec{p}, \vec{q}_1, \vec{q}_2) =$$

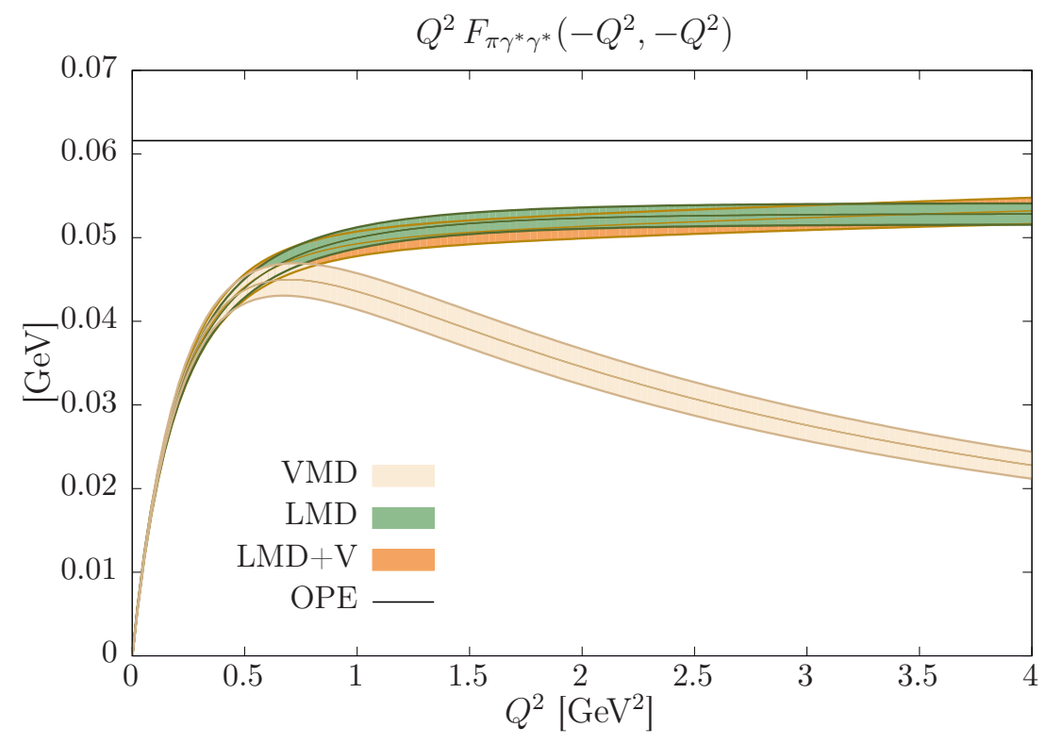
$$\sum_{\vec{x},\vec{z}} \left\langle T\left\{J_\nu(\vec{0}, \tau + t_\pi) J_\mu(\vec{z}, t_\pi) P(\vec{x}, 0)\right\}\right\rangle \mathrm{e}^{i\vec{p}\cdot\vec{x}} \mathrm{e}^{-i\vec{q}_1\cdot\vec{z}}$$

* Kinematics: $\vec{p} = 0, \quad q_1^2 = \omega_1^2 - |\vec{q}_1|^2, \quad q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$

Transition form factor $\pi^0 \to \gamma^*\gamma^*$

* Fit Q^2-dependence of $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ to several model *ansätze;* extrapolate to the physical point



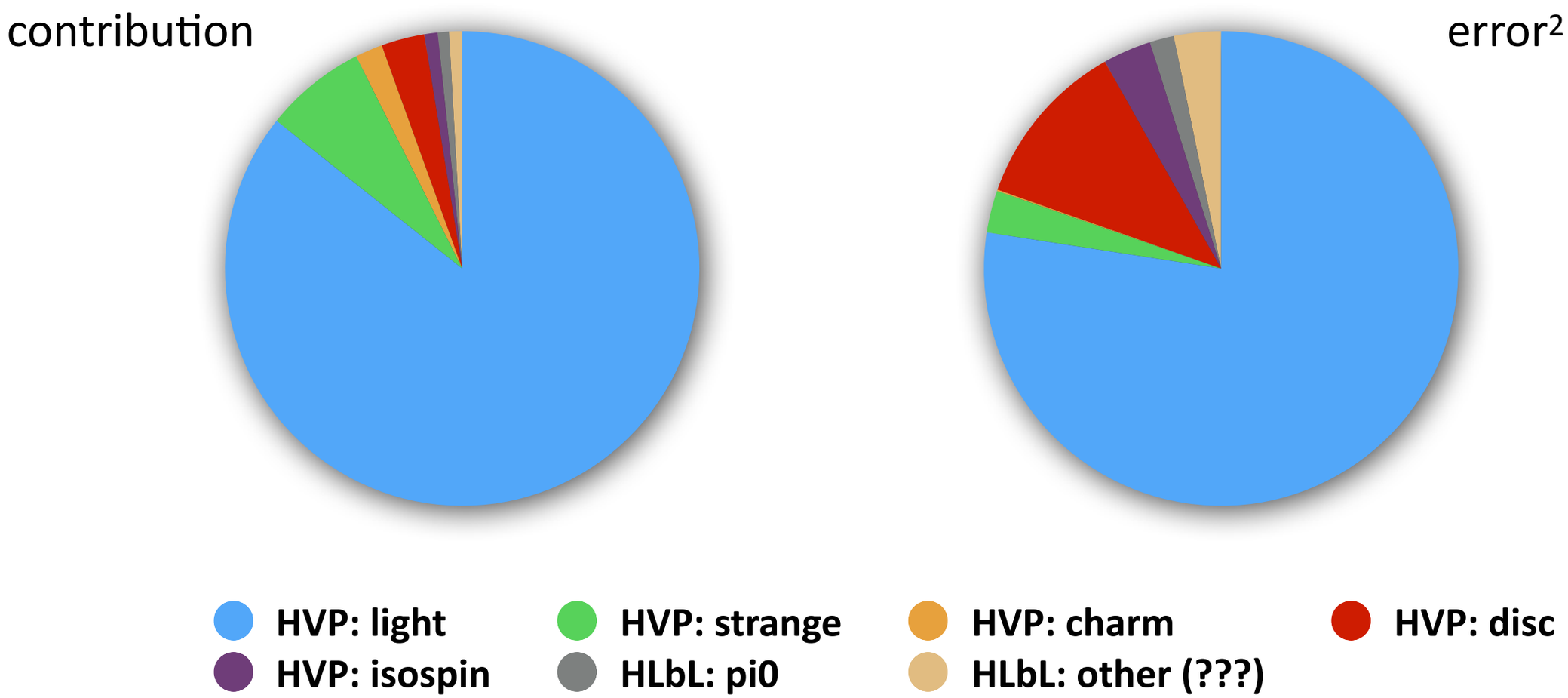


* Results for π^0 contribution to hadronic light-by-light scattering:

$$(a_\mu^{\text{hlbl}})_{\pi^0} = \begin{cases} (59.7 \pm 3.6) \cdot 10^{-11} & \text{Lattice QCD} \quad \textit{[Gérardin et al., arXiv:1903.09471]} \\ (62.6\,^{+3.0}_{-2.5}) \cdot 10^{-11} & \text{Dispersion theory} \quad \textit{[Hoferichter et al., arXiv:1903.09471]} \end{cases}$$

Total hadronic contribution

* Combine lattice estimates for the HVP and HLbL contributions
  — very preliminary!



Summary & Outlook

Muon anomalous magnetic moment

- One of the most promising hints for new physics
- Beautiful interplay between theory and experiment
- Numerous technical and computational challenges
- New experiments will significantly increase sensitivity
- Theory must keep pace

Lattice QCD

- Provides model-independent estimates for hadronic contributions
- HVP: difficult to reach sub-percent precision
- HLbL: 10–15% calculation will have great impact